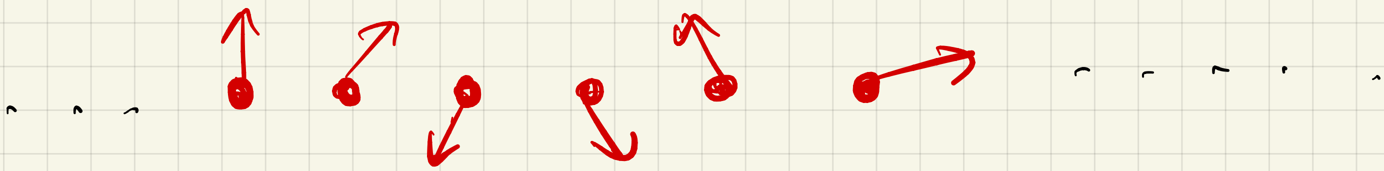
6. Higher Berry curvatures.

Immediate problem for $d > 0$:

$F \in \Omega^2(M)$ is ill-defined in infinite volume.

(decoupled spins)



$$\lim_{V \to \infty} F = \infty.$$

(Technical issue: GNS Hilbert spaces for different λ are not naturally isomorphic)

Field theory suggests that there must be a closed $(d+2)$-form $F^{(d+2)} \in \Omega^{d+2}(M)$ with quantized periods:

the <u>Wess-Zumino-Witten</u> form.

Promote λ to slowly varying classical field $\lambda(\vec{x}, t) : \mathbb{R}^{d+1} \to M$

Compactify $\mathbb{R}^{d+1}$ to S^{d+1}.

$$S_{eff} = \int \lambda^* \omega = \frac{1}{(d+1)!} \int \omega_{i_1 \dots i_{d+1}} d\lambda^{i_1} \dots d\lambda^{i_{d+1}}$$

$$= \int_{D^{d+2}} \lambda^* \Omega \quad , \quad \Omega \in \Omega^{d+2}(M).$$

$$\Omega = d\omega$$

ω need not be globally well-defined on M (it can be a "d-gerbe connection").

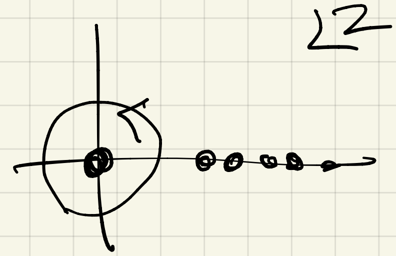
But Ω is a well-defined closed form.

$$\int_{S^{d+2}} \lambda^* \Omega \in 2\pi \mathbb{Z} \quad \forall \lambda : S^{d+2} \to M.$$

($\left[\frac{\Omega}{2\pi}\right]$ is "quantized on spherical cycles").

7. W2W classes for families of 1d lattice systems (A. Kitaev, unpublished. L. Spodyneiko & AK, 2001.03454)

$$H(\lambda) = \sum_p H_p(\lambda), \quad p \in \mathbb{Z}$$



Ordinary Berry curvature:

$$\Omega^{(2)} = \frac{i}{2} \oint \frac{dz}{2\pi i} \operatorname{Tr}\left(\frac{1}{z-H}\, dH \frac{1}{(z-H)^2}\, dH \right)$$

d: exterior derivative on M

$G(z) = \dfrac{1}{z-H}$ is a bounded operator if $z \notin$ spectrum of H

But $\Omega^{(2)}$ is not well-defined:

$$\Omega^{(2)} = \sum_{p,q \in \Lambda} \Omega^{(2)}_{pq}, \quad \text{where}$$

$$\Omega^{(2)}_{pq} = \frac{i}{2} \oint \frac{dz}{2\pi i} \operatorname{Tr}\left(G(z)\, dH_p\, G^2(z)\, dH_q \right)$$

Which infinite sums over $\Lambda = \mathbb{Z}$ are well-defined?

Let

$$\langle A ; B \rangle = \langle 0 | A B | 0 \rangle - \langle 0 | A | 0 \rangle \langle 0 | B | 0 \rangle$$

<u>Th</u> (Hastings-Koma, Nachtergaele-Sims)

$\langle A_p ; B_q \rangle$ decays rapidly as $|p-q| \to \infty$.

Let $\quad G_0 = (1-P) \frac{1}{H} (1-P), \quad P = |0\rangle\langle 0|$

<u>Th</u> (H. Watanabe)

$\langle 0 | A_p G_0^n B_q | 0 \rangle$ decays rapidly as $|p-q| \to \infty$

$n > 0$

$$\Omega^{(2)}_{pq} \sim \langle 0 | dH_p \, G_0^2 \, dH_q | 0 \rangle - (p \leftrightarrow q)$$

Hence $\Omega^{(2)}_{pq}$ decays rapidly as $|p-q| \to \infty$

$$\Omega^{(2)} = \sum_{p,q} \Omega^{(2)}_{pq}$$ is divergent, but

$$F^{(2)}_q = \frac{i}{2} \oint \frac{dz}{2\pi i} \operatorname{Tr}\left(G\, dH\, G^2\, dH_q\right) =$$

$$= \sum_p \Omega^{(2)}_{pq}$$

is well-defined.

It is not closed, but

$$dF^{(2)}_q = \sum_p F^{(3)}_{pq} \,.$$

Here $F^{(3)}_{pq}$ is a 3-form on M which satisfies:

- $F^{(3)}_{pq} = -F^{(3)}_{qp}$
- $F^{(3)}_{pq}$ decays rapidly as $|p-q| \to \infty$.

An unilluminating explicit formula:

$$F^{(3)}_{pq} = \frac{i}{6}\oint \frac{dz}{2\pi i} \mathrm{Tr}\,\big(G^2 dH\, G\, dH_p\, G\, dH_q - G\, dH\, G^2 dH_p\, G\, dH_q\big) - (p \leftrightarrow q)$$

The equation $dF^{(2)}_q = \sum_p F^{(3)}_{pq}$ was first observed by Kitaev for classical lattice systems.

$dF^{(3)}_{qr} = \sum_{p \in \Lambda} F^{(4)}_{pqr}$ for some 4-form $F^{(4)}_{pqr}$ which is

- completely anti-symmetric in p, q, r.
- decays rapidly away from $p=q=r$

Now we can construct the WZW 3-form as follows.

Let $f: \mathbb{Z} \to \mathbb{R}$ be a function such that $f(p) = 1$ for $p \gg 0$.

$f(p) = 0$ for $p \ll 0$.

Let

$$\Omega^{(3)}(f) = \frac{1}{2} \sum_{p,q} F^{(3)}_{pq} (f(q) - f(p)).$$

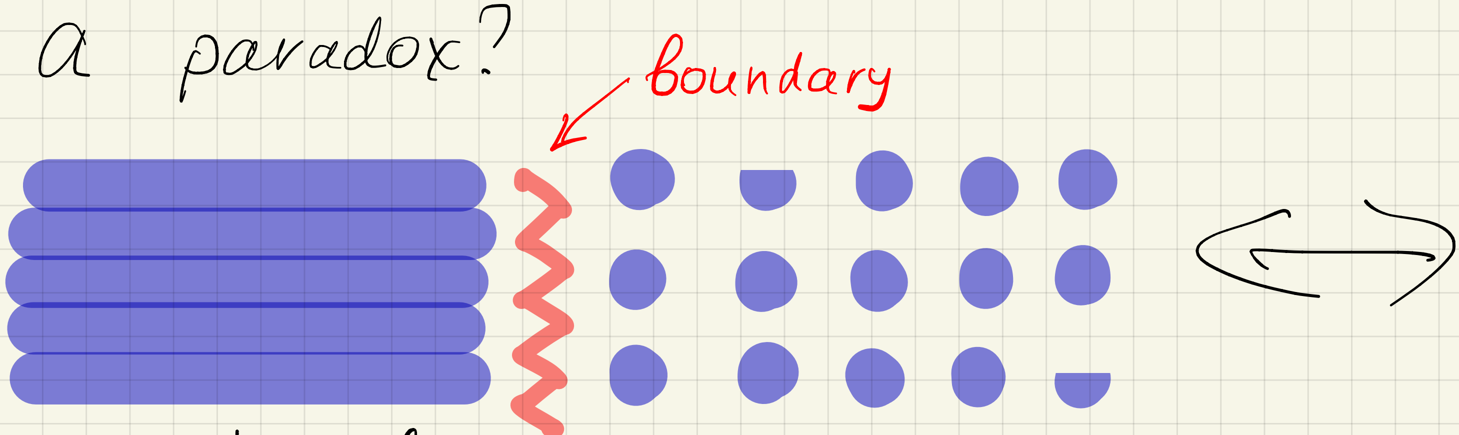
- $\Omega^{(3)}(f)$ is well-defined.
- $d\Omega^{(3)}(f) = 0$ (follows from the formula for $dF^{(3)}$)
- $\Omega^{(3)}(f) - \Omega^{(3)}(f') = d(\ldots)$ (follow from $dF^{(2)}_q = \sum_p F^{(3)}_{pq}$)

$[\Omega^{(3)}(f)]$ is the desired WZW class (up to a normalization factor).

Remark

- If f is constant except near $p = p_0$, $[\Omega^{(3)}]$ is sensitive only to the state of the system near $p = p_0$ ("Local compatibility")
- $[\Omega^{(3)}]$ does not depend on p_0.

a paradox?



a non-trivial family $H(\lambda)$ a trivial Hamiltonian $H(\lambda_0)$

Resolution: if $[\Omega^{(3)}] \neq 0$, the family $H(\lambda)$ cannot have a gapped interface with $H(\lambda_0)$.

Similar to Hall conductance!

8. Interlude: the coarse chain complex.

$F_p^{(2)}$, $F_{pq}^{(3)}$, $F_{pqr}^{(4)}$, ... are examples of chains in a certain chain complex.

Let $\Lambda \subset \mathbb{R}^d$ be a countable uniformly discrete subset. (i.e. $\inf_{\substack{p,q\in\Lambda \\ p\neq q}} |p-q| > 0$).

<u>Def.</u>
A subset $D \subset \Lambda \times \dots \times \Lambda$ is called controlled if $\exists\, \delta > 0$ s.t. $|p_i - p_j| < \delta$ $\forall\, i,j$ $\forall\, (p_1, \dots, p_n) \in D$.

<u>Def.</u> A coarse n-chain with values in a vector space V is a function

$$A : \underbrace{\Lambda \times \dots \times \Lambda}_{n+1 \text{ times}} \to V$$ which is

- skew-symmetric
- supported on a controlled subset.

Let $C_n(\Lambda; V)$ be the space of V-valued n-chains, $n = 0, 1, 2, \ldots$

The map $\partial : C_n(\Lambda; V) \to C_{n-1}(\Lambda; V)$ is defined as

$$(\partial A)_{p_1 \cdots p_n} = \sum_{p_0} A_{p_0 p_1 \cdots p_n} .$$

$\partial^2 = 0 \Rightarrow (C_\bullet(\Lambda; V), \partial)$ is a chain complex.

Its homology $H_\bullet(\Lambda; V)$ is called the coarse homology of $\Lambda \subset \mathbb{R}^d$.

Th. If Λ is uniformly filling (i.e. $\exists R > 0$ s.t. $\mathrm{dist}(x, \Lambda) \leq R \ \forall x \in \mathbb{R}^d$) then

$$H_n(\Lambda; V) = H_n^{BM}(\mathbb{R}^d; V) = \begin{cases} V, & \text{if } n = d \\ 0, & \text{if } n \neq d \end{cases}$$