Geometric tensors via spectral functionals

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Given a Laplace operator we use the noncommutative residue to define certain functionals of vector fields which yield metric and Einstein tensors. Alternatively, given a Dirac operator we define dual metric and Einstein functionals of differential forms, and also Ricci and torsion functionals. We generalise these concepts in non-commutative geometry and show e.g. that for the conformally rescaled noncommutative 2-torus the Einstein and the torsion functionals vanish. Also the Hodge-de Rham, Einstein-Yang-Mills and quantum SU(2) group spectral triples are torsion free, while the quantum 2-sheeted space has torsion. [Adv.Math. 427, 1091286, 2023; Commun.Math.Phys. 130, 2024 and DOI 10.4171/JNCG/573 (2024) with A. Sitarz and P. Zalecki].

ESI, Wien, 24 July 2024

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An eminent spectral scheme that generates geometric objects on Riemannian manifolds (volume, scalar curvature . . .) is $t \searrow 0$ asymptotic expansion of the trace of heat kernel

$$\mathrm{Tr}\, e^{-t\Delta} \approx \sum_{\ell=0}^{\infty} t^{\frac{\ell-n}{2}} a_\ell.$$

Here the scalar laplacian Δ for metric $g = \{g_{jk}\}$ reads

$$\Delta = -\frac{1}{\sqrt{\det(g)}} \partial_j \big(\sqrt{\det(g)} g^{jk} \partial_k \big). \tag{1}$$

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The coefficients a_ℓ can be transmuted into some values or residues of the zeta function of Δ, and in turn expressed using the noncommutative (Wodzicki) residue $\mathcal{W}$

$$\mathcal{W}(P) := \frac{1}{vol(S^{n-1})} \int_M \left(\int_{|\xi|=1} tr\, \sigma_{-n}(P)(x,\xi)\ \mathcal{V}_\xi \right)\ d^n x. \tag{2}$$

♠

→ residues

Geometry from residues:

Then, on closed oriented M of even dimension $n = 2m$

$$\mathcal{W}(\Delta^{-m}) = vol(M),$$

and in the *localized* form (as a functional of $f \in C^\infty(M)$)

$$\mathcal{V}(f) := \mathcal{W}(f\Delta^{-m}) = \int_M f \; vol_g.$$

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A. Connes divulged in 90s a startling result, confirmed by Kastler and by Kalau-Walze:

$$\mathcal{W}(\Delta^{-m+1}) = \frac{n-2}{12} \int_M R \; vol_g, \tag{3}$$

which is $\propto$ the Einstein-Hilbert action functional (of g) for the Riemannian general relativity (in vacuum).
Here R is the scalar curvature

$$R = R(g) = g^{jk} R_{jk} = g^{jk} R_{\ell j \ell k}.$$

A localised form of (3) is the *scalar curvature* functional on $C^\infty(M)$

$$\mathcal{R}(f) := \mathcal{W}(f\Delta^{-m+1}) = \frac{n-2}{12} \int_M f R \, vol_g. \tag{4}$$

$\looparrowright$ This is related to the asymptotic growth of eigenvalues of Δ; clear e.g. from the Connes "trace thm." that $\mathcal{W} = \mathrm{Tr}^+$. $\looparrowleft$

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$\looparrowright$ We have uncovered few new spectral 'localised' functionals, by placing some differential operators in place of f.
Let's start e.g. with a pair of vector fields V and W on M, viewed as derivations of $C^\infty(M)$:

Def/Thm: Metric functional

The functional

$$\mathrm{g}^{\Delta}(\mathrm{V},\mathrm{W}) := \mathcal{W}\big(\mathrm{V}\mathrm{W}\Delta^{-\mathrm{m}-1}\big)$$

is a bilinear, symmetric map, whose density is proportional to the metric g *evaluated on* V, W

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Def/Thm: Einstein functional

The functional

$$\mathrm{G}^{\Delta}(V,W) := \mathcal{W}\big(VW\Delta^{-m}\big)\,, \tag{5}$$

is a bilinear, symmetric map, whose density is proportional to the Einstein tensor $G := \mathit{Ric} - \frac{1}{2}R\, g$ *evaluated on V, W*

$$\mathrm{G}^{\Delta}(V,W) = \frac{1}{6}\int_M G(V,W)\, vol_g.$$

"Proof"

Algebra of symbols of pseudodifferential operators:

$$\sigma(PQ)(x,\xi) = \sum_{\beta} \frac{(-i)^{|\beta|}}{|\beta|!} \frac{\partial}{\partial \xi^{\beta}} \sigma(P)(x,\xi) \frac{\partial}{\partial x^{\beta}} \sigma(Q)(x,\xi). \qquad (6)$$

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Taylor expansion in <u>normal</u> coordinates x:

metric

$$g_{ab} = \delta_{ab} - \frac{1}{3} R_{acbd} x^c x^d + o(|x|^2), \qquad (7)$$

volume element

$$\sqrt{\mathrm{det}(g)} = 1 - \frac{1}{6} R_{ab} x^a x^b + o(|x|^2), \qquad (8)$$

and Levi-Civita symbol

$$\Gamma^a_{bc}(x) = -\frac{1}{3}(R_{abcd} + R_{acbd}) x^d + o(|x|^2). \qquad (9)$$

where R_{acbd} and R_{ab} are the values at $x = 0$.

"Proof" 2

Consequently, $\sigma(\Delta) = \mathfrak{a}_2 + \mathfrak{a}_1$, where

$$\begin{aligned} \mathfrak{a}_2 =& \big(\delta_{ab} + \frac{1}{3} R_{acbd} x^c x^d\big) \xi_a \xi_b + o(|x|^2), \\ \mathfrak{a}_1 =& \frac{2i}{3} R_{ab} x^a \xi_b + o(|x|^2). \end{aligned} \tag{10}$$

Next we compute the first three leading symbols of Δ^{-1}, and then of Δ^{-k}, $k > 0$, up to order resp. $o(|x|^2), o(|x|), o(1)$:

$$\sigma(\Delta^{-k}) = \mathfrak{c}_{2k} + \mathfrak{c}_{2k+1} + \mathfrak{c}_{2k+2} + \dots,$$

$$\begin{aligned} \mathfrak{c}_{2k} &= ||\xi||^{-2k-2} \left(\delta_{ab} - \frac{k}{3} R_{acbd} x^c x^d \right) \xi_a \xi_b + o(|x|^2), \\ \mathfrak{c}_{2k+1} &= \frac{-2ki}{3||\xi||^{2k+2}} R_{ab} x^b \xi_a + o(|x|), \\ \mathfrak{c}_{2k+2} &= \frac{k(k+1)}{3||\xi||^{2k+4}} R_{ab} \xi_a \xi_b + o(1). \end{aligned} \tag{11}$$

Now the composition with $\sigma(VW)$ shows the statements. $\square$

Laplace-type, Spin Laplacian, squared Dirac

More generally, we've treated Laplace-type operators

$$\Delta_{T,E} = -g^{ab}(\nabla_a\nabla_b - \Gamma^c_{ab}\nabla_c) + E$$

on a vector bundle Ξ with connection ∇ and $E \in End\ \Xi$.

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A particular interesting case is a $spin_c$ manifold M with Ξ a spinor bundle Σ of rank 2^m and the spin Laplacian

$$\Delta^{(s)} := \nabla^{(s)*}\nabla^{(s)} = -\nabla^{(s)}_{e_i}\nabla^{(s)}_{e_i} + \nabla^{(s)}_{\nabla_{e_i} e_i}, \tag{12}$$

where $\nabla^{(s)}$ is the spin connection and e_j is ON frame:

Proposition

$$\begin{aligned} \mathrm{g}^{\Delta^{(s)}}(V,W) &:= \mathcal{W}\big(\nabla^{(s)}_V \nabla^{(s)}_W (\Delta^{(s)})^{-m-1}\big) = 2^m \mathrm{g}^{\Delta}(V,W), \\ \mathrm{G}^{\Delta^{(s)}}(V,W) &:= \mathcal{W}\big(\nabla^{(s)}_V \nabla^{(s)}_W (\Delta^{(s)})^{-m}\big) = 2^m \mathrm{G}^{\Delta}(V,W) + 0. \end{aligned} \tag{13}$$

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or squared Dirac (coupled do $U(1)$-gauge 1-form A):

Proposition

$$g^{D_A^2}(V,W) := \mathcal{W}(\nabla^{(s)}_V \nabla^{(s)}_W |D_A|^{-n-2}) = 2^m g^{\Delta}(V,W),$$

$$G^{D_A^2}(V,W) := \mathcal{W}(\nabla^{(s)}_V \nabla^{(s)}_W |D_A|^{-n})$$

$$= 2^m\Big(G^{\Delta}(V,W) + 2^{-3}\int_M R\, g(V,W) vol_g\Big).$$

Go quantum (= noncommutative)

Noncommutative tori are prominent examples of quantum spaces. Their smooth algebra $A = C^\infty(\mathbb{T}^n_\theta)$, generated by n unitaries U_j,

$$U_j U_k = \delta_{jk} e^{i\theta} U_k U_j,$$

has a faithful state τ invariant under derivations δ_j, $\delta_j U_k = \delta_{jk} U_k$, which are interpreted as noncommutative vector fields.

One regards $\Delta = \sum_j \delta_j^2$ on $H = L^2(\mathbb{T}^2_\theta, \tau)$ as 'flat' Laplace operator, $D = \sum_j \gamma^j \delta_j$ on $H = L^2(\mathbb{T}^2_\theta, \tau) \otimes \mathbb{C}^{2^m}$ as 'flat' Dirac operator and the A-bimodule $\Omega_D(A)$ generated by $[D, A]$, as 1-forms. ♠

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For simplicity consider the *strictly irrational* $\mathbb{T}^n_\theta$ (i.e., $\mathcal{Z}(A) = \mathbb{C}$) with τ extended to $\hat{A} := A \otimes A^o$ as $\tau(a \otimes b^o) = \tau(a)\tau(b^o)$, where A^o is a copy of A in the commutant A' of A in $B(H)$.
Such τ is still invariant under the extended derivations.
We use it to define the tracial state $\mathcal{W}$ on $\hat{A}$-valued symbols $\sigma(\xi)$ (where $\delta_a \mapsto \xi_a$ much the same as for M).

Rescaled NC 2-torus: vector fields

Given $0 < h \in C^\infty(\mathbb{T}^2_\theta)$, by a conformally rescaled Δ on $\mathbb{T}^2_\theta$ we mean the selfadjoint operator on $H = L^2(\mathbb{T}^2_\theta, \tau)$:

♠
$$\Delta_h = h^{-1}\Delta\, h^{-1}.$$

Accordingly, as vector fields we take

$$V_h = \sum_{a=1,2} V^a h\delta_a h^{-1}, \quad V^a \in \mathbb{C}.$$

Proposition

$$\mathrm{g}^{\Delta_h}(V_h, W_h) = \mathcal{W}\big(V_h W_h \Delta_h^{-2}\big) = \pi\tau(h^4)V^a W^a,$$

whereas

$$\mathrm{G}^{\Delta_h}(V_h, W_h) = \mathcal{W}\big(V_h W_h \Delta_h^{-1}\big) = 0\,.$$

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We have also computed $\mathbb{T}^4_\theta$.

Can do also θ-deformed spaces, or NC spaces with derivations.

Alternatively ...

Spectral functionals on 1-forms

Now use D on spinors in a two-fold way to get (in terms of $\mathcal{W}$) certain "dual functionals" which are bilinear on 1-forms (co-vectors) and yield contravariant tensors (with "raised indices").

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For that need to represent 1-forms v as differential operators.
On a spin$_c$c manifold M use the Clifford representation of v as 0-order differential operators $\hat{\nu} \in \mathrm{End}(\Sigma)$.
As known they form a $C^\infty(M)$-bimodule $\Omega^1_D \simeq \Omega^1(M)$ generated by commutators of D with functions.
Thus the spinorial Dirac operator is self-sufficient for our purposes (and NCG-ready when assembled to a spectral triple of A. Connes), so comes now in its grandeur

→ fnls

Metric and Einstein functionals on 1-forms

Thm

The spectral functionals of one-forms on M

$$\mathrm{g}_D(v,w) := \mathcal{W}\big(\hat{v}\hat{w}D^{-n}\big),$$
$$\mathrm{G}_D(v,w) := \mathcal{W}\big(\hat{v}(D\hat{w}+\hat{w}D)D^{-n+1}\big) \qquad (14)$$
$$= \mathcal{W}\big((D\hat{v}+\hat{v}D)\hat{w}D^{-n+1}\big),$$

read

$$\mathrm{g}_D(v,w) = 2^m \int_M g(v,w)\ vol_g,$$
$$\mathrm{G}_D(v,w) = \frac{2^m}{6}\int_M G(v,w)\ vol_g, \qquad (15)$$

where $G = Ric - \frac{1}{2}Rg$ *is the contravariant Einstein tensor.*

They perfectly (dually) match g^Δ and G^Δ up to 2^m.

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They perfectly (dually) match g^Δ and G^Δ up to 2^m.

Actually,

$$\mathrm{Ric}_D(v,w) := \mathcal{W}\big(\hat v (D\hat w + \tfrac{n-4}{n-2}\hat w D) D^{-n+1}\big) = \tfrac{2^m}{6}\int_M Ric(v,w)\ vol_g.$$

→ torus

Rescaled noncommutative 2-torus: 1-forms

The above functionals extend to NC spaces:
As the conformal rescaling of D on $\mathbb{T}^n_\theta$ we take on H

$$D_k = kDk,$$

following Connes-Moscovici, however with $0 < k \in A^o \subset A'$,
which assures that (A, D_k, H) is a spectral triple and $\exists\, \Omega^1_{D_k}(A)$. ♠
In effect, $\Omega^1_{D_k}(A)$ is freely generated by $k^2\gamma^j$.

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For $n{=}2$, $\gamma^j{=}\sigma^j$, and for $\mathbb{T}^2_\theta$ we have

Proposition

For $v = k^2 v^j \sigma^j$ and $w = k^2 w^j \sigma^j$, $v^j, w^j \in A$,

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Spectral Torsion

In principle *connections* not needed for abstract Δ or D.
Thanks to our g_D we can now 'control' the *metricity* condition.
Instead what about the *zero-torsion* condition ?
Not clear if any (enigmatic & complicated) minimization procedure could be employed for that.
But the contribution of torsion can contaminate our g & G (!).

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But the contribution of torsion can contaminate our g & G (!).
Fortunately, for a n-summable regular $(\mathcal{A}, D, \mathcal{H})$, using $\mathcal{W}$ coming from the ΨDO calculus and tracial state by Connes-Moscovici'95, we found:

Def/Thm: Torsion functional

Torsion functional *is a trilinear functional of* $u, v, w \in \Omega^1_D(\mathcal{A})$,

$$\mathcal{T}_D(u, v, w) := \mathcal{W}(uvwD|D|^{-n}).$$

We say that D *is torsion-free if* $\mathcal{T}_D \equiv 0$. *For the Dirac operator* D_T *with torsion* T *on a closed spin manifold of dimension* n

$$\mathcal{T}_{D_T}(u, v, w) = -2^{[\frac{n}{2}]} i \int_M u_a v_b w_c T_{abc} vol_g. \tag{16}$$

Examples

$\boxed{\mathcal{T}=0}$ for:

• Hodge-de Rham: $\big(C^\infty(M), L^2(\Omega^\bullet_M), d+d^*\big)$.

• Einstein-Yang-Mills: $\big(C^\infty(M)\otimes M_N(\mathbb{C}), L^2(\Sigma)\otimes M_N(\mathbb{C})\big), \widetilde{D})$,
where $\widetilde{D} = D\otimes \mathrm{id}_N + A + JAJ^{-1}$ with $A = A^* \in \Omega^1_{\widetilde{D}}$ and
$J = C\otimes *$, with C being the charge conjugation on spinors in Σ.

• conformally rescaled noncommutative tori.

• quantum $SU(2)$: $\big(\mathcal{A}(SU_q(2)), \mathcal{H}, D\big)$, where $\mathcal{H}$ and D are
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$\boxed{\mathcal{T}\neq 0}$ for:

• almost commutative $M\times\mathbb{Z}_2$: $(C^\infty(M)\otimes\mathbb{C}^2, L^2(\Sigma)\otimes\mathbb{C}^2, \mathcal{D})$, where $\mathcal{D} = \begin{pmatrix} D & \chi\phi \\ \chi\phi^* & D\end{pmatrix}$, with D on Σ graded by χ, and $\phi\in\mathbb{C}$.

Now, $\Omega^1_{\mathcal{D}} \ni \omega = \begin{pmatrix} w^+ & \phi\chi f^+ \\ \phi^*\chi f^- & w^-\end{pmatrix}$ for $w^\pm \in \Omega^1(M)$, $f^\pm \in C^\infty(M)$.

Then, $\mathcal{W}\big(\omega_1^o\omega_2^o\omega_3^o\mathcal{D}\mathcal{D}^{-2m}\big) = \mathcal{W}\big(|\phi|^4(f_1^+f_2^-f_3^+ + f_1^-f_2^+f_3^-)D^{-2m}\big)$

$$= |\phi|^4 \int_M (f_1^+f_2^-f_3^+ + f_1^-f_2^+f_3^-) vol_g.$$

Outlook

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Conjecture: For a 2-dimensional regular spectral triple $\mathrm{G}_D = 0$.

- relation of $\mathcal{T}_D$ to other settings (algebraic, differential) for T and quantum analogues of Levi-Civita connection in the literature
- relation to W. Ugalde differential forms & conformal gometry

THANK YOU !