Algebraic classification of conformal invariants and conformal anomalies in arbitrary dimension

Nicolas Boulanger

Physique de l’Univers, Champs et Gravitation, Université de Mons (UMONS), Belgium

ESI programme on Geometry for Higher Spin Gravity: Conformal Structures, PDEs, and Q-manifolds, August 23 - September 17, 2021

Based on [hep-th/0405228], [hep-th/0412314], [0706.0340] and [1809.05445], the second one with Johanna ERDMENGER while the last one with Jordan FRANCOIS and Serge LAZZARINI
1 **Conformal Anomalies**
   - Introduction
   - Wess-Zumino consistency conditions
   - Solution of the WZ conditions for the anomaly
   - Results

2 **Conformal Invariants**
   - Another cohomological group
   - Type-A invariants
   - Type-B invariants
   - Action and field equations for pure Lorentz-Chern-Simons theories
Conformal Anomalies

1. Introduction
   - Wess-Zumino consistency conditions
   - Solution of the WZ conditions for the anomaly
   - Results

2. Conformal Invariants
   - Another cohomological group
   - Type-A invariants
   - Type-B invariants
   - Action and field equations for pure Lorentz-Chern-Simons theories
In 1973, Derek Capper and Michael J. Duff discovered that the invariance under Weyl rescaling of the metric tensor displayed by classical massless field systems in interaction with gravity no longer survives in the quantum theory.

\[ g_{\mu\nu}(x) \rightarrow \Omega^2(x) g_{\mu\nu}(x) \]

displayed by classical massless field systems in interaction with gravity no longer survives in the quantum theory.

\[\mapsto \text{ Weyl (or conformal) anomaly}\]
Conformal massless fields coupled to gravity

Examples of spin-1, spin-1/2 and spin-0 field theories:

- \( S[A_\mu, g_{\mu\nu}] = -\frac{1}{4} \int d^4x \sqrt{-g} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \)
  where \( F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \)

- \( S[\Psi, e_\mu^a] = -\frac{1}{2} \int d^n x e (\bar{\Psi} \gamma^a \nabla_a \Psi - \nabla_a \bar{\Psi} \gamma^a \Psi) \)

- \( S[\phi, g_{\mu\nu}] = -\frac{1}{2} \int \sqrt{-g} \left[ g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \xi(n) R \Phi^2 \right] d^n x \)
  with \( \xi(n) = \frac{1}{4} \left[ (n - 2)/(n - 1) \right] \).
Notation, definitions, conventions

- Spacetime indices → Greek letters, e.g. Riemann tensor
  \[ R^\mu_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} + \ldots, \]\[ \text{Christoffel symbols } \Gamma^\mu_{\nu\rho}, \text{ Ricci tensor } \]
  \[ \mathcal{R}_{\alpha\beta} = R^\mu_{\alpha\mu\beta} \] and scalar curvature \[ \mathcal{R} = g^{\alpha\beta} \mathcal{R}_{\alpha\beta}; \]
  Curvature two-form
  \[ R^\mu_{\nu} = \frac{1}{2} R^\mu_{\nu\rho\sigma} dx^\rho dx^\sigma. \]

- Frame (tangent bundle) indices → Latin letters.
  The frame fields are \( e_a = e^\mu_a \partial_\mu \) in coordinates \( x^\mu \).
  \( e = \det e^a_\mu \) where \( e^a_\mu e^\nu_a = \delta^\nu_\mu. \)

- For Dirac spinors: Clifford algebra \( \{ \gamma_a, \gamma_b \} = 2 \eta_{ab} \) where \( \gamma_a \) denote
  Dirac’s matrices and \( \eta = \text{diag}(-, +, +, +, +) \); \( \nabla_a \Psi = e^\mu_a (\partial_\mu - \frac{i}{2} \omega^{bc}_\mu \Sigma_{bc}) \Psi, \)
  where \( \Sigma_{bc} = \frac{i}{4} [\gamma_b, \gamma_c] \) and \( \omega^{bc}_\mu = \omega^{bc}_\mu(e) \) is the Levi-Civita
  spin-connection.
Trace of stress-tensor

These matter systems coupled to gravity are invariant under the local Weyl rescalings

\[
\begin{align*}
g_{\mu\nu} & \rightarrow \Omega^2(x) g_{\mu\nu} \\
e^{a}_\mu & \rightarrow \Omega e^{a}_\mu \\
\Psi & \rightarrow \Omega^{(1-n)/2} \Psi \\
\phi & \rightarrow \Omega^{(2-n)/2} \phi
\end{align*}
\]

This is reflected in the (on-shell) tracelessness of the corresponding symmetric stress-tensor: \( (1) \Rightarrow g^{\mu\nu} T_{\mu\nu} = 0 \).
Local symmetries

By construction these actions are also invariant under diffeomorphisms.

To summarize, the local symmetries of these conformally invariant massless systems coupled to gravity are

**Local symmetries:**

- Diffeomorphism invariance
- Local Weyl rescaling invariance
Both symmetries cannot survive

It turns out that, after regularization and renormalization, both symmetries cannot survive at the same time. One always chooses to maintain diffeomorphism invariance (conservation of energy-momentum). This is done at the price of a

\[ A = g^{\mu\nu} \langle T_{\mu\nu} \rangle_{\text{reg}} \neq 0 \]

Note: Weyl anomalies are also called “Trace anomalies” or “Conformal anomalies” for obvious reasons.
Some bits of QFT

- Generating functional of Green’s functions:
  \[
  Z[J] \equiv \int \mathcal{D}\Phi \, e^{\frac{i}{\hbar} \int d^n x \left[ \mathcal{L}(\Phi,\partial\Phi) + J(x)\Phi(x) \right]}
  \]

- Generating functional of connected Green’s functions:
  \[
  W[J] = -i \ln Z[J]
  \]

- The generating functional of 1PI Green’s functions
  \[
  \Gamma[\Phi_c] = W[J_{\Phi}] - \int d^nx \, \Phi_c(x) J_{\Phi}(x), \quad \Phi_c(x) := \frac{\delta W[J]}{\delta J(x)}.
  \]

The functional \( \Gamma \) is also called quantum action or effective action.
1. Conformal Anomalies
   - Introduction
   - Wess-Zumino consistency conditions
     - Solution of the WZ conditions for the anomaly
   - Results

2. Conformal Invariants
   - Another cohomological group
   - Type-A invariants
   - Type-B invariants
   - Action and field equations for pure Lorentz-Chern-Simons theories
The anomalies cannot be anything

1. An anomaly in QFT ...

Anomalies occur when quantization spoils symmetries of the classical action, i.e. if $\Gamma[\Phi]$ cannot be made invariant under infinitesimal transformations $s$ by a suitable choice of local counterterms.

2. ... is an infinitesimal variation

To lowest order in $\hbar$ the variation $A = s \Gamma[\Phi]$ is local. It is an anomaly if it cannot be written as $A = s C$ for any local functional $C$. 
Because an anomaly is a variation

\[ A = s \Gamma[\Phi] \]

it is not arbitrary but constrained to obey some consistency conditions. Similar to integrability conditions \( \vec{\nabla} \times \vec{F} = 0 \) which a gradient \( \vec{F} = \vec{\nabla} \varphi \) has to satisfy.

⇒ An anomaly must satisfy the

Wess-Zumino consistency conditions [1971]
The analysis of WZ consistency conditions simplifies in the Becchi-Rouet-Stora-Tyutin (BRST) formulation.

- one introduces a ghost for each gauge parameter;
- one suitably defines the transformations of the ghosts so that

\[ s^2 = 0 \]

Local cohomology of \( s \)

The WZ consistency conditions take the simple form

\[ sA = 0, \quad A \neq sC \]

where \( A \) and \( C \) are local functionals \( A = \int a^{1,n}([Φ], x) \), \( C = \int b^{0,n}([Φ], x) \) and \( s \) is the BRST differential.
1 Conformal Anomalies

- Introduction
- Wess-Zumino consistency conditions
- Solution of the WZ conditions for the anomaly
- Results

2 Conformal Invariants

- Another cohomological group
- Type-A invariants
- Type-B invariants
- Action and field equations for pure Lorentz-Chern-Simons theories
**Wess-Zumino Consistency Condition**

- Central equations for candidate anomalies in QFT: **Wess-Zumino (WZ) consistency conditions**. By using these conditions, the general structure of all the known anomalies (except the conformal one) had been determined by purely algebraic methods featuring **descent equations** à la Stora-Zumino.


- Determining the general solution of the WZ consistency conditions is tantamount to computing the **cohomology** of the corresponding Becchi-Rouet-Stora-Tyutin (BRST) differential in the space of local functionals with ghost number one.
Stora-Zumino descent of equations

- Letting $A = \int a^{1,n}$, the WZ conditions get translated to

$$\begin{align*}
as a^{1,n} + d a^{2,n-1} &= 0, \\
a^{1,n} &\sim a^{1,n} + s c^{0,n} + d c^{1,n-1}
\end{align*}$$

(2)

- With the total exterior derivative $d = dx^\mu \frac{\partial}{\partial x^\mu}$. One has

$$s^2 = 0, \quad d^2 = 0, \quad s d + d s = 0.$$ 

- Acting on (2) with $s$ and using the above relations:

$$d (s a^{2,n-1}) = 0 \quad \text{algebraic Poincaré lemma} \quad \implies \quad s a^{2,n-1} + d a^{3,n-2} = 0.$$ 

Apply $s$ again on this equations, ...
... one obtains the following descent equations

\[
\begin{align*}
sa_{1,n} + da_{2,n-1} &= 0, \\
&\vdots \\
sa_{q,n} - q + 1 + da_{q+1,n-q} &= 0, \\
sa_{q+1,n-q} &= 0 \quad (0 \leq q \leq n).
\end{align*}
\]

If \( q = 0 \), the descent is trivial: \( sa_{1,n} = 0 \).

**Dubois-Violette, Talon, Viallet (1985)**

- In order to find \( a_{1,n} \in H_{1,n}^1(s|d) \), find the \( a_{q+1,n-q} \in H_{q+1}^1(s) \) that can be lifted up to a top form.
Cohomological consideration, although without any descent equation analysis → pioneering works by Bonora, Cotta-Ramusino, Reina, Pasti and Bregola [1983–1985]. Results up to dimension $n = 6$.

They conjectured:

(i) Euler term times the Weyl parameter

$$e^{1,n} = \sqrt{-g} \omega \left( R_{\mu_1 \nu_1} \ldots R_{\mu_m \nu_m} \right) \varepsilon_{\mu_1 \nu_1 \ldots \mu_m \nu_m},$$

plus

(ii) strictly Weyl-invariant scalar densities times Weyl parameter. In $n = 4$, e.g.,

$$a^{1,4} = \omega \sqrt{-g} g^{\sigma \tau} g^{\lambda \kappa} W^\mu_{\rho \sigma \lambda} W^\rho_{\mu \tau \kappa} d^4 x$$

where $W^\mu_{\rho \sigma \lambda}$: conformally invariant Weyl tensor, traceless part of Riemann curvature tensor $R^\mu_{\rho \sigma \lambda}$. 
Using dimensional regularization, Deser and Schwimmer (1993) confirmed the structure obtained by Bonora et al. The Euler term from class (i) was called type-A Weyl anomaly, while the terms of (ii) were called type-B anomalies.

From the structure of the poles in the variation of the effective action, they observed that the type-A anomaly appears in a similar way to the non-Abelian chiral anomaly in Yang-Mills gauge theory. That the type-A anomaly should arise via some descent equations à la Stora-Zumino was therefore conjectured.
In the BRST formalism

- Apart from $g_{\mu\nu}$, the other fields of the problem are the Weyl ghost $\omega$ and the diffeomorphisms ghosts $\xi^\mu$, $gh(\xi^\mu) = gh(\omega) = 1$.

- The BRST transformations on the fields $\Phi^A = \{g_{\mu\nu}, \omega, \xi^\mu\}$ read

$$s_D g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\mu\rho}, \quad s_W g_{\mu\nu} = 2\omega g_{\mu\nu}$$

$$s_D \xi^\mu = \xi^\rho \partial_\rho \xi^\mu, \quad s_D \omega = \xi^\rho \partial_\rho \omega, \quad s_W \xi^\mu = 0 = s_W \omega,$$

where the BRST differentials $s_W$ and $s_D$ implement the Weyl transformations and the diffeomorphisms transformations, respectively.
**WZ conditions for WA**

Upon quantization one always chooses to preserve diffeomorphism invariance. With \( s = s_W + s_D \), decomposing \( s a + d b = 0 \), \( a \sim a + s c + d f \) w.r.t. the Weyl ghost degree gives the **WZ consistency conditions** for the Weyl anomalies in terms of local forms:

\[
\begin{cases}
    s_W a^{1,n} + d b^{2,n-1} = 0, & a^{1,n} \neq s_W p^{0,n} + d f^{1,n-1}, \\
    s_D a^{1,n} + d c^{2,n-1} = 0, & \forall p^{0,n} \text{ s.t. } s_D p^{0,n} + d h^{1,n-1} = 0.
\end{cases}
\]

N. Boulanger (UMONS)
**Stora’s Trick**

- Denoting $\tilde{s}_W = s_W + d$ and similarly for $s_D$, the problem (*) consists in determining the $\tilde{s}_D$-invariant $(n + 1)$-local total forms $\alpha(\mathcal{W})$ satisfying

$$\tilde{s}_W \alpha(\mathcal{W}) = 0, \quad \alpha(\mathcal{W}) \neq \tilde{s}_W \zeta(\mathcal{W}) + \text{constant},$$

where $\zeta(\mathcal{W})$ must be $\tilde{s}_D$-invariant.

- Using very general results obtained in [Friedemann Brandt, CMP 1996], we know that the solution of (3) will take the form

$$\alpha(\mathcal{W}) = 2\omega \tilde{C}^{N_1} \ldots \tilde{C}^{N_n} a_{N_1 \ldots N_n}(\mathcal{T}).$$
Elimination of trivial pairs in Jet space

**Lemma**

Suppose there is a set of local jet coordinates

\[ B = \{ U^\ell, V^\ell, W^\Lambda \} \]

such that the change of coordinates from \( J = \{ [\Phi^A], x^\mu, dx^\mu \} \) to \( B \) is local and locally invertible and

\[
\begin{align*}
\tilde{s} U^\ell &= V^\ell \quad \forall \ell, \\
\tilde{s} W^\Lambda &= R^\Lambda(W) \quad \forall \Lambda.
\end{align*}
\]

Then, locally the \( U \)'s and \( V \)'s can be eliminated from the \( \tilde{s} \)-cohomology, i.e., the latter reduces locally to the \( \tilde{s} \)-cohomology on total local forms depending only on the \( W \)'s.
In the case at hand

- Jet space $\mathcal{J} = \{[g_{\mu\nu}], [\omega], [\xi^\mu], x^\mu, dx^\mu\}$ and $\tilde{s} = s_D + s_W + d$ the differential acting on $\mathcal{J}$

- The $\{U, V, W\}$-decomposition of $\mathcal{J}$ corresponding to $\tilde{s}$:

  \[
  \begin{align*}
  \{U^\ell\} &= \{x^\mu, \partial_{(\mu_1\ldots\mu_k} \Gamma_{\mu_{k+1}\mu_{k+2})}^\nu, \nabla_{(\mu_1\ldots\mu_k} P_{\mu_{k+1}\mu_{k+2})}, k \in \mathbb{N}\}, \\
  \{V^\ell\} &= \{\tilde{s}U^\ell\}, \quad \{W^\Lambda\} = \{\mathcal{T}^i, \tilde{C}^N\}, \\
  \{W^\Lambda\} &= \{\mathcal{T}^i\} \cup \{\tilde{C}^N\}, \\
  \{\mathcal{T}^i\} &= \{g_{\mu\nu}, D_{(\alpha_k} \ldots D_{\alpha_1} W^\beta_{\gamma\delta)} e, k \in \mathbb{N}\}, \\
  \{\tilde{C}^N\} &= \{2\omega, \tilde{\xi}^\nu, \tilde{C}_\nu^\rho, \tilde{\omega}_\alpha\},
  \end{align*}
  \]

  \[
  \tilde{\xi}^\nu := \xi^\nu + dx^\nu, \quad \tilde{C}_\nu^\rho := \partial_\nu \xi^\rho + \tilde{\xi}^\alpha \Gamma_{\alpha\nu}^\rho, \quad \tilde{\omega}_\alpha := \omega_\alpha - \tilde{\xi}^\beta P_{\alpha\beta}.
  \]
Assignment of degrees:

\[ \text{totdeg}(\mathcal{T}^i) = 0, \quad \text{totdeg}(\hat{\mathcal{C}}^N) = 1, \quad \hat{\mathcal{C}}^N = \hat{\mathcal{C}}^N + \mathcal{A}^N, \]
\[ gh(\hat{\mathcal{C}}^N) = 1 = \text{formdeg}(\mathcal{A}^N), \quad gh(\mathcal{A}^N) = 0 = \text{formdeg}(\hat{\mathcal{C}}^N), \]

*covariant ghosts* and *connection 1-forms*:

\[
\begin{align*}
\{\hat{\mathcal{C}}^N\} &= \{2\omega, \xi^\nu, \hat{\mathcal{C}}_\nu^\rho := \partial_\nu \xi^\rho + \xi^\alpha \Gamma^\rho_{\alpha\nu}, \hat{\omega}_\alpha := \omega_\alpha - \xi^\mu P_{\mu\alpha}\}, \\
\{\mathcal{A}^N\} &= \{0, dx^\mu \delta_\mu, dx^\mu \Gamma_{\mu\nu}^\rho, -dx^\mu P_{\mu\alpha}\}.
\end{align*}
\]

The differential \(\tilde{s}\) raises the total degree by 1 unit, so

\[
\tilde{s}\mathcal{T}^i = \hat{\mathcal{C}}^N \Delta_N \mathcal{T}^i \iff \begin{cases} 
  s\mathcal{T}^i = \hat{\mathcal{C}}^N \Delta_N \mathcal{T}^i \\
  d\mathcal{T}^i = \mathcal{A}^N \Delta_N \mathcal{T}^i
\end{cases},
\]
\[ \{\Delta_N\} = \{\Delta^e_x, \mathcal{D}_\nu, \Delta_\rho^\nu, \Gamma^\alpha\}. \]
The $W$-tensors, by iteration:

$$W_{\Omega_k} = \mathcal{D}_{\alpha_k} W_{\Omega_{k-1}} = (\nabla_{\alpha_k} + P_{\beta \alpha_k} \Gamma^\beta) W_{\Omega_{k-1}}.$$ 

They transform as

$$s_w W_{\Omega_i} = \omega_\alpha \Gamma^\alpha W_{\Omega_i}$$

\textit{only the first derivative} $\omega_\alpha = \partial_\alpha \omega$ of the Weyl parameter \textit{appears} and

$$\Gamma^\alpha W_{\Omega_j} = [T^\alpha]_{\Omega_j}^{\Omega_{j-1}} W_{\Omega_{j-1}}.$$ 

The $[T^\alpha]_{\Omega_j}^{\Omega_{j-1}}$'s are built iteratively starting with $[T^\alpha]_{\Omega_j}^{\Omega_{j-1}} = 0 \ \forall \ j \leq 0$. 

The operators acting on the space $\mathcal{T}$ of tensors and connections:

$$\{\Delta_N\} = \{\Delta_{ex}^g, \mathcal{D}_\nu, \Delta^\nu_\rho, \Gamma^\alpha\},$$

$$\mathcal{D}_\mu := \partial_\mu - \Gamma_{\mu \nu}^\rho \Delta^\nu_\rho + P_{\mu \alpha} \Gamma^\alpha,$$

$\Delta_{ex}^g$ counts the number of metric tensors appearing in a given expression.
**Gauge covariant algebra**

With $\Delta^\mu_\nu$ the generators of $GL(n)$-transformations of world indices acting on a type-(1, 1) tensor $T^\beta_\alpha$ as $\Delta^\mu_\nu T^\beta_\alpha = \delta^\mu_\alpha T^\beta_\nu - \delta^\beta_\nu T^\mu_\alpha$, the gauge covariant algebra $G$ generated by \{\(\Delta_N\}\} = \{\Delta^g_\nu \ , D_\nu \ , \Delta^\mu_\nu \ , \Gamma^\alpha\}$ reads

\[
\begin{align*}
[\Delta^\mu_\nu, \Gamma^\alpha] &= -\delta^\alpha_\nu \Gamma^\mu, \\
[\Delta^\mu_\nu, D_\alpha] &= \delta^\mu_\alpha D_\nu, \\
[\Delta^\rho_\mu, \Delta^\sigma_\nu] &= \delta^\rho_\nu \Delta^\sigma_\mu - \delta^\sigma_\mu \Delta^\rho_\nu, \\
[D_\beta, \Gamma^\alpha] &= \mathcal{P}^\nu_\beta \Delta^\mu_\nu - \delta^\alpha_\beta \Delta^g_\nu, \\
[D_\rho, D_\sigma] &= -W^\mu_\nu \rho \sigma \Delta^\nu_\mu - C^\alpha_\rho \sigma \Gamma^\alpha,
\end{align*}
\]

where $C^\alpha_\rho \sigma := 2 \nabla^\nu [\nu P_\mu]_\alpha$ is the Cotton tensor and

$\mathcal{P}^\nu_\beta \mu := (-g^\nu_\alpha g_\beta_\mu + \delta^\nu_\beta \delta^\alpha_\mu + \delta^\alpha_\beta \delta^\nu_\mu).$ The operator $\Delta^g_\nu$ commutes with all the other generators.
• With \( \{ \Delta_N \} = \{ \Delta_g^e x , \mathcal{D}_\nu , \Delta^\mu \nu , \Gamma^\alpha \} \), the action of \( \tilde{s}_W \) on the tensor fields \( \{ \mathcal{T}^i \} \) and generalized connections \( \{ \tilde{\mathcal{C}}^i_N \} \) can be written as

\[
\tilde{s}_W \mathcal{T}^i = \tilde{\mathcal{C}}^N \Delta_N \mathcal{T}^i , \quad \tilde{s}_W \tilde{\mathcal{C}}^N = \frac{1}{2} \tilde{\mathcal{C}}^L \tilde{\mathcal{C}}^K \mathcal{F}_{KL}^N (\mathcal{T}) ,
\]

where \( \mathcal{F}_{KL}^N (\mathcal{T}) \) denote the structure functions of the gauge covariant algebra \( \mathcal{G} \) :

\[
[\Delta_M , \Delta_N] = \mathcal{F}_{MN}^L (\mathcal{T}) \Delta_L .
\]

• The relation \( \tilde{s}_W \tilde{\mathcal{C}}^i_N = \frac{1}{2} \tilde{\mathcal{C}}^L \tilde{\mathcal{C}}^K \mathcal{F}_{KL}^N (\mathcal{T}) \) generalizes what Stora coined the Russian formula.
From $\tilde{s}^2 \tilde{C}^N = 0$, get the following set of Bianchi identities

- $\tilde{s}^2 \omega = 0 \Rightarrow C_{[\mu \rho \sigma]} = 0$
- $\tilde{s}^2 \tilde{C}_\mu \nu = 0 \Rightarrow \nabla [\gamma W_{\delta \epsilon}]_{\alpha \beta} - C_{\alpha[\gamma \delta g_{\epsilon}]\beta} + C_{\beta[\gamma \delta g_{\epsilon}]\alpha} = 0$
- $\tilde{s}^2 \tilde{\xi}_\mu = 0 \Rightarrow \begin{cases} \mathcal{P}^{\alpha \mu}_{[\rho \nu]} = 0 \\ W^{\mu}_{[\nu \rho \sigma]} = 0 \end{cases}$
- $\tilde{s}^2 \tilde{\omega}_\alpha = 0 \Rightarrow \begin{cases} \Gamma^\alpha_{\beta \rho \sigma} + W^\alpha_{\beta \rho \sigma} = 0 \\ \mathcal{D}_{[\beta C_{\rho \sigma}]\alpha} = 0 \end{cases}$
Relation with conformal algebra

Introducing the new set of generators \( \{ P_\mu, K_\nu, M_{\mu\nu}, D \} \) via

\[
\{ \Delta_{\mu\nu}, \Gamma_\alpha, D \} = \{ g_{\mu\rho} \Delta^\rho_{\nu}, g_\alpha^\beta \Gamma^\beta_\mu, \delta^\mu_\nu \Delta^\nu_\mu - \Delta^{ex}_{\mu\nu} \},
\]

\[
\{ P_\mu, K_\nu, M_{\mu\nu} \} = \{ \frac{1}{4} \mathcal{D}_\mu, 2 \Gamma_\nu, -2 \Delta_{[\mu\nu]} \},
\]

one gets

\[
\begin{align*}
[P_\alpha, M_{\mu\nu}] &= 2 g_{\alpha[\mu} P_{\nu]} , \quad [K_\alpha, M_{\mu\nu}] = 2 g_{\alpha[\mu} K_{\nu]} , \\
[D, P_\mu] &= P_\mu , \quad [D, K_\mu] = -K_\mu , \\
[M_{\alpha\mu}, M_{\beta\nu}] &= 2 g_{\alpha[\beta} M_{\nu]\mu} - 2 g_{\mu[\beta} M_{\nu]\alpha} , \\
[P_\mu, K_\nu] &= 2 (g_{\mu\nu} D + M_{\mu\nu}) , \quad [K_\mu, K_\nu] = 0 , \\
[P_\mu, P_\nu] &= -\frac{1}{2} W^{\rho\sigma}_{\mu\nu} M_{\rho\sigma} - \frac{1}{2} C_{\alpha\mu\nu} K^\alpha
\end{align*}
\]

which is isomorphic to the conformal algebra \( \mathfrak{so}(2, n) \) when \( g_{\mu\nu} = \eta_{\mu\nu} \).
1 Conformal Anomalies

- Introduction
- Wess-Zumino consistency conditions
- Solution of the WZ conditions for the anomaly
- Results

2 Conformal Invariants

- Another cohomological group
- Type-A invariants
- Type-B invariants
- Action and field equations for pure Lorentz-Chern-Simons theories
**Lemma**: Let \( \psi_{\mu_1...\mu_2p} \) be the local total form

\[
\psi_{\mu_1...\mu_2p} := \frac{\omega}{\sqrt{-g}} \varepsilon^{\alpha_1...\alpha_r}_{\nu_1...\nu_r,\mu_1...\mu_2p} \tilde{\omega}_{\alpha_1} \ldots \tilde{\omega}_{\alpha_r} \ dx^{\nu_1} \ldots dx^{\nu_r},
\]

\[ p = m - r, \quad m = n/2, \quad 0 \leq r \leq m \]

and let \( W^{\mu\nu} \) denote the tensor-valued two-form \( W^{\mu\nu} = W^{\mu}_{\rho} \ g^{\rho\nu} \), then the local total forms \( \Phi_{r}^{[n-r]} \ (0 \leq r \leq m) \)

\[
\Phi_{r}^{[n-r]} := \frac{(-1)^p}{2^p} \frac{m!}{r! \ p!} \psi_{\mu_1...\mu_2p} \ W^{\mu_1\mu_2} \ldots W^{\mu_{2p-1}\mu_{2p}}
\]

obey a descent equations so that the following relations hold:

\[
\tilde{s}_{W} \alpha = 0 = \tilde{s}_{W} \beta
\]

with

\[
\alpha := \sum_{r=1}^{m} \Phi_{r}^{[n-r]}, \quad \beta := \Phi_{0}^{[n]}.
\]
**Theorem (A)**

The top form-degree component $a^{1,n}$ of $\alpha$ satisfies the WZ consistency conditions for the Weyl anomalies. The WZ conditions for $a^{1,n}$ give rise to a non-trivial descent and $a^{1,n}$ is the unique anomaly with such a property, up to the addition of trivial terms and anomalies satisfying a trivial descent.

**Theorem (B)**

The top form-degree component $e^{1,n}$ of $(\alpha + \beta)$ is proportional to the Euler density of the manifold $\mathcal{M}_n$:

$$e^n_1 = \frac{(-1)^m}{2^m} \sqrt{-g} \, \omega (R^\mu_1 \nu_1 \ldots R^\mu_m \nu_m) \, \varepsilon_{\mu_1 \nu_1 \ldots \mu_m \nu_m}.$$  

The anomaly $\beta = \Phi^{[n]}_0$ — a contraction of a product of Weyl tensors — satisfies a trivial descent. It is a type-B anomaly.
**Example for** $n = 6$

- From the definitions above, one gets for $n = 6$

$$\beta = \Phi_0^{[6]} = \frac{-\omega}{8} \sqrt{-g} \varepsilon_{\mu_1...\mu_6} W^{\mu_1\mu_2} W^{\mu_3\mu_4} W^{\mu_5\mu_6},$$

$$\Phi_1^{[5]} = \frac{3}{4} \omega \sqrt{-g} \varepsilon^\alpha_{\nu\mu_1...\mu_4} \tilde{\omega}_\alpha dx^n W^{\mu_1\mu_2} W^{\mu_3\mu_4},$$

$$\Phi_2^{[4]} = \frac{-3}{2} \omega \sqrt{-g} \varepsilon^{\alpha\beta}_{\mu\nu\rho\sigma} \tilde{\omega}_\alpha \tilde{\omega}_\beta dx^\mu dx^n W^{\rho\sigma},$$

$$\Phi_3^{[3]} = \omega \sqrt{-g} \varepsilon^{\alpha\beta\gamma}_{\mu\nu\rho} \tilde{\omega}_\alpha \tilde{\omega}_\beta \tilde{\omega}_\gamma dx^\mu dx^n dx^\rho.$$

- Extracting from $\alpha = \Phi_1^{[5]} + \Phi_2^{[4]} + \Phi_3^{[3]}$ its top form-degree component amounts to selecting everywhere the contribution $\mathcal{A}_\mu$ of $\tilde{\omega}_\mu = \omega_\mu + \mathcal{A}_\mu$. As a consequence, the top form-degree component of $(\alpha + \beta)$ reproduces the expression $e_1^6 = -\frac{\omega}{8} \sqrt{-g} \varepsilon_{\mu_1...\mu_6} R^{\mu_1\mu_2} R^{\mu_3\mu_4} R^{\mu_5\mu_6}$ making use of $R^{\mu\nu} = W^{\mu\nu} - 2 \mathcal{A}^{[\mu dx^n]}$ and $\mathcal{A}^{\mu} = -g^{\mu\nu} P_{\nu\rho} dx^\rho$. 

N. Boulanger (UMONS) Conformal Anomalies and Invariant 15 September 2021 35 / 57
A REGULARIZATION-FREE UNDERSTANDING

- Universal structure of Weyl anomalies established in a purely algebraic manner, independently of any regularization scheme and in arbitrary dimensions $n$. In particular, we do not resort to dimensional analysis. That the anomalies exist in even dimension $n = 2$, only is not an assumption but arises in the cohomological analysis. The type-A Weyl anomaly is the unique (up to trivial terms) Weyl anomaly satisfying a non-trivial descent of equations.

- The Weyl anomalies satisfying a trivial descent equations are all (integral) of product of the Weyl parameter times a strictly Weyl-invariant scalar density. They are called type-B Weyl anomalies.
1 Conformal Anomalies

- Introduction
- Wess-Zumino consistency conditions
- Solution of the WZ conditions for the anomaly
- Results

2 Conformal Invariants

- Another cohomological group
- Type-A invariants
- Type-B invariants
- Action and field equations for pure Lorentz-Chern-Simons theories
From anomalies to invariants

Conformal anomalies are related to *global conformal invariant*. The Deser-Schwimmer paper triggered the interest of some conformal geometers.

Global conformal invariants are given by the integral over a $n$-dimensional (pseudo) Riemannian manifold $\mathcal{M}_n(g)$ of linear combinations of strictly Weyl-invariant scalar densities and scalar densities that are invariant under Weyl rescalings *only up to* a total derivative.

What is the general structure of the latter?

$\rightarrow$ relevant for (quasi-)Weyl-invariant Lagrangians densities.
• By the assumption of *locality*, a **global** invariant is a ghost-zero scalar density whose Hodge dual $a^{0,n}$ obeys the **cocycle equation**

$$sa^{0,n} + db^{1,n-1} = 0.$$  

• The **local** conformal invariants are (the integral of) scalar densities that are strictly Weyl invariant. They can be built using various techniques, be them algebraic or geometric [tractor calculus].

• The global invariants are scalar densities that are Weyl invariant *only* up to a total derivative $\Rightarrow$ Produce a non-trivial **descent equations**.
Non-trivial descent equations:

\[
\begin{align*}
 s a^0,n + d a^1,n-1 &= 0, \\
 s a^1,n-1 + d a^2,n-2 &= 0, \\
 &\vdots \\
 s a^{p-1},n-p+1 + d a^p,n-p &= 0, \\
 s a^p,n-p &= 0
\end{align*}
\]

It stops either because \( p = n \) or because one encounters an \( s \)-cocycle \( a^{p,n-p} \).

Decomposing the first equation wrt Weyl-ghost degree:

\[
\begin{align*}
 s_D a^0,n + d f^1,n-1 &= 0, \\
 s_W a^0,n + d g^1,n-1 &= 0,
\end{align*}
\]

\( a^{0,n} \neq d b^{0,n-1} \).
The classification of global conformal invariants is also given by the cohomology of the associated BRST differential in top form degree \( n \), but this time, at ghost number \( \text{zero} \), i.e., \( H^{0,n}(s|d) \). The two cohomological groups \( H^{1,n}(s|d) \) (anomalies) and \( H^{0,n}(s|d) \) present some similarities but also important differences. The latter group is the larger!

The conjecture of Deser and Schwimmer on the structure of Weyl anomalies led the geometer Spyros Alexakis to study the problem of the classification of global conformal invariants.

Pursuing the cohomological analysis

From

\[
\begin{align*}
 s_D a^{0,n} + df^{1,n-1} &= 0, \\
 s_W a^{0,n} + dg^{1,n-1} &= 0, \\
 a^{0,n} &\neq db^{0,n-1},
\end{align*}
\]

Find the cocycles of the differential \( s_W \) modulo \( d \), in the cohomology of the diffeomorphism-invariant local \( n \)-forms.

The latter cohomology class already been worked out in [Brandt-Dragon-Kreuzer89] and [Barnich-Brandt-Henneaux95].

Denote by \( f_K := \text{Tr}(R^m(K)) \), \( K \in \{1, \ldots, r = [n/2]\} \), the invariant polynomials of the Lorentz algebra \( so(1, n - 1) \) and \( q^0_K \) the corresponding Chern-Simons \( (2m(K) - 1) \)-forms obeying \( dq^0_K = f_K \). The general solution of the first equation above decomposes into two main classes:
Two main classes:

\[ a^{0,n} = \sqrt{-g} L(\nabla, R, g) d^n x + \sum_{m} \sum_{K: m(K) = m} q^0_K \frac{\partial}{\partial f_K} P_m(f_1, \ldots, f_r) \]

The second class only contributes for spacetimes of dimensions \( n = 4p - 1 \), \( p \in \mathbb{N}^* \). Taking \( n = 7 \) as a definite example, the second class gives two structures

\[ \text{Tr}(\Gamma d\Gamma + \frac{2}{3} \Gamma^3)\text{Tr}(R^2) \equiv L^3_{CS} \text{Tr}(R^2) \text{ and } L^7_{CS} = \text{Tr}(I_7), \]

\[ I_7 = \Gamma(d\Gamma)^3 + \frac{8}{5} (d\Gamma)^2 \Gamma^3 + \frac{4}{5} \Gamma(\Gamma d\Gamma)^2 + 2 \Gamma^5 d\Gamma + \frac{4}{7} \Gamma^7, \]

where \( \Gamma \) denotes the matrix-valued 1-form \( dx^\mu \Gamma^\alpha_{\beta \mu} \) whose components \( \Gamma^\alpha_{\beta \mu} \) are the Christoffel symbols and \( \text{Tr}(\cdot) \) denotes the matrix trace.

\( \text{Tr}R^2 \equiv R^\alpha_\beta R^\beta_\alpha \) for \( R^\alpha_\beta = \frac{1}{2} dx^\mu dx^\nu R^\alpha_{\beta \mu \nu} \) the curvature 2-form.
1 Conformal Anomalies

- Introduction
- Wess-Zumino consistency conditions
- Solution of the WZ conditions for the anomaly
- Results

2 Conformal Invariants

- Another cohomological group
- Type-A invariants
- Type-B invariants
- Action and field equations for pure Lorentz-Chern-Simons theories
Lemma 1:
Let $\psi_{\mu_1...\mu_{2p}}$ be the local total form

$$\psi_{\mu_1...\mu_{2p}} = \frac{1}{\sqrt{-g}} \varepsilon^{\alpha_1...\alpha_r}_{\nu_1...\nu_r} \mu_1...\mu_{2p} \tilde{\omega}_{\alpha_1} ... \tilde{\omega}_{\alpha_r} \ dx^{\nu_1} ... dx^{\nu_r},$$

$$p = m - r, \quad m = n/2, \quad r \in \{0,...,m\}.$$ 

Then, the local total forms

$$\Phi^{[n-r]}_r = \frac{(-1)^p}{2^p} \frac{m!}{r! \ p!} \psi_{\mu_1...\mu_{2p}} \ W^{\mu_1\mu_2} ... \ W^{\mu_{2p-1}\mu_{2p}}$$

satisfy non-trivial descent equations and give solutions

$$\tilde{s}_W \alpha = 0 = \tilde{s}_W \beta \ \text{for}$$

$$\alpha = \sum_{r=1}^{m} \Phi^{[n-r]}_r \quad \text{and} \quad \beta = \Phi^{[n]}_0.$$
[Lemma 2 Invariants of class I]

The top form-degree component $a^{0,n}$ of $\alpha$ in Lemma 1 satisfies the cocycle condition for the conformal invariants. It gives rise to a non-trivial descent in $H(s_W|d)$. The invariant $\beta = \Phi_0^{[n]}$ satisfies a trivial descent and is obtained by taking contractions of products of Weyl tensors ($m$ of them in dimension $n = 2m$). The top form-degree component $e^{0,n}$ of $\alpha + \beta$ is proportional to the Euler density of the manifold $\mathcal{M}_n$:

$$e^{0,n} = \frac{(-1)^m}{2^m} \sqrt{-g} \varepsilon_{\alpha_1\beta_1...\alpha_m\beta_m} \left( R^{\alpha_1\beta_1} \wedge ... \wedge R^{\alpha_m\beta_m} \right)$$

It is the only conformal invariant of the class I that satisfies a non-trivial descent in $H(s_W|d)$. 
**Lemma 3 [Invariants of class II]**

Let $\alpha^{4p-1}_{[2m-1]}$ be the total $(4p - 1)$-form of degree $2m - 1$ in the connection 1-form $\Gamma$, defined by

$$
\alpha^{4p-1}_{[2m-1]} := -\frac{1}{2m-1} \text{Tr} ([\omega dx - R]^{2p-m} \Gamma^{2m-1}) , \quad m = 1, 2, \ldots, 2p ,
$$

$$
\alpha^{4p-1}_{[0]} := 2\omega (d\omega)^{2p-1} ,
$$

where $[\omega dx - R]$ stands for the matrix-valued total 2-form with components $\omega^\alpha dx_\beta - R^\alpha_\beta$ and $\Gamma$ denotes the matrix-valued 1-form with $\Gamma^\alpha_\beta$ for components. Then, the total form

$$
\tilde{\alpha}^{4p-1} := \alpha^{4p-1}_{[0]} + \sum_{m=1}^{2p} \alpha^{4p-1}_{[2m-1]}
$$

obeys the equation

$$
\tilde{s}_W \tilde{\alpha}^{4p-1} = \text{Tr} R^{2p} .
$$
By decomposing the equation $\tilde{s}_W \tilde{\alpha}^{4p-1} = \text{Tr} R^{2p}$ with respect to the form degree, we obtain, in dimension $n = 4p - 1$, the descent equations

\[
\text{Tr} R^{2p} = dL^n_{CS},
\]

\[
s_W L^n_{CS} + da^{1,n-1} = 0,
\]

\[
s_W a^{1,n-1} + da^{2,n-2} = 0,
\]

\[
\vdots
\]

\[
s_W a^{2p-1,2p} + da^{2p,2p-1} = 0,
\]

\[
s_W a^{2p,2p-1} = 0, \quad a^{2p,2p-1} \equiv \alpha^{4p-1}_{[0]}.
\]
Finally, descent equations associated with a product of the type 
\[ L^{4p-1}_{CS} f_{K_1} \cdots f_{K_m} \] will be exactly the same as the descent associated with 
\[ L^{4p-1}_{CS} \], where each element \( a^{q,n-q} \) is obtained from the corresponding one in the descent for \( L^{4p-1}_{CS} \) upon taking the wedge product with \( f_{K_1} \cdots f_{K_m} \). In other words, the products of the type \( f_{K_1} \cdots f_{K_m} \) are completely spectators in a descent of \( s_W \) modulo \( d \). That the \( f_K \)'s are \( s_W \)-closed is trivial once one realizes the identity \( \text{Tr}(R^m(K)) \equiv \text{Tr}(W^m(K)) \) that is obtained from the relation \( R^{ab} = W^{ab} + 2e^a [P^b] \) where \( e^a \) are the vielbein 1-forms and \( P^a \) is the Schouten 1-form.
Conformal Anomalies

- Introduction
- Wess-Zumino consistency conditions
- Solution of the WZ conditions for the anomaly
- Results

Conformal Invariants

- Another cohomological group
- Type-A invariants
- Type-B invariants
- Action and field equations for pure Lorentz-Chern-Simons theories
**Type B Global Conformal Invariants**

- The $W$-tensors $\{W_{\Omega_i}\}_{i \in \mathbb{N}}$ are the building blocks for the construction of Weyl invariants. They had been constructed earlier by Gerlach, Günther and Wünsch circa 1985 [R. Gerlach and V. Wünsch (1999)]. The Bach tensor is the double trace of $W_{\Omega_2}$:

$$B_{\mu\nu} := \nabla^{\alpha} C_{\mu\nu\alpha} - P^{\alpha\beta} W_{\alpha\mu\nu\beta} \equiv \frac{1}{(3-n)} g^{\alpha\rho} \mathcal{D}_{\alpha} \mathcal{D}_{\beta} W^\beta_{\mu\nu\rho}.$$ 

- In $n = 6$, the invariant found in [T. Parker and S. Rosenberg, *J. Diff. Geometry* 25 (1987) 199] writes as

$$\mathcal{I}_6 = \sqrt{-g} \left( W^{\alpha\beta\mu\nu} \mathcal{D}_\lambda \mathcal{D}^\lambda W_{\alpha\beta\mu\nu} + \frac{1}{2} \mathcal{D}^\lambda W^{\alpha\beta\mu\nu} \mathcal{D}_\lambda W_{\alpha\beta\mu\nu} + \frac{8}{9} \mathcal{D}^\lambda W_{\lambda\rho\mu\nu} \mathcal{D}_\sigma W^{\sigma\rho\mu\nu} \right).$$
**Type-B invariants in 8D**

The strictly Weyl-invariant scalar densities in 8D in 18-dimensional basis:

\[ I_8 = a_1 W_{\rho \gamma \mu \sigma} \mathcal{D}^\alpha \mathcal{D}_\alpha \mathcal{D}^\beta \mathcal{D}_\beta W^{\rho \gamma \mu \sigma} + \]
\[ b_1 \mathcal{D}_\beta W^{\beta}_{\gamma \mu \alpha} \mathcal{D}_\nu \mathcal{D}^\nu \mathcal{D}_\rho W^{\rho \gamma \mu \alpha} + b_2 \mathcal{D}_\alpha W_{\mu \beta \gamma \nu} \mathcal{D}_\rho \mathcal{D}^\rho \mathcal{D}^\alpha W^{\mu \beta \gamma \nu} + \]
\[ c_1 \mathcal{D}^\alpha \mathcal{D}^\beta W_{\gamma \alpha \beta \mu} \mathcal{D}_\nu \mathcal{D}_\rho W^{\gamma \nu \rho \mu} + c_2 \mathcal{D}^\gamma \mathcal{D}_\gamma W_{\alpha \mu \nu \beta} \mathcal{D}_\rho \mathcal{D}^\rho \mathcal{D}^\alpha W^{\alpha \mu \nu \beta} + \]
\[ c_3 \mathcal{D}_\alpha \mathcal{D}_\beta W_{\nu \gamma \mu \rho} \mathcal{D}^\alpha \mathcal{D}^\beta W^{\nu \gamma \mu \rho} + c_4 \mathcal{D}_\alpha \mathcal{D}_\gamma W^{\gamma}_{\mu \nu \beta} \mathcal{D}^\alpha \mathcal{D}_\rho \mathcal{D}^\rho W^{\gamma \mu \nu \beta} + \]
\[ d_1 \mathcal{D}^\rho \mathcal{D}^\sigma W^{\beta}_{\rho \sigma} W^{\alpha \gamma \mu \nu \alpha \gamma \mu} + d_2 \mathcal{D}^\beta \mathcal{D}^\gamma W^{\mu \nu \rho \sigma} W^{\beta}_{\mu \nu \rho \sigma} + \]
\[ d_3 \mathcal{D}^\sigma \mathcal{D}_\sigma W^{\nu \gamma \mu \rho \alpha \beta} W^{\nu \rho \beta \alpha \beta} + d_4 \mathcal{D}^\sigma \mathcal{D}_\sigma W^{\beta \nu \mu \rho \beta \gamma \nu \alpha \gamma W^{\alpha \mu \gamma \nu \rho \beta} + \]
\[ e_1 W^{\gamma}_{\alpha \beta} \mathcal{D}^\rho W^{\nu \gamma \mu \nu \alpha \beta} + e_2 W^{\gamma}_{\alpha \beta} \mathcal{D}^\rho W^{\rho \gamma \mu \sigma} \mathcal{D}^\sigma W^{\alpha \beta}_{\nu \sigma \mu} + \]
\[ e_3 W_{\alpha \beta \gamma \mu} \mathcal{D}_\nu W^{\alpha \beta \rho \sigma} \mathcal{D}^\nu W^{\gamma \mu}_{\rho \sigma} + e_4 W^{\gamma}_{\alpha \beta} \mathcal{D}^\rho W^{\nu \sigma \mu} \mathcal{D}^\sigma W^{\beta}_{\nu \sigma \gamma} + \]
\[ e_5 W_{\alpha \beta \gamma \mu} \mathcal{D}^\rho W^{\alpha \beta \nu \gamma \mu} + e_6 W^{\alpha}_{\mu \gamma \beta} \mathcal{D}_\alpha W^{\gamma \nu \rho \sigma} \mathcal{D}^\beta W^{\mu \nu \rho \sigma} + \]
\[ e_7 W^{\alpha}_{\mu \gamma \beta} \mathcal{D}^\nu W^{\rho \sigma \gamma \mu} \mathcal{D}_\nu W^{\mu \rho \beta}. \]
Besides the seven Weyl invariants of the type $\sqrt{-g} \, W W W W$ given in [Fulling et al.], there are

$$J_j = \sqrt{-g} \, I_j , \quad j = 1, \ldots, 5 .$$

The first one starts with the quadratic term $I_1 = W^{\mu \nu \rho \sigma} \Box^2 W_{\mu \nu \rho \sigma} + \cdots$ whereas the other four are at least cubic in the Riemann tensor.

The coefficients in the 18-dimensional basis given above:

\begin{align*}
I_1 & = (1, 48/25, 2, 42/125, 9/10, 3/5, 96/125, 74/25, 208/5, \\
& \quad -8, 16/5, -144/25, -104/5, 0, 0, -88/25, 0, 0) , \\
I_2 & = (0, 0, 0, 0, 0, 0, 0, 5, 0, 0, 0, 0, 5, 0, 12/5, 0, 0) , \\
I_3 & = (0, 0, 0, 0, 0, 0, 0, 1, 0, -20, 0, -48/5, 0, 0, 0, 0, 0, -20) , \\
I_4 & = (0, 0, 0, 0, 0, 0, 1, 12, -5/6, -5/24, 4/5, -28/5, -13/8, -12/5, -63/50, -1, 1/2) , \\
I_5 & = (0, 0, 0, 0, 0, 0, 0, 1, 8, -2/3, 5/6, 24/25, -16/5, 0, -16/5, -12/25, 0, 0) .
\end{align*}
1 Conformal Anomalies
   - Introduction
   - Wess-Zumino consistency conditions
   - Solution of the WZ conditions for the anomaly
   - Results

2 Conformal Invariants
   - Another cohomological group
   - Type-A invariants
   - Type-B invariants
   - Action and field equations for pure Lorentz-Chern-Simons theories
**Action and Field Equations for $L_{CS}$**

- Given a pseudo-Riemannian spacetime $\mathcal{M}_{4p-1}$ of dimension $n = 4p - 1$ with an orientation, consider the functional

  \[
  I[g_{\mu\nu}] = \frac{1}{2p} \int_{\mathcal{M}_{4p-1}} L_{CS}^{4p-1}.
  \]

- The Euler-Lagrange derivative (wrt the metric) of the functional is

  \[
  \mathcal{E}^{\mu\nu} := \frac{\delta I}{\delta g_{\mu\nu}} \equiv \frac{1}{2^{2p-1}} \nabla^\lambda \mathcal{A}^{(\mu|\nu)}_{\lambda},
  \]

  where

  \[
  \mathcal{A}^{\mu|\nu}_{\lambda} := \varepsilon^{\mu\nu_2\nu_3...\nu_{4p-1}} [R_{\nu_2\nu_3} \cdots R_{\nu_{4p-2}\nu_{4p-1}}]^\nu_{\lambda}.
  \]

  and $[R_{\nu_2\nu_3} \cdots R_{\nu_{4p-2}\nu_{4p-1}}]^\nu_{\lambda}$ denotes the $(2p - 1)$-fold product of the 2-form valued matrix $[R_{\nu_2\nu_3}]^\alpha_{\beta} \equiv R^\alpha_{\beta\nu_2\nu_3}$. 

Weyl and diffeomorphism invariances of the action $I[g_{\mu\nu}]$ get translated into the Noether identities

$$g_{\mu\nu}E^{\mu\nu} \equiv 0 , \quad \text{and} \quad \nabla_\mu E^{\mu\nu} \equiv 0 .$$

For the second identity, one must use

$$\varepsilon^{\nu_1...\nu_{4p-1}} \text{Tr}[R_{\nu_1\nu_2} \cdots R_{\nu_{4p-3}\nu_{4p-2}} R_{\nu_{4p-1}\nu}] \equiv 0 ,$$

(Schouten identity and cyclicity of the trace)

Finally, one has the strict invariance under Weyl transformations:

$$s_W E^{\mu\nu} = -2 \omega E^{\mu\nu} \Leftrightarrow s_W E^{\mu \nu} = 0 .$$

that can be seen by expressing

$$A^{\mu | \nu} = \varepsilon^{\mu \nu_2 \nu_3 ... \nu_{4p-1}} [W_{\nu_2 \nu_3} \cdots W_{\nu_{4p-2} \nu_{4p-1}}]^{\nu} \lambda .$$
Conclusions

- As a consequence of our decomposition, global conformal invariants are not in one-to-one correspondence with the conformal anomalies. Indeed, multiplying the Lorentz Chern-Simons densities by the Weyl parameter \( \sigma(x) \) does not produce any consistent conformal anomaly.

- Our work generalises the analyses devoted to the three-dimensional case \( p = 1 \) [Deser-Jackiw-Templeton, van Nieuwenhuizen] and completes the classification of Alexakis.