Three roads from tensors models to continuous geometry

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Motivation

We use the perspective

Quantizing Gravity $\simeq$ Randomizing Geometry

Functional integral quantization, in Euclidean setting

$$Z \simeq \sum_{S} \int Dg \;\; e^{\,-\int_S A_{EH}(g)}$$

where Dg and even S are to be defined...

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A fundamental difficulty is that the theory on a four dimensional flat space is perturbatively not renormalisable $\Longrightarrow$ non-UV complete.

In two dimensions, random matrix models are among the most successful ways to explore quantum gravity non perturbatively & ab initio.

The Tensor Track generalizes this success to use tensors to explore to quantum gravity in higher dimensions [VR '11, '12, '13, '16, '18, '20].

Renormalization

Physics is mathematics plus scales.

Since 1930's, the idea that physics also *depends* on the probing scale was independently exploited in particle physics and condensed matter:

- [Gell-Mann, Low, Dyson] "dress" an elementary particle with an effective (renormalized) charge;
- [Stueckelberg, Petermann, Kadanoff] block spin transformations to recover scaling laws near critical point.

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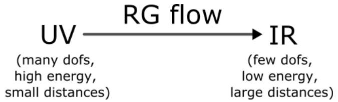
Wilson fused both points of view [Wilson '71]:

$$e^{-S_k[\phi_{<k}]} = \int_{k<k'<\Lambda} D\phi_{k'}\, e^{-S_\Lambda\left[\phi_{k'<k}+\phi_{k'>k}\right]}.$$

Fluctuations of higher energy scales are integrated out, generates a flow of the effective action in theory space.



Renormalization Group

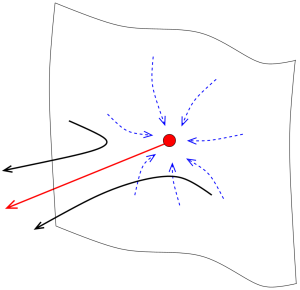
Given a QFT defined by a set of (dimensionless) couplings $\{g_i\}_{i=1,\ldots}$, after regularization, they flow with the probing scale μ as

$$\beta_i := \frac{dg_i}{d\log\mu} = f(g_1,\ldots).$$

UV/IR fixed points form universality classes of QFTs, characterized by

- symmetries,
- spacetime dimensions,
- number of degrees of freedom.

Relevant, irrelevant, marginal directions.
Asymptotic freedom: UV Gaussian fixed point.



[Credit: David Tong]

A theory is renormalizable if it has a finite number of relevant couplings.

Tensor Models in 0 dimensions

Generalising vector and matrix models, tensor models are:

Field: $T_{a_1 \dots a_r}$ rank r (unsymmetrized) tensor, transforms under $G^{\otimes r}$ (G of rank N):

$$T'_{b^1 \dots b^r} = \sum_a U^{(1)}_{b^1 a^1} \dots U^{(r)}_{b^r a^r} T_{a^1 \dots a^r} \ , \quad U^{(i)} \in G \ .$$

Action and Observables: $G^{\otimes r}$-invariants ($\mathcal{B}$ "bubbles").

$$\begin{aligned} S &= S_0 + S_{int}; \\ S_0(T, \bar{T}) &= \underbrace{\sum_a T_{a^1 \dots a^r} \bar{T}_{a^1 \dots a^r}}_{\text{propagator}}; \quad S_{int} = \underbrace{\sum_{r\text{-colored graphs } \mathcal{B}} t_{\mathcal{B}} \, \mathrm{Tr}_{\mathcal{B}}(T, \bar{T})}_{\text{interaction}} \ . \end{aligned}$$

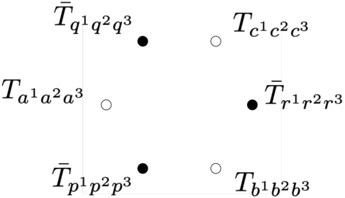
This action is invariant under the symmetry $G^{\otimes r}$.

Tensor invariants as Colored Graphs

Example ($r = 3,\ G = U(N)$):

$$\sum \delta_{a^1 p^1}\delta_{a^2 q^2}\delta_{a^3 r^3} \quad \delta_{b^1 r^1}\delta_{b^2 p^2}\delta_{b^3 q^3} \quad \delta_{c^1 q^1}\delta_{c^2 r^2}\delta_{c^3 p^3}$$
$$T_{a^1 a^2 a^3}\, T_{b^1 b^2 b^3}\, T_{c^1 c^2 c^3}\, \bar{T}_{p^1 p^2 p^3}\, \bar{T}_{q^1 q^2 q^3}\, \bar{T}_{r^1 r^2 r^3}$$

White (black) vertices for T ($\bar{T}$).



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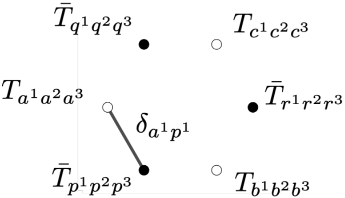
Edges for $\delta_{a^c q^c}$



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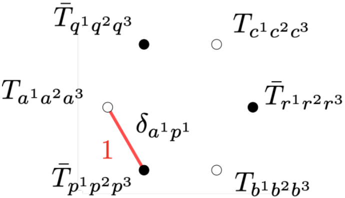

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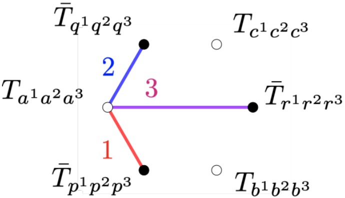

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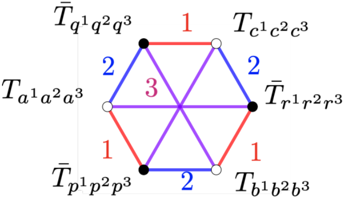

Tensor invariants as Colored Graphs

Example ($r = 3,\ G = U(N)$):

$$\mathrm{Tr}_{\mathcal{B}}(T, \bar{T}) = \sum \prod_{v} T_{a_v^1 \dots a_v^r} \prod_{\bar{v}} \bar{T}_{q_{\bar{v}}^1 \dots q_{\bar{v}}^r} \prod_{c=1}^{r} \prod_{e^c=(w,\bar{w})} \delta_{a_w^c q_{\bar{w}}^c}$$

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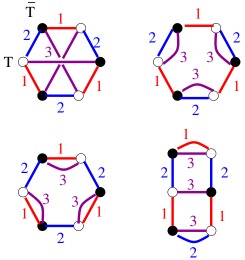


Feynman expansion

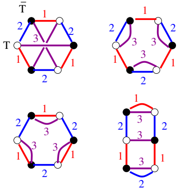
$$S(T,\bar{T}) = \sum T_{b^1\dots b^r}\,\bar{T}_{q^1\dots q^r}\prod_{c=1}^{r}\delta_{b^c q^c} + \sum_{r\text{-colored graphs }\mathcal{B}} t_{\mathcal{B}}\,\mathrm{Tr}_{\mathcal{B}}(T,\bar{T})\;,$$

$$Z(t_{\mathcal{B}}) = \int [d\bar{T}dT]\; e^{-N^{r-1}S(T,\bar{T})}$$

Feynman expansion:

- Taylor expand the interactions (*r*-colored graphs)

$$Z(\{t_{\mathcal{B}_i}\}) = \sum \int_{T,\bar{T}} e^{-N^{r-1}T\bar{T}}\, t_{\mathcal{B}_1}\,\mathrm{Tr}_{\mathcal{B}_1}(T,\bar{T})\, t_{\mathcal{B}_2}\,\mathrm{Tr}_{\mathcal{B}_2}(T,\bar{T})\dots$$



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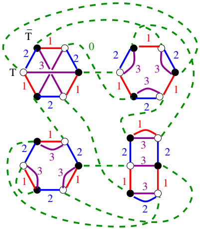
Feynman expansion:

- Taylor expand the interactions (r-colored graphs)
- Perform the Gaussian integrals by Wick theorem (($r+1$)-colored graphs)

$$Z(\{t_{\mathcal{B}_i}\}) = \sum\int_{T,\bar T} e^{-N^{r-1}T\bar T}\, t_{\mathcal{B}_1}\,\mathrm{Tr}_{\mathcal{B}_1}(T,\bar T)\, t_{\mathcal{B}_2}\,\mathrm{Tr}_{\mathcal{B}_2}(T,\bar T)\dots$$
$$= \sum_{(r+1)\text{-colored }\mathcal{G}} A(\mathcal{G})$$



$1/N$ expansion

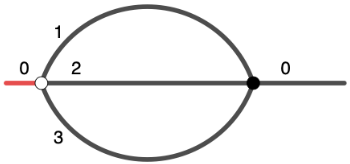
Without other rescaling of couplings, vacuum graphs indexed by Gurau degree

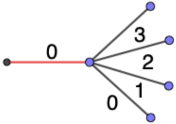
$$A(\mathcal{G}) \sim N^{r-\omega(\mathcal{G})} , \quad \omega(\mathcal{G}) = \frac{1}{2(r-1)!} \sum_{\mathcal{J}} g(\mathcal{J}) .$$

$\mathcal{J}$: embeddings of the colored graph on the plane (jackets), of genus $g(\mathcal{J})$.

$$\omega = 0 \iff g(\mathcal{J}) = 0 \ \forall \mathcal{J} \iff \text{melonic}.$$ [Gurau '10]

- Iterative self-insertion of the fundamental melon.
- Counted by edge-colored rooted $(r+1)$-ary trees.





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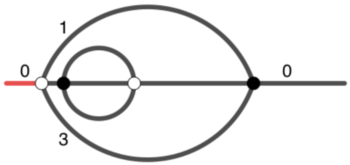

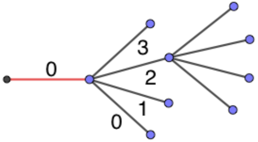

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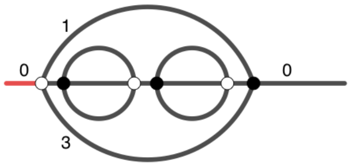

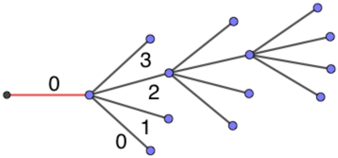

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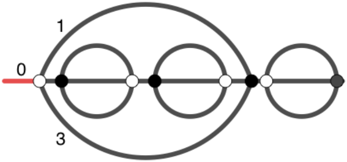
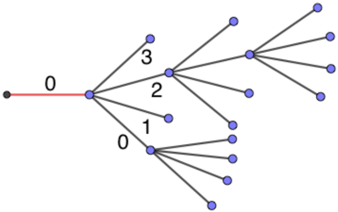


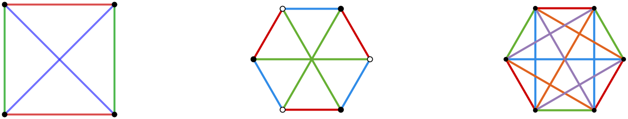


Optimal scalings

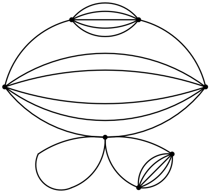
What scaling can allow an interaction to contribute infinitely at leading order?

$$S_N(T) = N^{r/2}\left(T\cdot T + \sum_{\mathcal{B}} t_{\mathcal{B}} N^{-\rho(\mathcal{B})} I_{\mathcal{B}}\right), \quad \rho(\mathcal{B}) = \frac{F_{\mathcal{B}}}{r-1} - \frac{r}{2},$$

for Maximally Single Trace interactions (1 face for each pair of colors):



allows generalized melonic diagrams [Carrozza, Tanasa '15, Ferrari, VR, Valette '17]:



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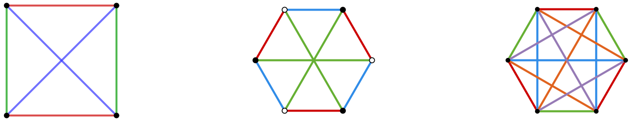

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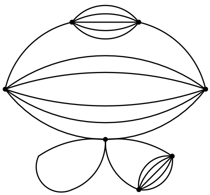
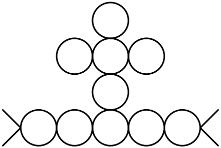
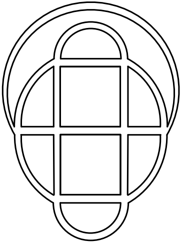
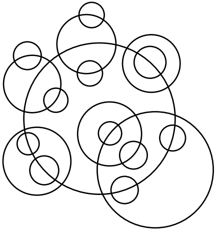


(still trees).

Large-N limits

Vectors v_i	Matrices M_{ij}	Tensors $T_{ij\dots k}$
		
Cyclomatic number	Genus	Gurau degree
Branched polymers ($d_H = 2,\ d_S = 4/3$)	Brownian sphere ($d_H = 4,\ d_S = 2$)	Branched polymers ($d_H = 2,\ d_S = 4/3$)
$(v_i v_i)$	$\mathrm{Tr}(M^n)$	$(2n)$-regular graphs $\sim n!$
Higher-spins	String theory	Unknown!

A surprise: the SYK model

The Sachdev-Ye-Kitaev model is a quantum system of N Majorana fermions at temperature $1/\beta$ with quenched disorder [Kitaev '15, Maldacena Stanford '16]

$$H = -\sum_{1\leq i<j<k<l\leq N} J_{ijkl}\chi_i\chi_j\chi_k\chi_l$$

$$\{\chi_i, \chi_j\} = \delta_{ij} \quad \left\langle J_{ijkl}^2 \right\rangle = \frac{3!}{N^3} J^2$$

whose large N and strong coupling limits ($1 \ll \beta J \ll N$) present

- approximate reparameterization symmetry,
- saturation of chaos bound [Maldacena et al. '15].

→ Simplest model of holography (AdS_2/CFT_1)
→ Recent progress regarding the black hole information paradox [Strings '20].

It is solvable because this limit is melonic.
1d tensor models present the same features, without disorder [Witten '16].
Motivated the study of $d \geq 1$ tensor models.

Tensor models: a (partial) timeline

- 2010: Colored models [Gurau]
- 2011: Single scaling limit [Gurau et al.]
  Universality [Gurau]
- 2012: Uncolored models [Bonzom et al.]
  Asymptotically safe and free models [Ben Geloun et al., Carrozza et al.]
- 2013: Melons are branched polymers [Gurau, Ryan]
  Double scaling limit [Dartois et al.] $\rightarrow$ cherry trees
  Counting invariants [Ben Geloun et.]
  Structure at all orders [Gurau, Schaeffer]
- 2014: Analyticity and Borel summability [Delepouve al.]
- 2015: Symmetry breaking [Delepouve, Gurau]
- 2016: Enhanced models: branched polymers, baby universes, Brownian map [Bonzom] (and later [Lionni '17] for many more bubble types)
- 2017: Melon dominance in irreps of $O(N)^3$ tensor models [Gurau, Benedetti et al., Carrozza] and later $Sp(N)$ [Carrozza, Pozsgay '18]
  Subleading corrections [Bonzom et al.], Crystallization theory [Casali et al.]
  Melonic CFTs [Giombi et al., Benedetti et al., etc.]
- 2018: Melonic limit in turbulence [Dartois et al.]
- 2020: Tensor eigenvalues [Evnin; Gurau], Data analysis [Lahoche et al.]

Double Scaling for Matrices

To go beyond the result of [Gurau, Ryan '13] that the melons are branched polymers and find more interesting geometries, one needs to incorporate the sub-leading contributions in 1/N.

One should try the *double scaling* limit. In the matrix case, let us consider the following partition function

$$Z(N,\lambda) = \int dM e^{-N\left(\frac{1}{2}\operatorname{tr} M^2 - \frac{\lambda}{4}\operatorname{tr} M^4\right)} ,$$

$$F(N,\lambda) = \log(Z) = \sum_{g\geq 0} N^{2-2g} F_g(\lambda) ,$$

where $F_g(\lambda)$ is the generating series of the genus g ribbon graphs.

Double Scaling for Matrices II

All F_g's are holomorphic in a certain domain of λ and meet a singularity at λ_c. The behaviour of F_g around λ_c is of the form

$$F_g(\lambda) \sim K_g(\lambda - \lambda_c)^{\frac{(2-\gamma)}{2}\chi(g)},$$

with $\gamma = -\frac{1}{m}$ for some $m \geq 2$, K_g is some constant and $\chi(g) = 2 - 2g$. Given the diverging point λ_c, the double scaling is when both $N \to \infty$ and $\lambda \to \lambda_c$ in a correlated way. Setting $x = N^{-1}(\lambda - \lambda_c)^{\frac{\gamma-2}{2}}$, we obtain

$$F(x) = \sum_{g \geq 0} x^{2g-2} K_g.$$

The K_g's behave as $(2g)!$ since the resulting series sums all Feynman graphs. Related to integrable minimal models.

[Brézin, Kazakov, Gross, Migdal, Douglas, Shenker, Miljokovic, Klebanov, Bleher, Eynard...]

Double Scaling for Tensors

Just as for matrix models, there is a single and double scaling limit.

For instance, in the quartic interacting model, of rank r, with coupling constant λ [Dartois, Gurau, VR '13] we introduce the variable

$$x = N^{r-2}\left[(4r)^{-1} + \lambda\right] \Rightarrow \lambda = -\frac{1}{4r} + \frac{x}{N^{r-2}},$$

and send $N \to \infty$ and $\lambda \to -\frac{1}{4r}$ while keeping x fixed. We obtain a power series in x

$$G_2 = N^{1-r/2} \sum_{p \geq 0} \frac{c_p}{x^{p-\frac{1}{2}}} + \mathcal{O}(N^{1/2-r/2}) \, ,$$

which has a new critical point in x at $x_c = 1/4(r-1)$. The corresponding double scaling-limit is

$$\bar{G}_{2,double}(N) = 2 - 4N^{1-r/2}\sqrt{r(x-x_c)} + \mathcal{O}(N^{1/2-r/2}).$$

A disappointment remains: the singularity stays of the branched polymer type for $r < 6$, but at a different location.

Multiple Scaling and Topological Recursion

- Contrary to matrix models, the double scaling limit still resums only triangulations of the sphere, so much less than general triangulations.
- In further contrast to matrix models, at least for $r = 6$, there is a triple scaling limit [Dartois '15].
- The Hubbard-Stratonovich transformation maps the quartic tensor model to a multi-matrix model which (after subtracting the leading order) satisfies the blobbed topological recursion [Borot and al, Bonzom and al '16]. [cf. R. Wulkenhaar's talk]
- This road, although mathematically the purest, is difficult to follow: requires fine analysis of subleading orders which gets quickly involved.

Breaking the Propagator

When there are no space available, breaking the isotropy of the covariance can be a useful device to generate a scale hierarchy between the degrees of freedom and to define the direction of the flow.

The ultra-violet corresponds to lowest covariance and to many degrees of freedom; the infra-red corresponds to highest covariance and fewer degrees of freedom.

The flow of the renormalisation group, as it should, is from ultra-violet to infra-red, averaging from the many degrees of freedom towards the fewer degrees of freedom.

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Precursor for matrices in [Brézin, Zinn-Justin '92].

Asymptotic freedom

Our tensors are still 0-dimensional, but let us distinguish the rank of the tensor from the space dimension.

Let us substitute a propagator of a Laplacian type (eventually some power of the Laplacian), which is independent (diagonal) but not identically distributed

$$S_0(T, \bar{T}) = \underbrace{\sum_{a} T_{a^1 \dots a^r} \Delta_{a^1 \dots a^r} \bar{T}_{a^1 \dots a^r}}_{\text{propagator}} \ ,$$

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Equipped with a quartic interaction S_{int}, it is renormalisable in rank 5 and surprisingly, it shares with the non-abelian gauge theories the property of being asymptotically free [Ben Geloun '13]. The large N limit consists of only melonic graphs. It is their combinatorics which are responsible for the phenomenon of asymptotic freedom so it is significant since it is protected by topological reasons [VR '15].

Finding New Fixed Points

It is tempting to launch a flow from the UV towards the IR to discover new fixed points which may be new geometries different from branched polymers.

- From truncated Wetterich equation one might find new fixed points with reasonable accuracy [Benedetti et al. '14].
- The fixed points and their associated geometries may share some universality, as it is reasonable to expect from fixed points of the renormalisation group.
- In A. Eichhorn's program a potential candidate for a continuum limit in such a model was found, which features two relevant directions [Eichhorn, Lumma et al. '19].

QFT on random trees [ND, VR '19]

If we can approximate the sub-dominant terms as matter fields living on the branched polymers (and it's a big "if"), we shall get in this approximation an SYK-type model on a random tree.

This motivates the study of quantum fields theories on trees, which is the third and newest road to discover interesting geometries in the tensor track.

Scalar field defined on *random trees* (equivalent to branched polymers) is the simplest QFT on an ensemble of interesting random geometry.

Preliminary results: on average, the standard power counting analysis for the superficial degree of divergence of amplitudes is consistent with $d = d_S$ $(= 4/3)$.

Random Walk Expansion [Symanzik '69]

Tools: Renormalization group flow + Random walks.

Idea: 2-point function as a sum over random walks, with precise heat kernel estimates of [Barlow, Kumagai '06],
$\rightarrow$ evaluate generic amplitudes and start an RG analysis.

Related works: Similar expansions (random walks, random currents, laces), on fixed geometry, allowed to analyze rigorously correlation functions for various statistical models (mostly scalar, Ising, Potts,...) [Aizenman, Fröhlich, Duplantier, Brydges, Duminil-Copin...].

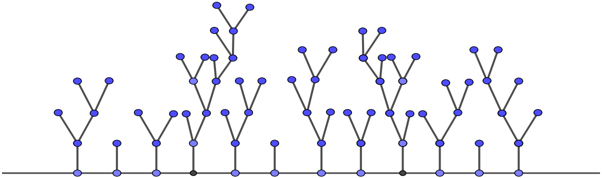
- triviality of the universality class of ϕ^4 in $d \geq 4$,
- prove relations between and bounds on critical exponents below critical dimensions,
- bounds on β-functions.

But seems hard to work on far-from-free models.

The "spacetime": Galton-Watson branching process

The ensemble $\mathcal{T}$ of rooted binary trees, conditioned on having an infinite spine $\mathcal{S}$ (criticality), can be seen as having side branches T (with $|T| = n$ vertices, $n < \infty$), with independent measure:

$$\mu(T) = 2^{-|T|}\,.$$



The probability measure on $\mathcal{T}$ is then:

$$\mathbb{P}[\tau] := \prod_{i \in \mathcal{S}} \mu(T_i), \quad \mathbb{E}[f] := \sum_{\tau \in \mathcal{T}} \mathbb{P}[\tau] f(\tau).$$

Spectral dimension d_S: if $p_t(x)$ is the probability for a random walk starting at x to be at x in a time t, then

$$p_t(x) \underset{t\to\infty}{\sim} \frac{1}{t^{d_S/2}}\,, \qquad d_S = 4/3 \text{ for } \mathcal{T} \text{ [Wheater et al. '06]}.$$

Propagating the matter field

On a fixed graph Γ, the (positive def) Laplacian is given by: $\mathcal{L}_\Gamma = D_\Gamma - A_\Gamma$
(D_Γ: degree matrix ; A_Γ: incidence matrix)
and its inverse kernel, the propagator, by a sum over random walks:

$$C_{\Gamma,m}(x,y) = \sum_{\omega: x\to y} \prod_{v\in\Gamma} \left[\frac{1}{d_v + m^2}\right]^{n_v(\omega)} \sim \int_0^\infty \mathrm{d}t \; e^{-m^2 t} p_t(x,y) \; ,$$

with an IR regulator m.
We then use the Euler β-function identity:

$$\mathcal{L}^{-\zeta} = \frac{\sin \pi\zeta}{\pi} \int_0^\infty \mathrm{d}m \frac{2m^{1-2\zeta}}{\mathcal{L} + m^2} \; ,$$

($0 < \zeta \leq 1$ to maintain positivity properties), for long-range propagator:

$$C_\Gamma^\zeta(x,y) = \frac{\sin \pi\zeta}{\pi} \int_0^\infty \mathrm{d}m \; 2m^{1-2\zeta} \sum_{\omega: x\to y} \prod_{v\in\Gamma} \left[\frac{1}{d_v + m^2}\right]^{n_v(\omega)} .$$

[analogous to a Källen-Lehmann representation]

Motivating the rescaling

With our convention for external legs [VR et al. '85], the IR degree of divergence for a scalar field of mass dimension $(d-2\zeta)/2$ and ϕ^q interaction:

$$\omega(G) = (d-2\zeta)E - d(V-1) = (d-2\zeta)(qV-N)/2 - d(V-1)\ ,$$
$$(V \text{ vertices, } E \text{ internal legs, } N \text{ external legs, } qV = 2E+N),$$

we tuned ζ to

$$\zeta = \frac{d}{2} - \frac{d}{q}\ ,$$

implying a just-renormalizable theory

$$\omega(G) = d\left(1 - \frac{N}{q}\right)\ .$$

We showed it is compatible with $d = d_S$.
For $q = 4$, 2- and 4-point functions need renormalization.

Field theory

Partition function (quenched):

$$Z(\Gamma;\lambda) = \int e^{-\lambda \sum_{x\in V_\Gamma} \phi^4(x)} d\mu_{C_\Gamma}(\phi) = \int d\nu_\Gamma(\phi).$$

Correlation functions (quenched):

$$S_N(\Gamma; z_1, ..., z_N) = \int \phi(z_1)...\phi(z_N)\ d\nu_\Gamma(\phi) = \sum_{V=0}^{\infty} \frac{(-\lambda)^V}{V!} \sum_G A_G(\Gamma; z_1, ..., z_N).$$

[Feynman graphs G on graphs Γ.]

For $\{z_1, ..., z_N\} \in \mathcal{S}$, we want the annealed quantity:

$$\mathbb{E}[S_N(\Gamma; z_1, ..., z_N)] = \sum_{\Gamma\in\mathcal{T}} \mathbb{P}[\Gamma] S_N(\Gamma; z_1, ..., z_N).$$

RG: multiscale analysis (in the IR)

(1) Decompose the propagators into "proper time" slices $I_j = [M^{2(j-1)}, M^{2j}]$:

$$C = \sum_{j=0}^{\rho} C^j; \quad A(G) = \sum_{\mu} A_\mu(G)$$

($j = 0$ is UV, ρ is IR; external propagators at scale ρ – "regularization").

(2) Identify superficial degree of divergence ω and divergent graphs.
Given G and μ, high subgraphs control the divergence:

HS : (scales of internal legs) $<$ (scales of external legs)

$$|A_\mu(G)| \leq \prod_{G_i \in HS} M^{\omega(G_i)}.$$

(3) Localization: expand the divergent subgraphs around reference point.
(need counterterms – "renormalization")

(4) RG flow: integrate out lower scales $j < i$ gives theory at scale i.

Probabilistic estimates [Barlow, Kumagai]

For a parameter $\lambda \geq 1$, the ball $B(x, r)$ is said **λ–good** if:

$$r^2 \lambda^{-2} \leq |B(x, r)| \leq r^2 \lambda.$$

Crucially, they showed that it occurs more often, for larger and larger λ:

$$\mathbb{P}[B(x, r) \text{ is not } \lambda\text{–good}] \leq \mathcal{O}(1) e^{-\mathcal{O}(1)\lambda}.$$

Then, they obtained the quenched bounds:
Given $r > 0$ and that $B(x, r)$ is λ–good, if $t \in [r^3 \lambda^{-6}, r^3 \lambda^{-5}]$, then

- for any $K \geq 0$ and any $y \in T$ with $d(x, y) \leq K t^{1/3}$

$$p_t(x, y) \leq \mathcal{O}(1) \left(1 + \sqrt{K}\right) t^{-2/3} \lambda^3 ,$$

- for any $y \in T$ with $d(x, y) \leq \mathcal{O}(1) r \lambda^{-19}$

$$p_t(x, y) \geq \mathcal{O}(1) t^{-2/3} \lambda^{-17}.$$

Our results: Propagators

Slicing the propagator into proper time slices $I_j = [M^{2(j-1)}, M^{2j}]$:

$$C_T^{\zeta,j}(x,y) \underset{u=m^2}{=} \int_0^\infty \mathrm{d}u \; u^{-\zeta} \int_{I_j} \mathrm{d}t \; p_t(x,y) e^{-ut} = \Gamma(1-\zeta) \int_{I_j} \mathrm{d}t \; p_t(x,y) t^{\zeta-1} \,,$$

Lemma (Single Line)

- $\mathbb{E}\left[C_T^{\zeta,j}(x,x)\right] \leq \mathcal{O}(1) M^{-2j/3}, \quad \mathbb{E}\left[\sum_y C_T^{\zeta,j}(x,y)\right] \leq \mathcal{O}(1) M^{2j/3}$
- $\mathbb{E}\left[C_T^{\zeta,j}(x,x)\right] \geq \mathcal{O}(1) M^{-2j/3}, \quad \mathbb{E}\left[\sum_y C_T^{\zeta,j}(x,y)\right] \geq \mathcal{O}(1) M^{2j/3}.$

Interpretation: a typical volume integration corresponds to $d = 2$; while in proper time t, the propagator scales as $t^{-1/3}$.

Our results: Convergent graphs

Theorem ($N > 4$)

For a completely convergent graph (no 2- or 4-point subgraphs) G of order $V(G) = n$, the limit as $\lim_{\rho\to\infty} \mathbb{E}(A_G)$ of the averaged amplitude exists and obeys the uniform bound

$$\mathbb{E}(A_G) \leq K^n (n!)^\beta$$

where $\beta = \frac{52}{3}$[a].

[a] Not optimal

Comment: the proof uses Cauchy-Schwarz, the preceding bounds and slicing the space into rings that are asked to be λ-good; however intersecting rings don't have independent probabilities (which we assumed) and lead to the factorial growth.

Our results: Divergent graphs I

We want to know how an amplitude changes when moving an external leg from one point z to a close point y:

Lemma

Defining $\Delta_T^{\zeta,j}(x;y,z) := \left| C_T^{\zeta,j}(x,y) - C_T^{\zeta,j}(x,z) \right|$, *we obtain*

$$\mathbb{E}[\Delta_T^{\zeta,j}(x;y,z)] \leq \mathcal{O}(1) M^{-2j/3} M^{-j/3} \sqrt{d(y,z)}.$$

Comment: uniform in $x \in \mathcal{S}$ and the factor $M^{-j/3}\sqrt{d(y,z)}$ is the gain, provided $d(y,z) \ll r_j = M^{2j/3}$. The inequality for $y, z \in \tau$

$$|f(y) - f(z)|^2 \leq d(y,z)\mathcal{E}(f,f),$$

proved very useful ($\mathcal{E}(f,f) \sim \sum_{x \sim y \in \tau} (f(x) - f(y))^2$).

Our results: Divergent graphs II

For $j_m \ll j_M$, we want to compare the "bare" amplitude

$$A_T^{bare}(x,z) := \sum_{y \in T} C_T^{j_M}(x,y) C_T^{j_m}(y,z)$$

to the "localized" amplitude at z

$$A_T^{loc}(x,z) := C_T^{j_M}(x,z) \sum_{y \in T} C_T^{j_m}(y,z).$$

Lemma

Introducing the averaged "renormalized" amplitude $\bar{A}_{ren}(x,z) := \mathbb{E}[A_T^{bare}(x,z) - A_T^{loc}(x,z)]$, *we have*

$$\left|\bar{A}_{ren}(x,z)\right| \leq cM^{-2(j_M - j_m)/3 - (j_M - j_m)/3}.$$

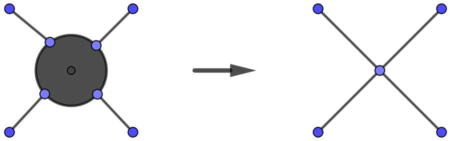
Our results: Divergent graphs III

The previous lemma allows to write 4-point subgraphs as a local 4-vertex, plus corrections unseen by the external scale, defining hence a renormalized amplitude A^{ren}:

Theorem ($N \geq 4$)

For a graph G with $N(G) \geq 4$ and no 2-point subgraph G of order $V(G) = n$, the averaged renormalized amplitude $\mathbb{E}[A^{ren}_G] = \lim_{\rho\to\infty} \mathbb{E}[A^{ren}_{G,\rho}]$ is convergent as $\rho \to \infty$ and obeys the same uniform bound than in the completely convergent case, namely

$$\mathbb{E}(A^{ren}_G) \leq K^n (n!)^\beta .$$



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Thank you!