Gopakumar–Vafa invariants

– what they mean to a physicist, and how to compute them –

Part I: GV-invariants are BPS-invariants

- important physics data: given QFT (+ gravity), what is the spectrum of masses & charges of particles?

- Wigner's class.: particle $\simeq$ unitary irrep. of

  Poincaré group $\times$ compact Lie group

  [cf. Coleman–Mandula theorem '67]

  algebra:
  $$[P_\mu, P_\nu] = 0$$
  $$[P_\mu, M_{\nu\rho}] = i(\eta_{\mu\nu} P_\rho - \eta_{\mu\rho} P_\nu)$$
  $$[M_{\mu\nu}, M_{\rho\sigma}] = i(M_{\mu\rho}\eta_{\nu\sigma} + \dots)$$

- labeled by (mass, spin, R)

- $\underline{\text{mass}}$: $P_\mu P^\mu |\psi\rangle = -m^2 |\psi\rangle$

w.l.o.g. $\rightsquigarrow$ $P_\mu |\psi\rangle = \underbrace{(-m, \vec{0}_{d-1})}_{\text{invariant under } SO(d-1)} |\psi\rangle$

"massive little group"

- $\underline{\text{spin}}$: $SO(d-1)$ - rep. (or really $Spin(d-1)$)

$SO(d-1) \equiv$ "massive little group"

$\underline{d=4}$: double cover $Spin(3) = SU(2)$

$\rightsquigarrow$ spin $= j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$

$\dim = 2j+1$

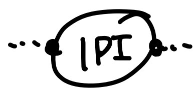
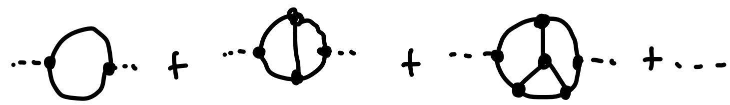
$\underline{d=5}$: $Spin(4) = SU(2)_L \times SU(2)_R$

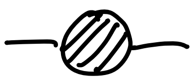
$\Rightarrow$ spin $= (j_L, j_R)$

$j_{L,R} \in \{0, \frac{1}{2}, \dots\}$

scalar: $(0,0)$
spinor: $(\frac{1}{2}, 0), (0, \frac{1}{2})$
vector: $(\frac{1}{2}, \frac{1}{2})$ etc.

- masses are usually impossible to compute exactly in interacting QFTs due to self-energy:

$-i\Sigma(p^2) \equiv$  $=$ 

 $= \dfrac{-i}{p^2+m^2+\Sigma(p^2)}$

- in SUSY theories, situation is sometimes better: SUSY-multiplets are irreps of

$$(\text{SuperPoincaré}) \times (\text{compact Lie group})$$

Poincaré-algebra

in 4d:

$I, J = 1, \dots, N$

$N \leq 8$

$$+ \ \{Q^I_\alpha, \bar{Q}^{\bar{J}}_{\dot\beta}\} = -2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^{I\bar{J}}$$

$$[M^{\mu\nu}, Q^I_\alpha] = i(\sigma^{\mu\nu})_\alpha{}^\beta Q^I_\beta$$

$$\{Q^I_\alpha, Q^J_\beta\} = 2\epsilon_{\alpha\beta} Z^{IJ}$$

central extension for $N \geq 2$ in 4d

for $N=2$: $Z^{IJ} = \begin{pmatrix} 0 & Z \\ -Z & 0 \end{pmatrix}$,

BPS-bound $m \geq |Z|$

key fact: multiplets that saturate it,

$$m = |Z|,$$

are annihilated by half of the Q_α^I

$\leadsto$ # of states is halved.

in 5d $\mathcal{N}=1$: M-theory on CY_3!

- $Spin(4) = SU(2)_L \times SU(2)_R$ (massive little group)
- 8 real supercharges $Q \cong$ Dirac spinor (better: symplectic-Majorana)

$$= \left(\tfrac{1}{2},0\right) \oplus \left(0,\tfrac{1}{2}\right)$$

under $Spin(4) \subseteq Spin(1,4)$

$\Rightarrow$ two independent $SU(2)$-spinors.

- generic "long" multiplet:

$$\left[2(0,0) + \left(\tfrac{1}{2},0\right)\right] \otimes \left[2(0,0) + \left(0,\tfrac{1}{2}\right)\right] \otimes (j_L, j_R)$$

ex.: non-BPS vectormultiplet, $j_L = j_R = 0$,

$$\underbrace{4\cdot(0,0)}_{4 \text{ scalars}} \oplus \underbrace{2\cdot\left(\tfrac{1}{2},0\right) \oplus 2\cdot\left(0,\tfrac{1}{2}\right)}_{2 \text{ Dirac spinors}} \oplus \underbrace{\left(\tfrac{1}{2},\tfrac{1}{2}\right)}_{1 \text{ vector}}$$

- BPS-multiplet (annihilated by Q^R_α):

$$\left[2(0,0) + (\tfrac{1}{2},0)\right] \otimes (j_L, j_R)$$

e.g.: $j_L = j_R = 0$: $2(0,0) \oplus (\tfrac{1}{2},0)$

"BPS half-hypermultiplet"

$j_L = 0$, $j_R = 1/2$: $2(0,\tfrac{1}{2}) \oplus (\tfrac{1}{2},\tfrac{1}{2})$

"BPS vectormultiplet"

actually: CRT theorem requires symm. under $j_L \leftrightarrow j_R$,

$\rightsquigarrow$ hypermultiplet: $\underbrace{4(0,0)}_{\text{4 scalars}} \oplus \underbrace{(\tfrac{1}{2},0) \oplus (0,\tfrac{1}{2})}_{\text{Dirac spinor}}$

$\rightsquigarrow$ vectormultiplet: $\underbrace{2(\tfrac{1}{2},0) \oplus 2(0,\tfrac{1}{2})}_{\text{2 Dirac spinors}} \oplus \underbrace{2(\tfrac{1}{2},\tfrac{1}{2})}_{\text{2 vectors}}$

- In principle, BPS spectrum is much better to assess because $m = |Z|$ central extension of susy-algebra (known)

and a single BPS multiplet cannot just become non-BPS under varying Kähler & CS moduli (non BPS-multiplets have more states!)

- Caveat:

  $2\times$ (non-BPS vector)

  $\to 8\cdot(0,0) + 4(\tfrac{1}{2},0) + 4\cdot(0,\tfrac{1}{2}) + 2\cdot(\tfrac{1}{2},\tfrac{1}{2})$

  $= 2\times$(hypermultiplet) $\oplus$ (BPS vectormultiplet)

  $\Rightarrow$ BPS vectors & hypers can disappear from BPS spectrum in pairs

- Only BPS-indices $\simeq n_H - 2n_V$ are invariant under small def.

- geometric point of view:

  $\to$ BPS-state from M2-branes on holomorphic curves $\mathcal{C}$ in CY_3 X.

  $\to$ Mori-cone enhances at special loci in CS moduli space, where additional algebraic curves arise

  $\to$ BPS spectrum jumps, but BPS indices are invariant.

Def.: Gopakumar-Vafa invariants

- Given a SUSY-rep. $\mathcal{R}$,

$$\mathcal{R} \overset{BPS}{=} \left[2(0,0) + (\tfrac{1}{2},0)\right] \otimes \sum_{j_L, j_R} N^{j_L, j_R}_{[e]} (j_L, j_R)$$

from M2s wrapped on curves in fixed class $[e]$,

$$\mathrm{Tr}_R\Big(\underbrace{(-1)^{2j_R}}_{\text{s.t.}} \mathcal{R} \Big) = \sum_{g=0}^{\infty} \boxed{n^g_e} \left[2(0) + (\tfrac{1}{2})\right]^{g+1}$$

s.t. non-BPS multiplets contribute nothing!

genus g GV-invariants

$n^g: H_2(X,\mathbb{Z}) \longrightarrow \mathbb{Z}$