Interval Translation Maps: Renormalization and Weak Mixing

Silvia Radinger
based on joint work with Henk Bruin

universität wien

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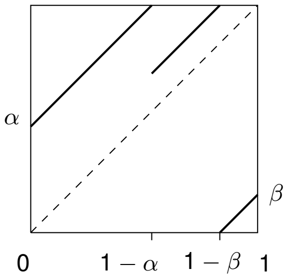
Two-parameter family of Interval Translation Maps (ITMs)

introduced by Bruin, Troubetzkoy in 2003

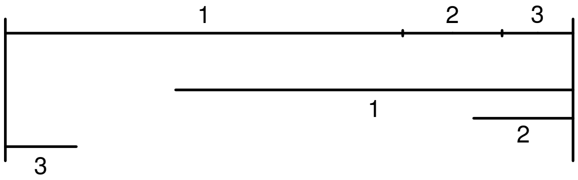
$$T_{\alpha,\beta}(x) = \begin{cases} x+\alpha, & x \in [0, 1-\alpha), \\ x+\beta, & x \in [1-\alpha, 1-\beta), \\ x-1+\beta, & x \in [1-\beta, 1] \end{cases}$$

on the parameter space $U = \{(\alpha, \beta) : 0 \leq \beta \leq \alpha \leq 1\}$.



Renormalization

Analyse the first return map to $[1-\alpha, 1]$:



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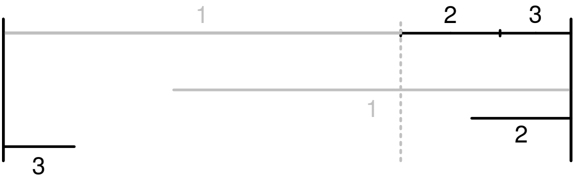

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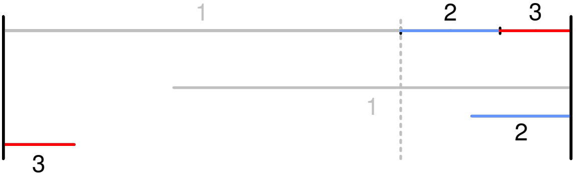

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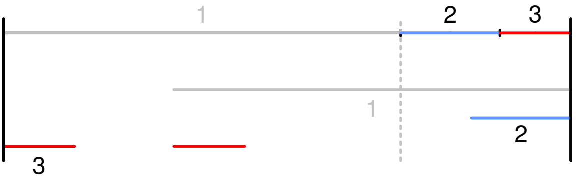

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On parameter space $U = \{(\alpha, \beta) : 0 \le \beta \le \alpha \le 1\}$ function G transforms $T_{\alpha,\beta}$ into $T_{\alpha',\beta'}$ with

$$(\alpha', \beta') = G(\alpha, \beta) = \left(\frac{\beta}{\alpha}\, ,\, \frac{\beta - 1}{\alpha} + \left\lfloor \frac{1}{\alpha} \right\rfloor \right).$$

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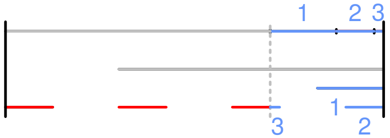

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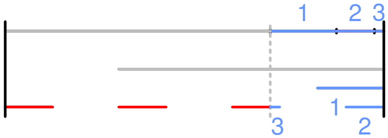
Two types of parameters

- **Finite Type:** $T_{\alpha,\beta}$ reduces to interval exchange transformation
- **Infinite Type:** $\Omega := \bigcap_{n\geq 0} \overline{T^n_{\alpha,\beta}([0,1])}$ is a Cantor set with $T_{\alpha,\beta}$ a minimal endomorphism.

The set of parameters (α, β) with $T_{\alpha,\beta}$ is of infinite type has Lebesgue measure zero.

S-adic Subshift



Symbolically, one renormalization step is given by the substitution

$$\chi_k : \begin{cases} 1 \to 2 \\ 2 \to 31^k \\ 3 \to 31^{k-1} \end{cases} \qquad \text{for } k = \left\lfloor \frac{1}{\alpha} \right\rfloor \in \mathbb{N}$$

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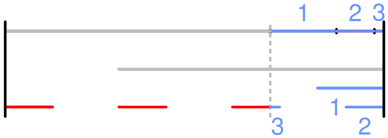

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with incidence matrix

$$A_k = \begin{pmatrix} 0 & k & k-1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \text{ and } \det(A_k) = -1.$$

We define a S-adic subshift based on a sequence of substitutions χ_{k_i}, $k_i \in \mathbb{N}$. The itinerary of the point 1 is

$$\rho = \lim_{i \to \infty} \chi_{k_1} \circ \chi_{k_2} \circ \chi_{k_3} \circ \cdots \circ \chi_{k_i}(3).$$

The **subshift** X is the closure of $\{\sigma^n(\rho)\}_{n \in \mathbb{N}}$ where σ is the **left-shift**.

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The **subshift** X is the closure of $\{\sigma^n(\rho)\}_{n \in \mathbb{N}}$ where σ is the **left-shift**.

Every *ITM of infinite type* in this family is uniquely characterised by a sequence $(k_i)_{i \in \mathbb{N}} \subset \mathbb{N}$ such that

$k_{2i} > 1$ for infinitely many $i \in \mathbb{N}$ and $k_{2j-1} > 1$ for infinitely many $j \in \mathbb{N}$.

Proposition

The S-adic subshift (X, σ), based on substitutions $(\chi_{k_i})_{i \in \mathbb{N}}$ from an ITM of infinite type, is

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- aperiodic
- left-proper

Proof.

Left-proper.

$$\chi_{k_i} \circ \chi_{k_{i+1}} : \begin{cases} 1 \to 31^{k_i} \\ 2 \to 31^{k_i - 1} 2^{k_{i+1}} \\ 3 \to 31^{k_i - 1} 2^{k_{i+1} - 1}. \end{cases}$$

□

Proposition

The S-adic subshift (X, σ), based on substitutions $(\chi_{k_i})_{i \in \mathbb{N}}$ from an ITM of infinite type, is

- aperiodic
- left-proper
- combinatorially recognizable

Proof.

Combinatorial Recognizability.
For example

$$x = \ldots \mid 2 \mid 3\ 1\ 1 \mid 3\ 1 \mid 2 \mid 2 \mid 3\ 1\ 1 \mid \ldots$$
$$= \ldots \chi_2(1)\ \chi_2(2)\ \chi_2(3)\ \chi_2(1)\ \chi_2(1)\ \chi_2(2) \ldots$$

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Proposition

The S-adic subshift (X, σ), based on substitutions $(\chi_{k_i})_{i \in \mathbb{N}}$ from an ITM of infinite type, is

- aperiodic
- left-proper
- combinatorially recognizable
- primitive ($\Rightarrow (X, \sigma)$ is minimal).

Proof.

Primitivity.

$$\tilde{A}_i = \underbrace{A_1 \cdots A_1}_{r_{i,1} \geq 0} \cdot A_{k_{i,1}} \cdot \underbrace{A_1 \cdots A_1}_{r_{i,2} \text{ odd}} A_{k_{i,2}} \cdots \cdots A_{k_{i,m}} \cdot \underbrace{A_1 \cdots A_1}_{r_{i,m+1} \text{ even}} \cdot A_{k_{i,m+1}} \cdot A_{k_{i,m+2}},$$

is a full matrix for

- $k_{i,j} \geq 2$ for $1 \leq j \leq m+1$, $k_{i,m+2} \geq 1$.

□

Linearly Recurrent Subshift

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A subshift (X, σ) is linearly recurrent if there is $L \in \mathbb{N}$ such that for every $x \in X$, every subword w reappears in x with gap $\leq L|w|$.

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Theorem

The subshift (X, σ) associated to an ITM of infinite type is linearly recurrent if and only if

- $(k_i)_{i \in \mathbb{N}}$ *is bounded and*
- *the sets* $\{i : k_{2i} > 1\}$ *and* $\{i : k_{2i-1} > 1\}$ *have bounded gaps.*

Proof idea: Show $\exists$ telescoping $(\chi_{k_i})_i$ into finitely many, left-proper substitutions with full incidence matrices.

Weakly Mixing

Definition

A minimal dynamical system (X, T) is called *weakly mixing* if the Koopman operator

$$U_T(f) = f \circ T$$

has 1 as its only eigenvalue.

If an eigenfunction f is

- in L^2, then its eigenvalue is called *measurable*,
- continuous, then its eigenvalue is called *continuous*.

Eigenvalue Conditions - Periodic Case

Theorem (Host in 1986)

For a primitive substitution system a sufficient condition to have an eigenvalue $e^{2\pi it}$ for some $t \in (0,1)$ is

$$\sum_{n=1}^{\infty} \|\vec{t}A^n\| < \infty, \qquad \vec{t} = (t,t,t),$$

where $\|x\|$ is the distance of a vector to the nearest integer lattice point.

This condition was later expanded to hold for linearly recurrent S-adic shifts and their continuous eigenvalues.

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Define $A^n = A_{k_1}\cdots A_{k_n} > 0$ with period n.

- A^n is irreducible matrix, eigenvalues are cubic numbers.

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- A^n is irreducible matrix, eigenvalues are cubic numbers.
- Stable space is one-dimensional in direction $E_3 = (u, v, -1)$.
- For $\vec{t}$ to be in integer translation of E_3:

$$\begin{pmatrix} t \\ t \\ t \end{pmatrix} + s \begin{pmatrix} u \\ v \\ -1 \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix} \text{ for } p, q, r \in \mathbb{Z} \text{ and reals } u, v > 0.$$

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- Find λ_3 is a quadratic number, a contradiction.

□

General Case - Continuous Eigenvalues

Theorem (Durand, Frank, Maass in 2019)

Let (X, σ) be a subshift based on a proper Bratteli diagram. Then $e^{2\pi i t}$ is a continuous eigenvalue if and only if

$$\sum_{n=1}^{\infty} \max_{x \in X} \| \langle s_n(x), \vec{t}\tilde{A}_1 \cdots \tilde{A}_n \rangle \| < \infty, \qquad \vec{t} = (t, t, t),$$

where

$$(s_n(x))_v = \sharp\{e \in E_{n+1} \ : \ e \succ x_{n+1}, s(e) = v\},$$

the vector $s_n(x)$ counts the number of incoming edges that are higher in the order than edge x_{n+1} of the path x. **It depends on $\tilde{A}_{n+1}$.**

Lyapunov Exponents

Proposition

For every ITM of infinite type with sequence $(k_i)_{i\in\mathbb{N}}$ the sequences of eigenvalues $(\lambda_{n,i})_{n\geq 1}$, $i = 1, 2, 3$, of $\prod_{i=1}^{n} A_{k_i}$ satisfy

$$\limsup_{n\to\infty} \frac{1}{n} \log \lambda_{n,3} < 0 \leq \liminf_{n\to\infty} \frac{1}{n} \log |\lambda_{n,2}| \leq \limsup_{n\to\infty} \frac{1}{n} \log |\lambda_{n,2}| \leq \liminf_{n\to\infty} \frac{1}{n} \log \lambda_{n,1}.$$

Further there are vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^3$, and $C > 0$ such that for all n, such that

$$\frac{1}{C}\lambda_{n,i} \leq \|\vec{v}_i \prod_{i=1}^{n} A_{k_i}\| \leq C\lambda_{n,i},$$

for $i = 1, 2, 3$ and for all $n \geq 1$.

Proof.

- A_k preserves positive octant $\mathcal{Q}^+$.
  By primitivity, $\tilde{A}_i$ is a positive integer matrix, thus $\log(\lambda_1) > 0$ and unstable space in interior of $\mathcal{Q}^+$.

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- A_k^{-1} preserves octant $\mathcal{Q}^- = \{x_1, x_2 \geq 0 \geq x_3\}$,
  change coordinates to $\mathcal{Q}^+$

$$B_k = UA_k^{-1}U^{-1} = \begin{pmatrix} 0 & 1 & k-1 \\ 1 & 0 & 0 \\ 0 & 1 & k \end{pmatrix}$$

$\Rightarrow \log(\lambda_3) < 0$.

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- By

  $$\log(\lambda_1) + \log(\lambda_2) + \log(\lambda_3) = 0$$

  if we can show $\log(\lambda_1) + \log(\lambda_3) \leq 0$, then $\log(\lambda_2) \geq 0$.

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Proposition

Every linearly recurrent ITM of infinite type has two strictly positive (as $\liminf$) and one strictly negative (as $\limsup$) Lyapunov exponent.

Direction of the Stable Space

Follow $(u, v, 1 - (u + v))B_k$ normalised to unit length, indicate the first two coordinates

$$H_k : (u, v) = \frac{1}{D_k}(v\ ,\ 1 - v) \quad \text{for} \quad D_k = k(1 - v) + 1 - u,$$

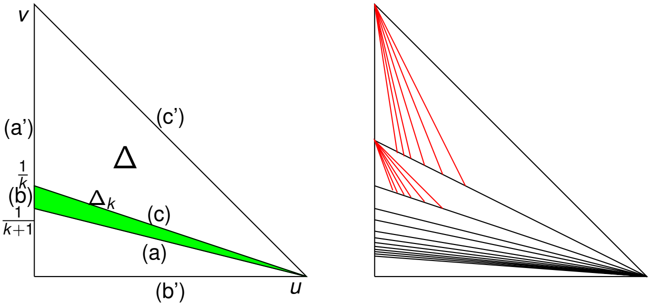
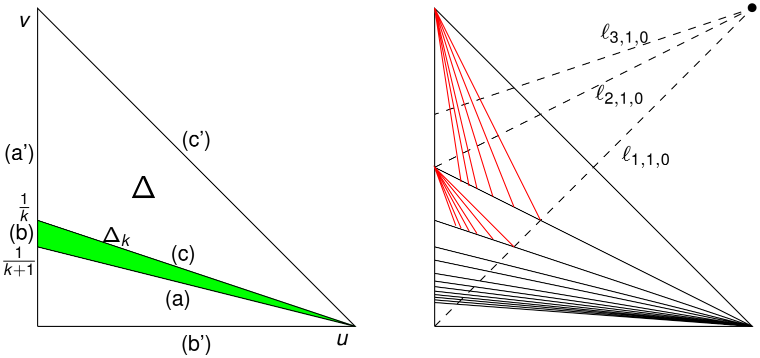


Figure: The simplex Δ and image $\Delta_k = H_k(\Delta)$ (left) and further images $H_{k_1}(H_{k_2}(\Delta))$ (right).

Direction of the Stable Space

To have $\vec{t}$ in stable direction

$$(u, v) \in \ell_{p,q,r} = \{(u, v) \in \Delta : u(q - r) = v(p - r) + q - p\}.$$



Results for Continuous Eigenvalues

Theorem

Let $T_{(\alpha,\beta)}$ be a ITM of infinite type with the stable space W^s. If $\vec{t} \notin W^s \mod 1$ for all $t \in (0,1)$, then $e^{2\pi i t}$ is not a continuous eigenvalue of the Koopman operator.

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Theorem

There exist parameters (α, β) such that $\vec{t} \in W^s \mod 1$ and $e^{2\pi it}$ is not a continuous eigenvalue of the Koopman operator.

$$\sum_{n=1}^{\infty} \max_{x \in X} \| \langle s_n(x), \vec{t}\tilde{A}_1 \cdots \tilde{A}_n \rangle \| = \infty \text{ for } \vec{t} = (t, t, t),$$

Measurable Eigenvalues

Conditions for measurable eigenvalues are more difficult to compute:

Theorem (Bressaud, Durand, Maass in 2005)

A necessary and sufficient condition for $e^{2\pi it}$ to be a measurable eigenvalue is the following:
There is a sequence of functions $\rho_n : V_{n+1} \to \mathbb{R}$ such that

$$g_n(x) := t\left(\tilde{S}_n(x) + \rho_n(w)\right) \bmod 1$$

converges for μ-a.e. $x \in X_{BV}$ as $n \to \infty$,

where $\tilde{S}_n(x) = \sum_{j=1}^{n} \langle \tilde{s}_j(x), \vec{1}\tilde{A}_1 \cdots \tilde{A}_j \rangle$ is the minimal number of steps to the base of a tower.

Results for Measurable Eigenvalues

Theorem

Let $T_{(\alpha,\beta)}$ be a ITM of infinite type with $\liminf_n k_n < \infty$. If $\vec{t}$ does not belong to the stable space $W^s \bmod 1$ for all $t \in (0,1)$, then the corresponding ITM is weakly mixing.

Remark: $\liminf_n k_n < \infty$ implies unique ergodicity

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Remark: $\liminf_n k_n < \infty$ implies unique ergodicity

Corollary

A linearly recurrent ITM of infinite type is

$$\text{weakly mixing if and only if } \vec{t} \notin W^s \mod 1 \text{ for all } t \in (0,1).$$

Furthermore, any measurable eigenvalue is continuous.

Thank you for your attention!

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