Noncrossing partitions of a marked surface

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21 February 2025

Early work was joint with Laura Brestensky, including her thesis research.

Representation theory connections are joint with Eric Hanson.

The noncrossing partition poset

(W, S): a Coxeter system with reflections T.

Coxeter element: $c =$ product of S in any order.

Absolute order $u \leq_T w$ is prefix order for T.

The noncrossing partition poset is $[1, c]_T$.

$W \implies [1, c]_T$ is a lattice (Bessis, Brady-Watt).

Classical finite types (A, B, D): $\exists$ planar models.

Heuristic: Project a small orbit to Coxeter plane $\rightarrow$ planar model.

One motivation: Garside structures for Artin groups
(Barbara Baumeister's talk)

The prototypical example is type A (Kreweras/Biane):

$W = S_{n+1}, \qquad s_i = (i \;\; i+1).$

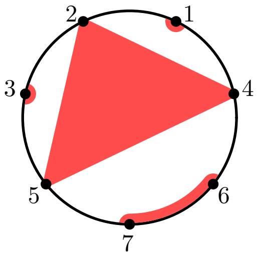
i is a down element if s_{i-1} precedes s_i in c

i is an up element if s_{i-1} follows s_i in c.

$[1, c]_T \leftrightarrow$ nc partitions of the $(n+1)$-cycle

(1 down elements, increasing $n+1$ up elements, decreasing).

Example: $c = s_3 s_5 s_2 s_1 s_6 s_4 = (1 \;\; 4 \;\; 6 \;\; 7 \;\; 5 \;\; 3 \;\; 2)$



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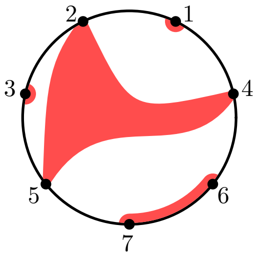

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Type B (Reiner/Athanasiadis-Reiner):

W is the group B_n of signed permutations.

The choice of c is a choice of a signing of $\{1, 2, \ldots n-1\}$ ($n-1$ elements of $\{\pm 1, \ldots, \pm(n-1)\}$, distinct absolute values).

$[1, c]_T \leftrightarrow$ centrally symmetric nc partitions of the $(2n)$-cycle

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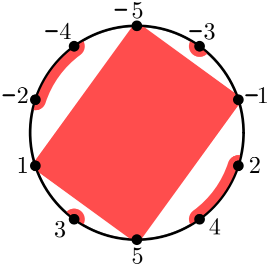

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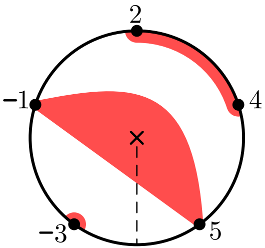


Or: Noncrossing partitions of a disk with an orbifold point

$\longleftrightarrow$ $(-3)\,(3)\,(2\ \ 4)\,(-4\ \ -2)\,(5\ \ 1\ \ -5\ \ -1)$

Type D (Athanasiadis-Reiner):

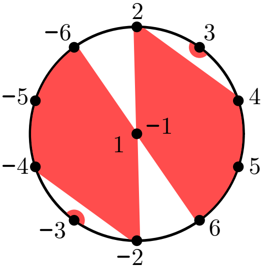
W is the group D_n of even-signed permutations.

Project a smallest W-orbit to the Coxeter plane: two points land in the center.

. . . some combinatorial way to encode Coxeter elements. . .

$[1, c]_T \leftrightarrow$ centrally sym. nc partitions of disk with a double point.

Example:



$$\longleftrightarrow \quad (-1\ \ 2\ \ 4\ \ 5\ \ 6)\,(3)$$
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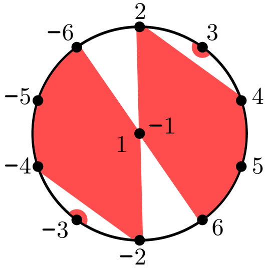

It isn't possible to "mod out" by the symmetry without recording some additional information.

$$\longleftrightarrow \quad (-1\ \ 2\ \ 4\ \ 5\ \ 6)\,(3)$$
$$(1\ \ -2\ \ -4\ \ -5\ \ -6)\,(-3)$$

Affine Coxeter groups (Euclidean Artin groups)

W not finite $\implies$ $[1, c]_T$ need not be a lattice.

Affine Coxeter group: a Cox. group generated by affine reflections.

McCammond and Sulway extended the affine Coxeter group W to a larger group, by "factoring translations", thus extending $[1, c]_T$ to a lattice (Garside structure for a supergroup of the Artin group).

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Recent research: Planar diagrams for $[1, c]_T$ and the larger lattice in classical affine types (jointly with Laura Brestensky as part of her thesis research).

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Type $\widetilde{A}$ is as nice as one could hope:

- A natural combinatorial construction of the larger lattice.
- An easy combinatorial restriction obtains $[1, c]_T$.

Noncrossing partitions of classical affine types

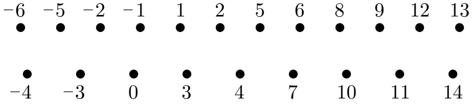
Run the heuristic from finite type: Project a “small” orbit to the “Coxeter plane”, then mod out by some symmetries.

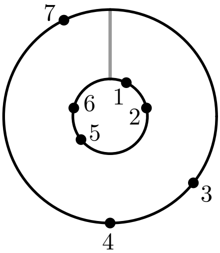
Classical cases:

- The orbit is indexed $\{\mathbf{e}_i : i \in \mathbb{Z}\}$.
- The projection is an infinite strip with translational symmetry.
- This becomes an annulus.

Example: Affine type $\widetilde{A}_6$

$c = s_6 s_5 s_2 s_1 s_3 s_4 s_7$





Translation $\longleftrightarrow$ mod-7 symmetry.

Type $\widetilde{A}$: Affine permutations and periodic permutations

Type $\widetilde{A}_{n-1}$ affine Coxeter group $\widetilde{S}_n$ is affine permutations π of $\mathbb{Z}$:

- $\pi(i+n) = \pi(i) + n$ for all $i \in \mathbb{Z}$
- $\sum_{i=1}^{n} \pi(i) = \binom{n+1}{2}$.

Larger group $S_{\mathbb{Z}}(\mathrm{mod}\ n)$: $\pi(i+n) = \pi(i) + n \quad \forall i$.

Cycle notation:

$(a_1\ \ a_2\ \cdots\ a_k)_n$ means $\prod_{q\in\mathbb{Z}}(a_1 + qn\ \ a_2 + qn\ \cdots\ a_k + qn)$.

Infinite cycles are $(\cdots\ a_1\ \ a_2\ \cdots\ a_k\ \ a_i + qn\ \cdots)$, $\quad q \neq 0$.

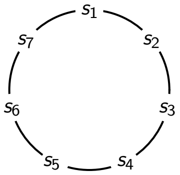
Reflections: $T = \{(i\ \ j)_n : i < j, i \not\equiv j \pmod{n}\}$.

Loops: $\ell_i = (\cdots\ i\ \ \ i+n\ \cdots) \qquad L = \{\ell_i^{\pm 1} : i \in 1, \ldots, n\}$

Generators: $\widetilde{S}_n$ generated by T. $\quad S_{\mathbb{Z}}(\mathrm{mod}\ n)$ generated by $T \cup L$.

Affine type $\widetilde{A}$: Coxeter elements

The Coxeter diagram for $\widetilde{S}_n$ is an n-cycle.
Example: $n = 6$

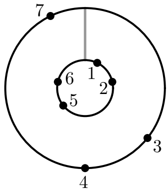


The choice of Coxeter element is recorded by placing of numbers on the annulus:

Place $1, \ldots, n$ in clockwise order.

- i on the outer boundary iff s_{i-1} is before s_i.
- i on the inner boundary iff s_{i-1} is after s_i.

Example: $c = s_6 s_5 s_2 s_1 s_3 s_4 s_7$



Noncrossing partitions of an annulus

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Compare Josue Vazquez-Becerra's talk.

Noncrossing partitions of an annulus

Noncrossing partitions of an annulus

Take the annulus, numbered pts. on the inner and outer boundary.

Noncrossing partitions: Set partitions plus additional topology.

An embedded block is

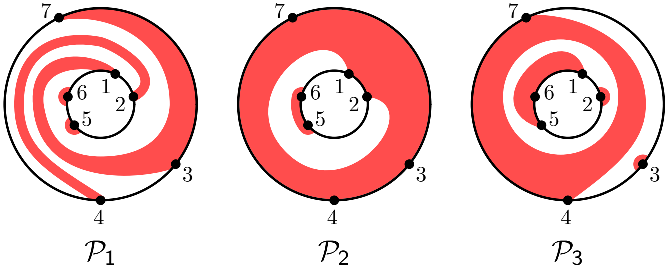
- a disk block, a closed disk containing at least one numbered point.
- a dangling annular block, a closed annulus with one boundary component containing numbered points, the other a nontrivial closed curve.
- a nondangling annular block, a closed annulus with each component of its boundary containing numbered points.

A noncrossing partition is a collection of embedded blocks:

- the embedded blocks are disjoint;
- every numbered point is in some block;
- there is at most one annular block.

Considered up to isotopy.

Noncrossing partitions of an annulus (continued)



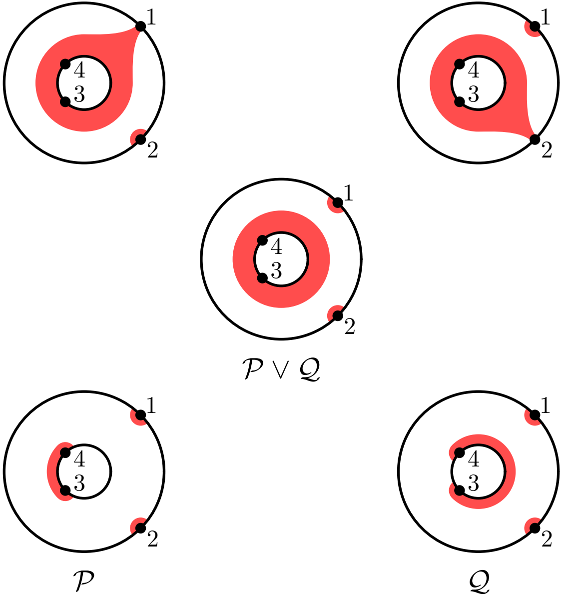
Noncrossing partition lattice $\widetilde{NC}_c^A$:

$\mathcal{P} \leq \mathcal{Q}$ iff every block of $\mathcal{P}$ is contained in a block of $\mathcal{Q}$.

Theorem. $\widetilde{NC}_c^A$ is a graded lattice, with rank function given by n minus the number of non-annular blocks.

Proof idea. The partial order is containment of curve sets. The meet is intersection of curve sets.

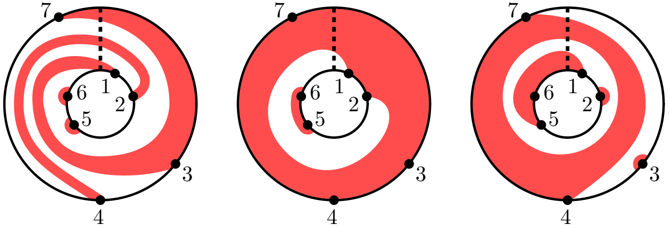
The lattice property needs dangling annular blocks



Isomorphisms

Define a map perm : $\widetilde{NC}_c^A \to S_{\mathbb{Z}}(\mathrm{mod}\ n)$: Read boundaries of blocks as cycles (keep the interior of the block on the right).

Add n each time you cross the date line clockwise, or subtract n when crossing counterclockwise.



$\mathcal{P}_1$ $\mathcal{P}_2$ $\mathcal{P}_3$

$$\mathrm{perm}(\mathcal{P}_1) = (1\ \ -7\ \ -4)_7\,(2\ \ -3)_7\,(5)_7\,(6)_7$$
$$\mathrm{perm}(\mathcal{P}_2) = (\cdots\ 2\ \ 1\ \ -5\ \cdots)\,(\cdots\ 3\ \ 4\ \ 7\ \ 10\ \cdots)\,(5\ \ 6)_7$$
$$\mathrm{perm}(\mathcal{P}_3) = (1\ \ -1\ \ -2)_7\,(2)_7\,(3)_7\,(\cdots\ 4\ \ 7\ \ 11\ \cdots)$$

Isomorphisms (continued)

perm : $\widetilde{NC}_c^A \to S_{\mathbb{Z}}(\mathrm{mod}\ n)$ reads boundaries of blocks as cycles (interior on the right), $\pm n$ when crossing the date line.

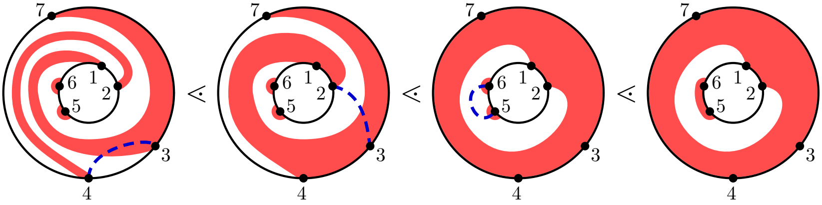
$\widetilde{NC}_c^{A,\circ}$: Noncrossing partitions with no dangling annular blocks.

Theorem.

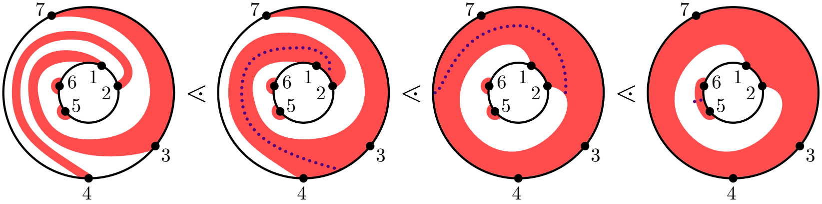
- perm : $\widetilde{NC}_c^A \to S_{\mathbb{Z}}(\mathrm{mod}\ n)$ is an isomorphism from $\widetilde{NC}_c^A$ to $[1, c]_{T \cup L}$ in $S_{\mathbb{Z}}(\mathrm{mod}\ n)$.
- It restricts to an isomorphism from $\widetilde{NC}_c^{A,\circ}$ to $[1, c]_T$ in $\widetilde{S}_n$ (the noncrossing partition poset).

Cover relations in $\widetilde{NC}_c^A$

Covers in $\widetilde{NC}_c^A$ are described by simple connectors



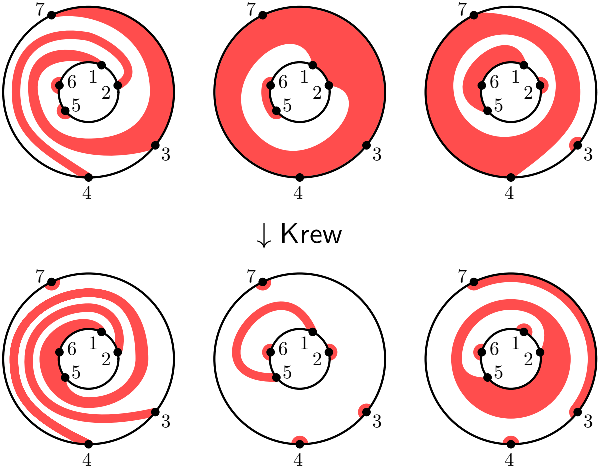
or by cutting curves.



Kreweras complements

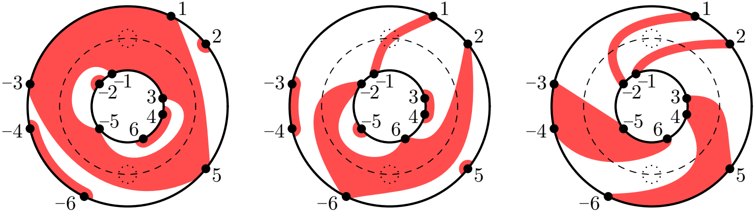
Kreweras complementation:

- an antiautomorphism of $\widetilde{NC}_c^A$
- restricts to an antiautomorphism of $\widetilde{NC}_c^{A,\circ}$.



Affine type $\widetilde{C}$: affine signed permutations

$[1, c]_T$ is the lattice of symmetric n.c. partitions of an annulus



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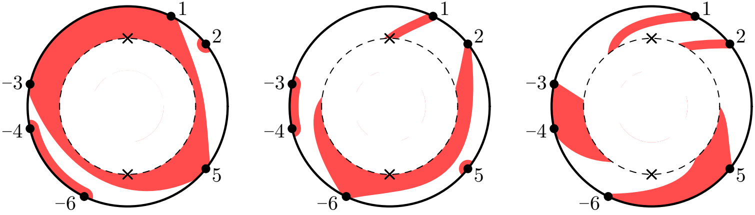

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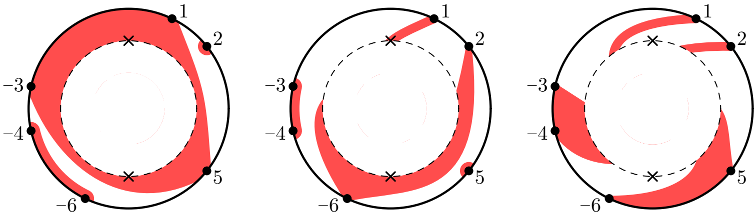
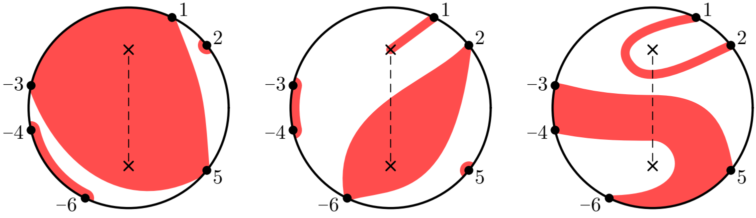


or noncrossing partitions of a disk with 2 orbifold points.



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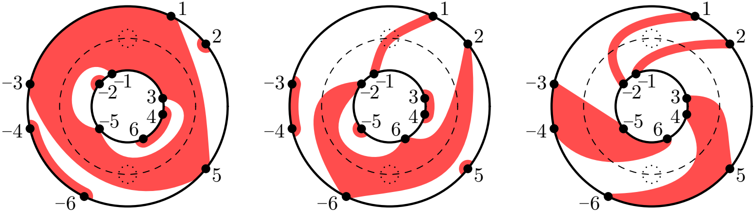

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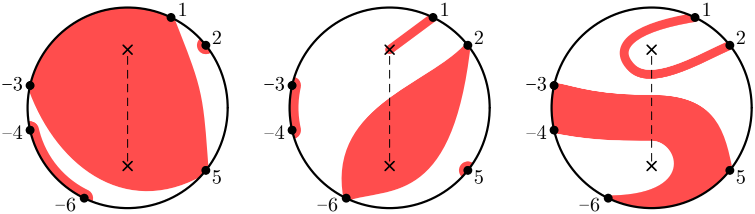
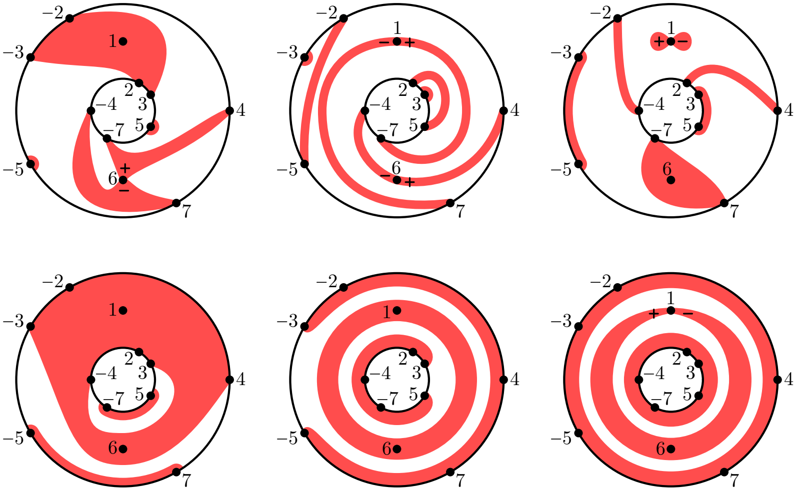


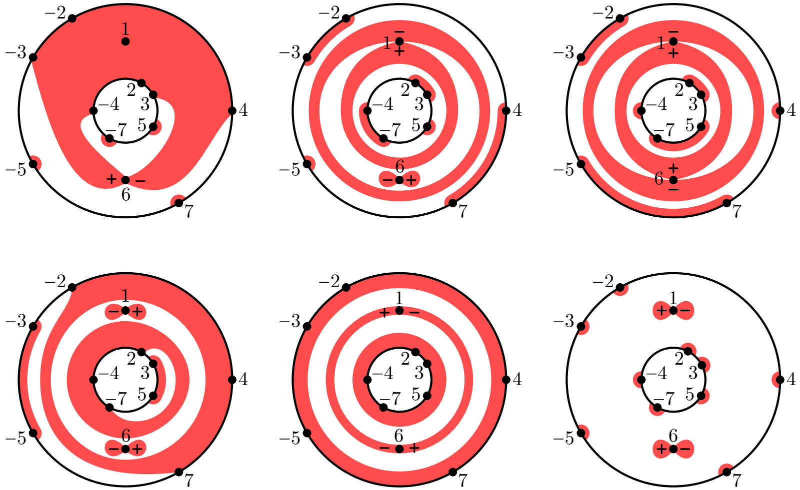
Affine type $\widetilde{D}$

Symmetric n.c. partitions of an annulus with two double points



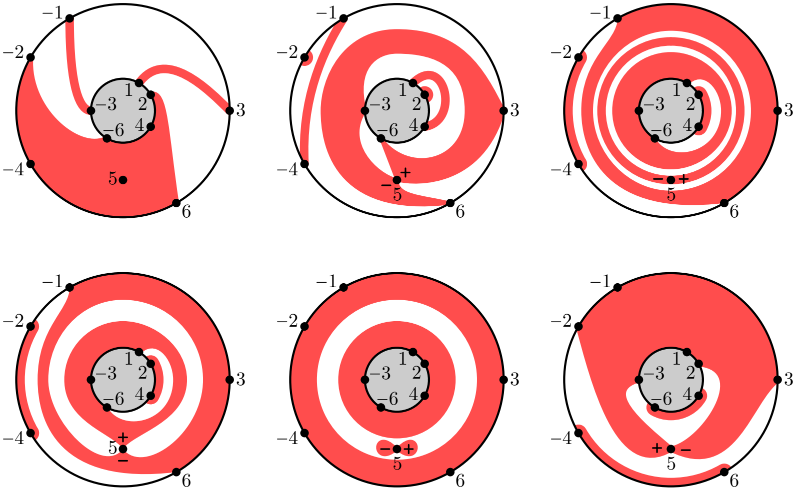
Affine type $\widetilde{D}$ (continued)

Symmetric n.c. partitions of an annulus with two double points



Affine type $\widetilde{B}$

Symmetric n.c. partitions of an annulus with one double point



Summing up the affine models

Classical affine cases:
The planar model suggested by projecting an orbit to the Coxeter plane captures the noncrossing partition poset $[1, c]_T$.
(Exclude dangling annular blocks.)

Types $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$:
The planar model captures the larger lattice that McCammond and Sulway defined. (Allow dangling annular blocks.)

Type $\widetilde{D}$:
To understand the McCammond-Sulway lattice, you also need a small amount of algebraic information.

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Marked surface $(\mathbf{S}, \mathbf{M})$:

- a compact surface **S** with boundary and
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In cluster algebras language: a marked surface without punctures.

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An embedded block in $(\mathbf{S}, \mathbf{M})$: A closed subset E of $\mathbf{S}$ such that $(E, E \cap \mathbf{M})$ is a marked surface (+ some conditions).

A noncrossing partition of $(\mathbf{S}, \mathbf{M})$: A collection of disjoint embedded blocks such that every point in $\mathbf{M}$ is contained in some block (+ some conditions). (Consider these up to ambient isotopy.)

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Theorem. $\mathrm{NC}(\mathbf{S}, \mathbf{M})$ is a graded lattice with rank function

$$\mathrm{rank}(\mathcal{P}) = |\mathbf{M}| + b_1(\mathcal{P}) - b_0(\mathcal{P}).$$

(Betti numbers)

Symmetric noncrossing partitions with double points

$\mathbf{D}\pm$: Double points in the interior of a surface $\mathbf{S}$ (two "copies" of each point in $\mathbf{D}$)

$\mathbf{S}^{\pm}$: The surface with these points

$\mathbf{B}$: Marked points on the boundary

φ: An involutory symmetry of $\mathbf{S}$ with finite fixed-point set containing $\mathbf{D}$. Acts on $\mathbf{S}^{\pm}$ by also swapping double points.

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Cluster algebra intuition: We should "mod out" by φ and turn double points into "punctures". But that loses information about noncrossing partitions.

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A symmetric noncrossing partition of $(\mathbf{S}^{\pm}, \mathbf{B}, \mathbf{D}^{\pm}, \varphi)$: A collection of disjoint embedded blocks, such that the action of φ permutes the blocks of $\mathcal{P}$, every point in $\mathbf{M}$ is contained in some block of $\mathcal{P}$ (+ conditions).

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Theorem. NC($\mathbf{S}, \mathbf{M}$) is a graded ~~lattice~~ poset with rank function

$$\text{rank}(\mathcal{P}) = \frac{1}{2}|\mathbf{B}| + |\mathbf{D}| + b_1^{\varphi}(\mathcal{P}) - b_0^{\varphi}(\mathcal{P}).$$

(something like Betti numbers)

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*The two double points are different points.

Theorem. $\mathrm{NC}(\mathbf{S}, \mathbf{M})$ is a graded poset with rank function

$$\mathrm{rank}(\mathcal{P}) = \frac{1}{2}|\mathbf{B}| + |\mathbf{D}| + b_1^{\varphi}(\mathcal{P}) - b_0^{\varphi}(\mathcal{P}).$$

(something like Betti numbers)

Factored translations and the McCammond-Sulway lattice

McCammond and Sulway build the larger interval (in their larger group) by factoring the translations in $[1, c]_T$.

Let F be the set of all factors that arise.

The set T of reflections and the set F together generate a group larger than the Coxeter group W.

$[1, c]_{T \cup F}$: The interval (analogous to the noncrossing partition poset) in this larger group.

Theorem (McCammond-Sulway). $[1, c]_{T \cup F}$ is a lattice (and furthermore a Garside structure).

Corollary (McCammond-Sulway). ...long-conjectured facts about the corresponding Euclidean Artin groups...

Factored translations and dangling annular blocks

Recall in type $\widetilde{A}$:

$[1,c]_T \cong \widetilde{NC}_c^{A,\circ}$ (n.c. partitions, with no dangling annular blocks).

Translations in $[1,c]_T$ are

$$(\cdots\ i\ \ i+n\ \cdots)(\cdots\ j\ \ j-n\ \cdots) \text{ for } i \text{ outer and } j \text{ inner}$$

$$\longleftrightarrow$$

Noncrossing partitions with only one nontrivial block—
an annulus with one numbered point on each boundary component.

The obvious factorization is $\ell_i \cdot \ell_j^{-1}$.

$\ell_i \longleftrightarrow$ the dangling annular block containing only i.

$\ell_j^{-1} \longleftrightarrow$ the dangling annular block containing only j.

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(There are similar ideas in other classical types.)

A combinatorial abstraction for factored translations

(with Eric Hanson and guided by representation theory)

Let c be a Coxeter element

U^c: A hyperplane in the root space such that a root γ has finite c-orbit iff $\gamma \in U_c$. (For rep theorists: "roots in the tubes".)
Roots in U^c form a "sub root system" Υ^c.

Formal symbols:

- f_β for every simple root in the subsystem (factored translations).
- r_γ for every positive root $\gamma \notin \Upsilon^c$ and every positive root $\gamma \in \Upsilon^c$ with non-full support. (reflections in $[1, c]_T$).

The roots in Υ^c with non-full support will be $r_{\beta,k} = r_{\beta_{(k)}}$.

$$\beta_{(k)} = \beta + c(\beta) + \cdots + c^{k-1}(\beta)$$

with β simple (+ conditions).

Connections with representation theory

Results of Hubery-Krause and Igusa-Schiffler amount to a characterization of sequences of formal symbols r_γ that correspond to maximal chains in the noncrossing partition poset $[1, c]_T$.

We characterize combinatorially the sequences of formal symbols r_γ and f_β that correspond to maximal chains in the McCammond-Sulway lattice $[1, c]_{T \cup F}$.

Some consequences.:

- Maximal chains in the McCammond-Sulway lattice are in bijection with something like exceptional sequences (entries in the sequences are any non-homogeneous bricks).
- The McCammond-Sulway lattice $\cong$ a poset of certain wide subcategories.
- A representation-theory proof of the lattice property.

Thanks for listening!

L. Brestensky and N. Reading, **Noncrossing partitions of an annulus.**

E. Hanson and N. Reading **Title TBD** (in preparation).

J. McCammond and R. Sulway, **Artin groups of Euclidean type.**

N. Reading. **Noncrossing partitions of a marked surface.**

N. Reading, **Symmetric noncrossing partitions of an annulus with double points.**