Remarks on Infinite dimensional symplectic and Poisson geometry

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[KM97] Andreas Kriegl, Peter W. Michor: The Convenient Setting of Global Analysis. Mathematical Surveys and Monographs, Volume: 53, Amer. Math. Soc., 1997.

See also:
Wikipedia [https://en.wikipedia.org/wiki/Convenient_vector_space]

[BIM24] Martin Bauer, Sadashige Ishida, Peter W. Michor. Symplectic structures on the space of space curves. arXiv:2407.19908.

Review

For a finite dimensional symplectic manifold (M, ω) we have the following exact sequence of Lie algebras:

$$0 \to H^0(M) \to C^\infty(M, \mathbb{R}) \xrightarrow{\text{grad}^\omega} \mathfrak{X}(M, \omega) \to H^1(M) \to 0.$$

$H^*(M)$ De Rham cohomology of M with 0 bracket.
$C^\infty(M, \mathbb{R})$ is equipped with the Poisson bracket $\{\quad,\quad\}$,
$\mathfrak{X}(M, \omega)$ all vector fields ξ with $\mathcal{L}_\xi \omega = 0$ with usual Lie bracket.

Furthermore, $\text{grad}^\omega f$ is the Hamiltonian vector field for $f \in C^\infty(M, \mathbb{R})$ given by $i(\text{grad}^\omega f)\omega = df$ and $\gamma(\xi) = [i_\xi \omega]$.

Consider a symplectic right action $r : M \times G \to M$ of a connected Lie group G on M; we use the notation $r(x, g) = r^g(x) = r_x(g) = x.g$. By $\zeta_X(x) = T_e(r_x)X$ we get a mapping $\zeta : \mathfrak{g} \to \mathfrak{X}(M, \omega)$ which sends each element X of the Lie algebra $\mathfrak{g}$ of G to the fundamental vector field ζ_X. This is a Lie algebra homomorphism (for right actions!).

$$
\begin{array}{ccccccc}
H^0(M) & \xrightarrow{\ i\ } & C^\infty(M,\mathbb{R}) & \xrightarrow{\ \mathrm{grad}^\omega\ } & \mathfrak{X}(M,\omega) & \xrightarrow{\ \gamma\ } & H^1(M) \\
 & & \nwarrow^{j} & & \nearrow_{\zeta} & & \\
 & & & \mathfrak{g} & & &
\end{array}
$$

A linear lift $j : \mathfrak{g} \to C^\infty(M,\mathbb{R})$ of ζ with $\mathrm{grad}^\omega \circ j = \zeta$ exists if and only if $\gamma \circ \zeta = 0$ in $H^1(M)$. This lift j may be changed to a Lie algebra homomorphism if and only if the 2-cocycle $\bar{\jmath} : \mathfrak{g} \times \mathfrak{g} \to H^0(M)$, given by $(i \circ \bar{\jmath})(X,Y) = \{j(X), j(Y)\} - j([X,Y])$, vanishes in the Lie algebra cohomology $H^2(\mathfrak{g}, H^0(M))$, for if $\bar{\jmath} = \delta\alpha$ then $j - i \circ \alpha$ is a Lie algebra homomorphism.

If $j : \mathfrak{g} \to C^\infty(M,\mathbb{R})$ is a Lie algebra homomorphism, we may associate the *momentum mapping* $J : M \to \mathfrak{g}' = L(\mathfrak{g},\mathbb{R})$ to it, which is given by $J(x)(X) = \chi(X)(x)$ for $x \in M$ and $X \in \mathfrak{g}$. It is G-equivariant for a suitably chosen (in general affine) action of G on $\mathfrak{g}'$.

Infinite dimensional weak symplectic manifolds

Let M be a manifold, in general is infinite dimensional, Hausdorff, in the sense of convenient calculus.
A 2-form $\omega \in \Omega^2(M)$ is called a *weak symplectic structure* on M if the following three conditions holds:

1. ω is closed, $d\omega = 0$.
2. The associated vector bundle homomorphism $\check{\omega} : TM \to T^*M$ is injective.
3. The gradient of ω with respect to itself exists and is smooth; this can be expressed most easily in charts, so let M be open in a convenient vector space E. Then for $x \in M$ and $X, Y, Z \in T_xM = E$ we have $d\omega(x)(X)(Y, Z) = \omega(\Omega_x(Y, Z), X) = \omega(\tilde{\Omega}_x(X, Y), Z)$ for smooth $\Omega, \tilde{\Omega} : M \times E \times E \to E$ which are bilinear in $E \times E$.

A 2-form $\omega \in \Omega^2(M)$ is called a *strong symplectic structure* on M if it is closed ($d\omega = 0$) and if its associated vector bundle homomorphism $\check{\omega} : TM \to T^*M$ is invertible with smooth inverse.

In this case, the vector bundle TM has reflexive fibers T_xM: Let $i : T_xM \to (T_xM)''$ be the canonical mapping onto the bidual. Skew symmetry of ω is equivalent to the fact that the transposed $(\check{\omega})^t = (\check{\omega})^* \circ i : T_xM \to (T_xM)'$ satisfies $(\check{\omega})^t = -\check{\omega}$. Thus, $i = -((\check{\omega})^{-1})^* \circ \check{\omega}$ is an isomorphism.

Cotangent bundles

Every cotangent bundle T^*Q, viewed as a manifold, carries a canonical weak symplectic structure $\omega_Q \in \Omega^2(T^*Q)$, which is defined as follows. Let $\pi_Q^* : T^*Q \to Q$ be the projection. Then the *Liouville form* $\theta_Q \in \Omega^1(T^*Q)$ is given by $\theta_Q(X) = \langle \pi_{T^*Q}(X), T(\pi_Q^*)(X) \rangle$ for $X \in T(T^*Q)$, where $\langle \quad , \quad \rangle$ denotes the duality pairing $T^*Q \times_Q TQ \to \mathbb{R}$. Then the symplectic structure on T^*Q is given by $\omega_Q = -d\theta_Q$, which of course in a local chart looks like $\omega_E((v,v'),(w,w')) = \langle w', v \rangle_E - \langle v', w \rangle_E$. The associated mapping $\check{\omega} : T_{(0,0)}(E \times E') = E \times E' \to E' \times E''$ is given by $(v,v') \mapsto (-v', i_E(v))$, where $i_E : E \to E''$ is the embedding into the bidual. So the canonical symplectic structure on T^*Q is strong if and only if all model spaces of the manifold Q are reflexive.

Towards the Hamiltonian mapping

Let M be a weak symplectic manifold. The first thing to note is that the Hamiltonian mapping $\text{grad}^\omega : C^\infty(M,\mathbb{R}) \to \mathfrak{X}(M,\omega)$ does not make sense in general, since $\check{\omega} : TM \to T^*M$ is not invertible. Namely, $\text{grad}^\omega f = (\check{\omega})^{-1} \circ df$ is defined only for those $f \in C^\infty(M,\mathbb{R})$ with $df(x)$ in the image of $\check{\omega}$ for all $x \in M$. A similar difficulty arises for the definition of the Poisson bracket on $C^\infty(M,\mathbb{R})$.

For a weak symplectic manifold (M,ω) let $T_x^\omega M$ denote the real linear subspace $T_x^\omega M = \check{\omega}_x(T_xM) \subset T_x^*M = L(T_xM,\mathbb{R})$, and let us call it the *ω-smooth cotangent space* with respect to ω of M at x. The convenient structure on $T_x^\omega M$ is the one from T_xM. All $T_x^\omega M$ together form a subbundle of T^*M isomorphic to TM via $\check{\omega} : TM \to T^\omega M \subseteq T^*M$. It is in general not a splitting subbundle.

Note that only for strong symplectic structures the mapping $\check{\omega}_x : T_xM \to T_x^*M$ is a diffeomorphism onto $T_x^\omega M$ with the structure induces from T_x^*M.

Definition of $C^\infty_\omega(E,\mathbb{R}) \subset C^\infty(E,\mathbb{R})$.

For a weak symplectic vector space (E,ω) we consider linear subspace $C^\infty_\omega(E,\mathbb{R}) \subset C^\infty(E,\mathbb{R})$ consisting of all smooth functions $f : E \to \mathbb{R}$ such that

- each iterated derivative $d^k f(x) \in L^k_{\text{sym}}(E;\mathbb{R})$ has the property that
$$d^k f(x)(\quad, y_2, \dots, y_k) \in E^\omega$$
is actually in the smooth dual $E^\omega \subset E'$ for all $x, y_2, \dots, y_k \in E$,
- and that the mapping $\prod^k E \to E$
$$(x, y_2, \dots, y_k) \mapsto (\check{\omega})^{-1}(df(x)(\quad, y_2, \dots, y_k))$$
is smooth. By the symmetry of higher derivatives, this is then true for all entries of $d^k f(x)$, for all x.

This makes sense even if (E,ω) is a weak symplectic manifold which happens to be a convenient vector space since $T^\omega E \cong TE = E \times E =: E \times E^\omega \subset T^*E = E \times E'$.

Lemma. [KM97, 48.6] *For $f \in C^\infty(E, \mathbb{R})$ the following assertions are equivalent:*

1. *$df : E \to E'$ factors to a smooth mapping $E \to E^\omega$.*
2. *f has a smooth ω-gradient $\operatorname{grad}^\omega f \in \mathfrak{X}(E) = C^\infty(E, E)$ which satisfies $df(x)y = \omega(\operatorname{grad}^\omega f(x), y)$.*
3. *$f \in C^\infty_\omega(E, \mathbb{R})$.*

Definition of $C^\infty_\omega(M, \mathbb{R}) \subset C^\infty(M, \mathbb{R})$:
For a weak symplectic manifold (M, ω) the space $C^\infty_\omega(M, \mathbb{R})$ is the linear subspace consisting of all smooth functions $f : M \to \mathbb{R}$ such that the differential $df : M \to T^*M$ factors to a smooth mapping $M \to T^\omega M$. It follows that these are exactly those smooth functions on M which admit a smooth ω-gradient $\operatorname{grad}^\omega f \in \mathfrak{X}(M)$.

Theorem [KM97, Thm 48.8] with gap closed in [BIM24, appendix]

Let (M, ω) be a weak symplectic manifold. The Hamiltonian mapping $\operatorname{grad}^\omega : C^\infty_\omega(M, \mathbb{R}) \to \mathfrak{X}(M, \omega)$, *which is given by*

$$i_{\operatorname{grad}^\omega f}\omega = df \quad \text{or} \quad \operatorname{grad}^\omega f := (\check{\omega})^{-1} \circ df$$

is well defined. Also the Poisson bracket

$$\begin{aligned} \{\ ,\ \} &: C^\infty_\omega(M, \mathbb{R}) \times C^\infty_\omega(M, \mathbb{R}) \to C^\infty_\omega(M, \mathbb{R}) \\ \{f, g\} &:= i_{\operatorname{grad}^\omega f} i_{\operatorname{grad}^\omega g}\omega = \omega(\operatorname{grad}^\omega g, \operatorname{grad}^\omega f) = \\ &= dg(\operatorname{grad}^\omega f) = (\operatorname{grad}^\omega f)(g) \end{aligned}$$

is well defined and gives a Lie algebra structure to the space $C^\infty_\omega(M, \mathbb{R})$, *which also fulfills*

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

Theorem, continued.

We equip $C^\infty_\omega(M,\mathbb{R})$ with the initial structure with respect to the the two following mappings:

$$C^\infty_\omega(M,\mathbb{R}) \xrightarrow{\subset} C^\infty(M,\mathbb{R}), \qquad C^\infty_\omega(M,\mathbb{R}) \xrightarrow{\text{grad}^\omega} \mathfrak{X}(M).$$

Then the Poisson bracket is bounded bilinear on $C^\infty_\omega(M,\mathbb{R})$.

We have the following long exact sequence of Lie algebras and Lie algebra homomorphisms:

$$0 \to H^0(M) \to C^\infty_\omega(M,\mathbb{R}) \xrightarrow{\text{grad}^\omega} \mathfrak{X}(M,\omega) \xrightarrow{\gamma} H^1_\omega(M) \to 0,$$

where $H^0(M)$ is the space of locally constant functions, and

$$H^1_\omega(M) = \frac{\{\varphi \in C^\infty(M \leftarrow T^\omega M) : d\varphi = 0\}}{\{df : f \in C^\infty_\omega(M,\mathbb{R})\}}$$

is the first symplectic cohomology space of (M,ω), a linear subspace of the De Rham cohomology space $H^1(M)$.

The Diez-Rudolph topology

In [DR24, 5.3: T.Diez, G.Rudolph: Symplectic Reduction in Infinite Dimensions, arXiv:2409.05829], for a weak symplectic vector space (E, ω), a locally convex topology τ on E is called *compatible with* ω if the dual $(E, \tau)' = \check{\omega}(E) = E^\omega \subset E'$.

Proposition. [DR24,5.4] *For a convenient weak symplectic vector space the bornological topology on E is compatible with ω*

- *in the Bastiani setting: iff E is a reflexive Banach space and ω is strong.*
- *here: iff E is reflexive and ω is strong.*

Note that $L^p \times L^{p'}$ is symplectic, Banach, but i,g, not Hilbert. Namely: If we take $E' \times E \to \mathbb{R}$ is given by $(x', x) \mapsto \omega(\check{\omega}^{-1}(x'), x)$ as duality reflexivity follows.

How does this notion fit into the convenient framework?

Example: Let $E = \ell^2 \times \ell^2$ with the weak symplectic structure $\omega((x,y),(x',y')) = \sum_n c_n(x_n y'_n - y_n x'_n)$ for a sequence $0 < c_n \searrow 0$ sufficiently fast.

Then any l.c. topology on E compatible with ω is NOT convenient: Namely, let $0 < b_n \nearrow \infty$ with $b_n c_n \searrow 0$. Then for suitable $x \in \ell^2$ the sequence $X_k := (b_n x_n)_{n=1}^k \in \ell^2$ is a Mackey-Cauchy sequence for the weak $\sigma(E, E^\omega)$-topology but its limit $X = (b_n x_n)$ is i.g. not in ℓ^2.

Smooth Curves into (E,τ). [KM97, Section 1] Since (E,τ) is not Mackey complete in general, we define $c : \mathbb{R} \to (E,\tau)$ to be smooth if $\lambda \circ c : \mathbb{R} \to \mathbb{R}$ is smooth **and** each iterated derivative $c^{(n)}(t)$ lies in E (a priori only in the c^∞-completion of E). We denote this space by $C^\infty(\mathbb{R},(E,\tau))$, and by $c^\infty(\tau)$ we denote the final topology on E with respect to $C^\infty(\mathbb{R},(E,\tau))$.

Question. Let (E, ω) be a convenient weak symplectic vector space and let τ be any l.c. topology compatible with ω. Under which conditions do we have $C^\infty(\mathbb{R}, (E, \tau)) = C^\infty(\mathbb{R}, E)$?

Proposition. *Let (E, ω) be a convenient weak symplectic vector space and let τ be any l.c. topology compatible with ω. Suppose that the bornology of E has a basis of $\sigma(E, \check{\omega}(E))$-closed sets (i.e., each bounded set is contained in a $\sigma(E, \omega(E))$-closed bounded set). This is he case if (E, ω) is a convenient weak symplectic vector space which is a dual space $E = F'$ such that*
$\check{\omega}(E) \subseteq F \subseteq E' = E''$.
Then we have $C^\infty(\mathbb{R}, (E, \tau)) = C^\infty(\mathbb{R}, E)$.

This includes the the $\ell^2 \times \ell^2$ example from above.

In the convenient spirit, under this condition we then have $C^\infty_\omega(E, \mathbb{R}) = C^\infty((E, \tau), \mathbb{R})$, although (E, τ) is NOT a convenient space.

Proof. This is a special case of the following theorem.

Theorem[KF88, Theorem 4.1.19] *Let $c : \mathbb{R} \to E$ be a curve in a convenient vector space E. Let $\mathcal{F} \subseteq E'$ be a subset of bounded linear functionals such that the bornology of E has a basis of $\sigma(E, \mathcal{F})$-closed sets. Then the following are equivalent:*

1. *c is smooth*
2. *There exist locally bounded curves $c^k : \mathbb{R} \to E$ such that $\lambda \circ c$ is smooth $\mathbb{R} \to \mathbb{R}$ with $(\lambda \circ c)^{(k)} = \lambda \circ c^k$, for each $\lambda \in \mathcal{F}$ and each k.*

If $E = F'$ is the dual of a convenient vector space F, then for any point separating subset $\mathcal{F} \subseteq F$ the bornology of E has a basis of $\sigma(E, \mathcal{F})$-closed subsets, by [FK88 4.1.22].

[FK88] Frölicher, A.; Kriegl, A., Linear spaces and differentiation theory, Pure Appl. Math., J. Wiley, Chichester, 1988.

Weakly symplectic group actions.

An infinite dimensional regular Lie group G with Lie algebra $\mathfrak{g}$ acts from the right on a weak symplectic manifold (M, ω) by $r : M \times G \to M$ (notation $r(x, g) = r^g(x) = r_x(g)$), so that each r^g is a symplectomorphism. Some immediate consequences:

(1) *The space $C^\infty_\omega(M)^G$ of G-invariant smooth functions with ω-gradients is a Lie subalgebra for the Poisson bracket,* since for each $g \in G$ and $f, h \in C^\infty(M)^G$ we have $(r^g)^*\{f, h\} = \{(r^g)^* f, (r^g)^* h\} = \{f, h\}$.

(2) *For $x \in M$ the pullback of ω to the orbit $x.G$ is a 2-form, invariant under the action of G on the orbit.* In finite dimensions the orbit is an initial submanifold. Here this has to be checked directly in each example. There is a tangent bundle $T_x(x.G) = T(r_x)\mathfrak{g}$. If $i : x.G \to M$ is the embedding of the orbit then $r^g \circ i = i \circ r^g$, so that $i^*\omega = i^*(r^g)^*\omega = (r^g)^* i^*\omega$ holds for each $g \in G$ and thus $i^*\omega$ is invariant.

(3) The infinitesimal action $\zeta : \mathfrak{g} \to \mathfrak{X}(M, \omega)$, given by $\zeta_X(x) = T_e(r_x)X$ for $X \in \mathfrak{g}$ and $x \in M$, is a homomorphism of Lie algebras (for a left action we get an anti homomorphism of Lie algebras). We have the exact sequence of Lie algebra homomorphisms

$$\begin{array}{ccccccccc}
0 \longrightarrow H^0(M) & \xrightarrow{\alpha} & C^\infty_\omega(M) & \xrightarrow{\text{grad}^\omega} & \mathfrak{X}(M,\omega) & \xrightarrow{\gamma} & H^1_\omega(M) \longrightarrow 0 \\
 & & & {\overset{j}{\nwarrow}} & \Big\uparrow{\scriptstyle \zeta} & & \\
 & & & & \mathfrak{g} & &
\end{array}$$

(4) If $H^1_\omega(M) = 0$ then any symplectic action on (M, ω) is a Hamiltonian action.

(5) *If the Lie algebra* $\mathfrak{g}$ *is equal to its commutator subalgebra* $[\mathfrak{g}, \mathfrak{g}]$*, the linear span of all* $[X, Y]$ *for* $X, Y \in \mathfrak{g}$ *(true for all full diffeomorphism groups), then any infinitesimal symplectic action* $\zeta : \mathfrak{g} \to \mathfrak{X}(M, \omega)$ *is a Hamiltonian action,* since then any $Z \in \mathfrak{g}$ can be written as $Z = \sum_i [X_i, Y_i]$ so that $\zeta_Z = \sum [\zeta_{X_i}, \zeta_{Y_i}] \in \text{im}(\text{grad}^\omega)$ since $\gamma : \mathfrak{X}(M, \omega) \to H^1_\omega(M)$ is a homom.into the zero Lie bracket.

(6) If $j : \mathfrak{g} \to (C^\infty_\omega(M), \{\quad,\quad\})$ happens to be not a homomorphism of Lie algebras then $c(X, Y) = \{j(X), j(Y)\} - j([X, Y])$ lies in $H^0(M)$, and indeed $c : \mathfrak{g} \times \mathfrak{g} \to H^0(M)$ is a cocycle for the Lie algebra cohomology: $c([X, Y], Z) + c([Y, Z], X) + c([Z, X], Y) = 0$. If c is a coboundary, i.e., $c(X, Y) = -b([X, Y])$, then $j + \alpha \circ b$ is a Lie algebra homomorphism. If the cocycle c is non-trivial we can use the central extension $H^0(M) \times_c \mathfrak{g}$ with bracket $[(a, X), (b, Y)] = (c(X, Y), [X, Y])$ in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(M) & \xrightarrow{\alpha} & C^\infty_\omega(M) & \xrightarrow{\text{grad}^\omega} & \mathfrak{X}(M, \omega) & \xrightarrow{\gamma} & H^1_\omega(M) & \longrightarrow & 0 \\
 & & & & \uparrow{\scriptstyle \bar{\jmath}} & & \uparrow{\scriptstyle \zeta} & & & & \\
 & & & & H^1(M) \times_c \mathfrak{g} & \xrightarrow{\text{pr}_2} & \mathfrak{g} & & & &
\end{array}$$

where $\bar{\jmath}(a, X) = j(X) + \alpha(a)$. Then $\bar{\jmath}$ is a homomorphism of Lie algebras.

Momentum mapping

For an infinitesimal symplectic action $\zeta : \mathfrak{g} \to \mathfrak{X}(M, \omega)$ we can find a linear lift $j : \mathfrak{g} \to C^\infty_\omega(M, \mathbb{R})$ iff there exists $J \in C^\infty_\omega(M, \mathfrak{g}^*) := \{f \in C^\infty(M, \mathfrak{g}^*) : \langle f(\quad), X\rangle \in C^\infty_\omega(M) \text{ for all } X \in \mathfrak{g}\}$ such that

$$\operatorname{grad}^\omega(\langle J, X\rangle) = \zeta_X \quad \text{for all } X \in \mathfrak{g}.$$

$J \in C^\infty_\omega(M, \mathfrak{g}^*)$ is called the *momentum mapping*for the infinitesimal action $\zeta : \mathfrak{g} \to \mathfrak{X}(M, \omega)$.

Basic properties of the momentum mapping

(1) *For $x \in M$, the transposed mapping of the linear mapping $dJ(x) : T_xM \to \mathfrak{g}^*$ is*

$$dJ(x)^\top : \mathfrak{g} \to T_x^*M, \qquad dJ(x)^\top = \check{\omega}_x \circ \zeta$$

(2) *The closure of the image $dJ(T_xM)$ of $dJ(x) : T_xM \to \mathfrak{g}*$ is the annihilator $\mathfrak{g}_x^\circ$ of the isotropy Lie algeba $\mathfrak{g}_x := \{X \in \mathfrak{g} : \zeta_X(x) = 0\}$ in $\mathfrak{g}^*$*, since the annihilator of the image is the kernel of the transposed mapping,

(3) *The kernel of* $dJ(x)$ *is the symplectic orthogonal*

$$(T(r_x)\mathfrak{g})^{\perp,\omega} = (T_x(x.G))^{\perp,\omega} \subseteq T_xM.$$

(4) *If* G *is connected,* $x \in M$ *is a fixed point for the* G*-action if and only if* x *is a critical point of* J*, i.e.* $dJ(x) = 0$*.*

(5) (Emmy Noether's theorem) *Let* $h \in C^\infty_\omega(M)$ *be a Hamiltonian function which is invariant under the Hamiltonian* G *action. Then* $dJ(\text{grad}^\omega(h)) = 0$*. Thus the momentum mapping* $J : M \to \mathfrak{g}^*$ *is constant on each trajectory (if it exists) of the Hamiltonian vector field* $\text{grad}^\omega(h)$*.*

Predualed convenient vector spaces and manifolds

A predualed convenient vector space E is a convenient vector space with a convenient predual E^p so that E is the dual (of bounded linear functionals) of E^p. Any reflexive convenient vector space is predualed.

A smooth mapping

$$E \underset{c^\infty\text{-open}}{\hookleftarrow} U \xrightarrow{f} F$$

between between predualed convenient vector spaces is called *predual preserving* if $df(x)^* : F^* \to E^*$ maps the canonically embedded predual $F^p \subseteq F^*$ to $E^p \subseteq E^*$ for each $x \in U$, and that $df : U \to L(F^p, E^p)$ is smooth.

Note that a function $f : U \to \mathbb{R}$ is predual preserving if and only if $df(x) : E \to \mathbb{R}$ is given by an element of E^p and that $df : U \to E^p$ is smooth.

Lemma *For a predual preserving mapping* $E \supset U \xrightarrow{f} F$ *between predualed convenient vector spaces the following properties are equivalent:*

1. *For each* $\alpha \in F'$ *(equivalently, in a subset* $\mathcal{V} \subset E'$ *which recognizes bounded subsets in* E*)* $d(\alpha \circ f)(x)$ *lies in* E^p *for all* $x \in U$*, i.e.,* $\alpha \circ f$ *is predual preserving.*
2. *For every* $\alpha \in F'$ *(equivalently, in* $\mathcal{V} \subset E'$*) and each* $k \geq 1$ *the mapping* $d^k(\alpha \circ f)(x)(\ ,y_2,\dots,y_k)$ *is in* E^p *for all* $x \in U$*, and all* $y_2,\dots,y_k \in E$*. By symmetry of* $df(x)$ *this is then true for any entry.*

Proof. Clearly (2) implies (1). Conversely, if (1) holds, then $d(\alpha \circ f)(x) \in E^p$ for all $x \in U$ so $d(\alpha \circ f)(\ ) : U \to E^p$ is smooth thus

$$d^k(\alpha \circ f)(x)(\ ,y_2,\dots,y_k) = d^{k-1}(d(\alpha \circ f)(\ ))(x)(y_1,\dots,y_k) \in E^p\,. \quad \square$$

A predualed smooth manifold

M is a smooth manifold modelled on a predualed convenient vector space E with an atlas $M \supset U_\alpha \xrightarrow{u_\alpha} u_\alpha(U\alpha) \subset E$ such that all chart changings $u_{\alpha\beta} : u_\beta(U_\alpha \cap U_\beta) \to u_\alpha(U_\alpha \cap U_\beta)$ are predual preserving. Then there exists the predual bundle $T^p M \subset T^* M$ with the property that $T^p_y M^* = T_y M$.

Summable predual differential forms on predualed manifolds
Let M be a predualed smooth manifold. A *summable predual differential form* ω in M is a smooth section sections of the bundle of skew symmetric tensors $\bigwedge^k_{\mathrm{sum},\beta} T^p M \subset \bar{\otimes}^k_\beta T^p M = T^p M \bar{\otimes}_\beta T^p M \bar{\otimes}_\beta \ldots \bar{\otimes}_\beta T^p M \to M$. Let us denote by $\Omega^k_{\mathrm{pred,sum}}(M)$ the space of all summable predualed differential forms. Note that exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ does not map $\Omega^k_{\mathrm{pred,sum}}(M)$ into $\Omega^{k+1}_{\mathrm{pred,sum}}(M)$; both summability of a form and preduality are destroyed by the exterior derivative.

Skew multi vector fields

For any predualed smooth manifold M, a *skew multi vector field P* of degree k is a smooth section of the bundle $L^k_{\text{skew}}(T^pM;\mathbb{R}) \to M$. Let us denote by $\mathrm{SM}^k(M)$ the space of all skew multi vector fields of degree k. If we denote by $TM\bar{\otimes}_\varepsilon TM$ the closure of the fiberwise algebraic tensor product in $L^2(T^pM;\mathbb{R})$ (also called the ε-tensor product), then the space of smooth sections of the bundle $\bigwedge^k_{\text{sum},\varepsilon} TM \subset \bar{\otimes}^k_\varepsilon TM = TM\bar{\otimes}_\varepsilon TM\bar{\otimes}_\varepsilon \dots \bar{\otimes}_\varepsilon TM \to M$ is a closed linear subspace of $\mathrm{SM}^k(M)$; we shall call it the space of

For preduialed 1-forms $\alpha_1, \dots, \alpha_k$ and a skew multi vector field P we get a function $P(\alpha_1, \dots, \alpha_k) \in C^\infty(M,\mathbb{R})$. Since our model spaces are not smoothly normal in general, we cannot assert that any skew k-linear operator $(\alpha_1, \dots, \alpha_k) \mapsto P(\alpha_1, \dots, \alpha_k)$ which is bounded $\Omega^1_{\text{pre}}(M)^k \to C^\infty(M,\mathbb{R})$ and which $C^\infty(M,\mathbb{R})$-multilinear, is a skew multi vector field. We will have to rely on coordinate formulas for this. Moreover, and this is the operation that we will use most often, $P(\alpha_1, \dots, \alpha_{k-1}) \in \mathfrak{X}(M)$ is a vector field on M.

Easy Theorem.

Schouten-Nijenhuis bracket for summable multi vector fields. *Let M be a smooth manifold. We consider the space $\Gamma(\bigwedge_{sum,\varepsilon} TM)$ of multivector fields on M. This space carries a graded Lie bracket for the grading $\Gamma(\bigwedge_{sum,\varepsilon}^{*+1} TM), * = -1, 0, 1, 2, \dots,$ called the Schouten-Nijenhuis bracket, which is given by*

$$[X_1 \wedge \cdots \wedge X_p, Y_1 \wedge \cdots \wedge Y_q]$$
$$= \sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \widehat{X_i} \cdots \wedge X_p \wedge Y_1 \wedge \cdots \widehat{Y_j} \cdots \wedge Y_q,$$
$$[f, U] = -\bar{\imath}(df)U,$$

where $\bar{\imath}(df)$ is the insertion operator $\bigwedge_{sum,\varepsilon}^{k} TM \to \bigwedge_{sum,\varepsilon}^{k-1} TM$, the adjoint of $df \wedge (\quad) : \bigwedge_{sum,\varepsilon}^{l} T^*M \to \bigwedge_{sum,\varepsilon}^{l+1} T^*M$.

Easy Theorem continued

Let $U \in \Gamma(\bigwedge^u_{sum,\varepsilon} TM)$, $V \in \Gamma(\bigwedge^v_{sum,\varepsilon} TM)$, $W \in \Gamma(\bigwedge^w_{sum,\varepsilon} TM)$, *and* $f \in C^\infty(M, \mathbb{R})$. *Then we have:*

$$[U, V] = -(-1)^{(u-1)(v-1)}[V, U].$$

$$[U, [V, W]] = [[U, V], W] + (-1)^{(u-1)(v-1)}[V, [U, W]].$$

$$[U, V \wedge W] = [U, V] \wedge W + (-1)^{(u-1)v} V \wedge [U, W].$$

$$[X, U] = \mathcal{L}_X U.$$

Let $P \in \Gamma(\bigwedge^2_{sum,\varepsilon} TM)$. *Then the product* $\{f, g\} := \frac{1}{2}\langle df \wedge dg, P\rangle$ *on* $C^\infty(M)$ *satisfies the Jacobi identity if and only if* $[P, P] = 0$.

Maybe wrong first try. Schouten-Nijenhuis bracket for multi vector fields.

Let M be a predualed smooth manifold. We consider the space

$$\mathrm{SN}^{\circ}(M) = \Gamma(L^{\circ}_{skew}(TM;\mathbb{R})) := \bigoplus_{k\geq 0} \Gamma(L^{k}_{skew}(TM;\mathbb{R}))$$

of bounded multivector fields on M. This space carries a graded Lie bracket for the grading $\mathrm{SN}^{\circ+1}(M), \circ = -1, 0, 1, 2, \dots,$ called the Schouten-Nijenhuis bracket, which is given for $P \in \mathrm{SN}^{p+1}(M)$ and $Q \in \mathrm{SN}^{q+1}(M)$ and for predualed 1-forms $\alpha_i \in \Omega^1_{pre}(M)$ by

$$\begin{aligned}
&[P, Q](\alpha_1, \dots, \alpha_{p+q}) \\
&\qquad = \tfrac{1}{p!\,q!} \sum_{\sigma} \sigma\ [P(\alpha_{\sigma 1}, \dots, \alpha_{\sigma p}), Q(\alpha_{\sigma(p+1)}, \dots, \alpha_{\sigma(p+q)})] \\
&\qquad\quad + \tfrac{-1}{p!\,(q-1)!} \sum_{\sigma} \sigma\ Q(\mathcal{L}_{P(\alpha_{\sigma 1},\dots,\alpha_{\sigma p})}\alpha_{\sigma(p+1)}, \alpha_{\sigma(p+2)}, \dots) \\
&\qquad\quad + \tfrac{(-1)^{pq}}{(p-1)!\,q!} \sum_{\sigma} \sigma\ P(\mathcal{L}_{Q(\alpha_{\sigma 1},\dots,\alpha_{\sigma q})}\alpha_{\sigma(q+1)}, \alpha_{\sigma(q+2)}, \dots) \\
&\qquad\quad + \dots
\end{aligned}$$

Thank you for listening.