Algebraic classification of conformal invariants and conformal anomalies in arbitrary dimension

Nicolas Boulanger

Physique de l'Univers, Champs et Gravitation, Université de Mons (**UMONS**), Belgium

ESI programme on *Geometry for Higher Spin Gravity: Conformal Structures, PDEs, and Q-manifolds*, August 23 - September 17, 2021

Based on [hep-th/0405228], [hep-th/0412314], [0706.0340] and [1809.05445], the second one with Johanna ERDMENGER while the last one with Jordan FRANCOIS and Serge LAZZARINI

Plan

1. Conformal Anomalies
   - Introduction
   - Wess-Zumino consistency conditions
   - Solution of the WZ conditions for the anomaly
   - Results
2. Conformal Invariants
   - Another cohomological group
   - Type-A invariants
   - Type-B invariants
   - Action and field equations for pure Lorentz-Chern-Simons theories

1 Conformal Anomalies

- Introduction
- Wess-Zumino consistency conditions
- Solution of the WZ conditions for the anomaly
- Results

2 Conformal Invariants

- Another cohomological group
- Type-A invariants
- Type-B invariants
- Action and field equations for pure Lorentz-Chern-Simons theories

History

In 1973, Derek Capper and Michael J. Duff discovered that the invariance under Weyl rescaling of the metric tensor

$$g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu}(x)$$

displayed by classical massless field systems in interaction with gravity no longer survives in the quantum theory.

$\hookrightarrow$ Weyl (or conformal) anomaly

Conformal massless fields coupled to gravity

Examples of spin-1, spin-1/2 and spin-0 field theories :

- $S[A_\mu, g_{\mu\nu}] = -\frac{1}{4}\int d^4x\sqrt{-g}\,F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}$

  where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$

- $S[\Psi, e^a_\mu] = -\frac{1}{2}\int d^nx\,e(\bar{\Psi}\gamma^a\nabla_a\Psi - \nabla_a\bar{\Psi}\gamma^a\Psi)$

- $S[\phi, g_{\mu\nu}] = -\frac{1}{2}\int\sqrt{-g}\,\left[g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \xi(n)\,\mathscr{R}\,\Phi^2\right]\,d^nx$

  with $\xi(n) = \dfrac{1}{4}\,[(n-2)/(n-1)]\,.$

Notation, definitions, conventions

- Spacetime indices → Greek letters, e.g. Riemann tensor $R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}{}_{\nu\sigma} + \ldots$, Christoffel symbols $\Gamma^{\mu}{}_{\nu\rho}$, Ricci tensor $\mathscr{R}_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta}$ and scalar curvature $\mathscr{R} = g^{\alpha\beta}\mathscr{R}_{\alpha\beta}$; Curvature two-form $R^{\mu}{}_{\nu} = \frac{1}{2}\, R^{\mu}{}_{\nu\rho\sigma} dx^\rho dx^\sigma$.
- Frame (tangent bundle) indices → Latin letters.
  The frame fields are $e_a = e^{\mu}_{a}\partial_\mu$ in coordinates x^μ. $e = \det e^{a}_{\mu}$ where $e^{a}_{\mu} e^{\nu}_{a} = \delta^{\nu}_{\mu}$.
- For Dirac spinors : Clifford algebra $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$ where γ_a denote Dirac's matrices and $\eta =$diag$(-,+,+,+)$; $\nabla_a \Psi = e^{\mu}_{a}(\partial_\mu - \frac{i}{2}\,\omega_{\mu}{}^{bc}\Sigma_{bc})\Psi$, where $\Sigma_{bc} = \frac{i}{4}\,[\gamma_b, \gamma_c]$ and $\omega_{\mu}{}^{bc} = \omega_{\mu}{}^{bc}(e)$ is the Levi-Civita spin-connection.

Trace of stress-tensor

- These matter systems coupled to gravity are invariant under the local Weyl rescalings

$$\left.\begin{array}{ll} g_{\mu\nu} & \to \Omega^2(x)\, g_{\mu\nu} \\ e^a_\mu & \to \Omega\, e^a_\mu \\ \Psi & \to \Omega^{(1-n)/2}\, \Psi \\ \phi & \to \Omega^{(2-n)/2}\, \phi \end{array}\right\} \quad (1)$$

- This is reflected in the (on-shell) tracelessness of the corresponding symmetric stress-tensor : (1) $\Rightarrow g^{\mu\nu}T_{\mu\nu} = 0\,.$

Local symmetries

By construction these actions are also invariant under diffeomorphisms.

To summarize, the local symmetries of these conformally invariant massless systems coupled to gravity are

Local symmetries :

- Diffeomorphism invariance
- Local Weyl rescaling invariance

Both symmetries cannot survive

It turns out that, after regularization and renormalization, both symmetries cannot survive at the same time. One always chooses to maintain diffeomorphism invariance (conservation of energy-momentum). This is done at the price of a

Weyl anomaly

$$\hookrightarrow A = g^{\mu\nu} \langle T_{\mu\nu} \rangle_{reg} \neq 0$$

Note : Weyl anomalies are also called "Trace anomalies" or "Conformal anomalies" for obvious reasons.

Some bits of QFT

- Generating functional of Green's functions :

$$Z[J] \;=\; \int \mathscr{D}\Phi \, e^{\frac{i}{\hbar} \int d^n x \, [\mathscr{L}(\Phi, \partial\Phi) + J(x)\Phi(x)]}$$

- Generating functional of *connected* Green's functions :

$$W[J] \;=\; -i \ln Z[J]$$

- The generating functional of 1PI Green's functions

$$\Gamma[\Phi_c] \;=\; W[J_\Phi] - \int d^n x \, \Phi_c(x) J_\Phi(x) \,, \quad \Phi_c(x) := \frac{\delta W[J]}{\delta J(x)} \,.$$

The functional Γ is also called *quantum action* or *effective action*.

1. Conformal Anomalies
   - Introduction
   - Wess-Zumino consistency conditions
   - Solution of the WZ conditions for the anomaly
   - Results
2. Conformal Invariants
   - Another cohomological group
   - Type-A invariants
   - Type-B invariants
   - Action and field equations for pure Lorentz-Chern-Simons theories

The anomalies cannot be anything

1. An anomaly in QFT ...

Anomalies occur when quantization spoils symmetries of the classical action, i.e. if $\Gamma[\Phi]$ cannot be made invariant under infinitesimal transformations s by a suitable choice of local counterterms.

2. ... is an infinitesimal variation

To lowest order in $\hbar$ the variation $A = s\,\Gamma[\Phi]$ is local. It is an anomaly if it cannot be written as $A = s\,C$ for any local functional C .

Consistency conditions

Because an anomaly is a variation

$$\boxed{A = s\,\Gamma[\Phi]}$$

it is not arbitrary but constrained to obey some consistency conditions. Similar to integrability conditions $\vec{\nabla} \times \vec{F} = 0$ which a gradient $\vec{F} = \vec{\nabla}\varphi$ has to satisfy.

$\Rightarrow$ An anomaly must satisfy the

Wess-Zumino consistency conditions [1971]

BRST-cohomological rephrasing

The analysis of WZ consistency conditions simplifies in the

Becchi-Rouet-Stora-Tyutin (BRST) formulation.

$\hookrightarrow$ one introduces a ghost for each gauge parameter ;

$\hookrightarrow$ one suitably defines the transformations of the ghosts so that

$$\boxed{s^2 = 0}$$

Local cohomology of s

The WZ consistency conditions take the simple form

$$s\,A = 0\,, \quad A \neq s\,C$$

where A and C are local functionals $A = \int a^{1,n}([\Phi], x)\,,\; C = \int b^{0,n}([\Phi], x)$ and s is the BRST differential.

1 Conformal Anomalies
- Introduction
- Wess-Zumino consistency conditions
- Solution of the WZ conditions for the anomaly
- Results

2 Conformal Invariants
- Another cohomological group
- Type-A invariants
- Type-B invariants
- Action and field equations for pure Lorentz-Chern-Simons theories

Wess-Zumino Consistency condition

- Central equations for candidate anomalies in QFT : Wess-Zumino (WZ) consistency conditions. By using these conditions, the general structure of all the know anomalies (except the conformal one) had been determined by purely algebraic methods featuring descent equations à la Stora-Zumino.
  $\hookrightarrow$ Book by R. Bertlmann (U. Wien) at Oxford U.P. (1996) on that topic.

- Determining the general solution of the WZ consistency conditions is tantamount to computing the cohomology of the corresponding Becchi-Rouet-Stora-Tyutin (BRST) differential s in the space of local functionals with ghost number one.

Stora-Zumino descent of equations

- Letting $A = \int a^{1,n}$, the WZ conditions get translated to

$$\boxed{s\,a^{1,n} + d\,a^{2,n-1} = 0\,, \quad a^{1,n} \sim a^{1,n} + s\,c^{0,n} + d\,c^{1,n-1}} \tag{2}$$

- With the total exterior derivative $d = dx^\mu \frac{\partial}{\partial x^\mu}$. One has

$$s^2 = 0\ , \qquad d^2 = 0\ , \qquad s\,d + d\,s = 0\ .$$

- Acting on (2) with s and using the above relations :

$$d\,(s\,a^{2,n-1}) = 0 \quad \overset{\text{algebraic Poincaré lemma}}{\Longrightarrow} \quad \boxed{s\,a^{2,n-1} + d\,a^{3,n-2} = 0}\ .$$

Apply s again on this equations, ...

A ladder of equations

... one obtains the following descent equations

$$
\begin{aligned}
s\,a^{1,n} + d\,a^{2,n-1} &= 0 \quad , \\
s\,a^{2,n-1} + d\,a^{3,n-2} &= 0 \quad , \\
&\vdots \\
s\,a^{q,n-q+1} + d\,a^{q+1,n-q} &= 0 \quad , \\
s\,a^{q+1,n-q} &= 0 \quad (0 \leqslant q \leqslant n)\,.
\end{aligned}
$$

If $q = 0$, the descent is *trivial* : $s\,a^{1,n} = 0\,$.

Dubois-Violette, Talon, Viallet (1985)

- In order to find $a^{1,n} \in H^{1,n}(s|d)$, find the $a^{q+1,n-q} \in H^{q+1}(s)$ that can be lifted up to a top form.

Bonora et al.

Cohomological consideration, although without any descent equation analysis
↪ pioneering works by Bonora, Cotta-Ramusino, Reina, Pasti and Bregola [1983–1985]. Results up to dimension $n = 6$.

They conjectured :

(i) Euler term times the Weyl parameter

$$e^{1,n} \quad = \quad \sqrt{-g}\,\omega\,(R^{\mu_1\nu_1}\dots R^{\mu_m\nu_m})\,\varepsilon_{\mu_1\nu_1\dots\mu_m\nu_m}\ ,$$

plus

(ii) strictly Weyl-invariant scalar densities times Weyl parameter. In $n = 4$, e.g.,

$$a^{1,4} = \omega\,\sqrt{-g}\,g^{\sigma\tau}g^{\lambda\kappa}W^{\mu}{}_{\rho\sigma\lambda}W^{\rho}{}_{\mu\tau\kappa}\,d^4x$$

where $W^{\mu}{}_{\rho\sigma\lambda}$: conformally invariant Weyl tensor, traceless part of Riemann curvature tensor $R^{\mu}{}_{\rho\sigma\lambda}$.

- Using dimensional regularization, Deser and Schwimmer (1993) confirmed the structure obtained by Bonora et al.
  The Euler term from class (i) was called type-A Weyl anomaly, while the terms of (ii) were called type-B anomalies.

- From the structure of the poles in the variation of the effective action, they observed that the type-A anomaly appears in a similar way to the non-Abelian chiral anomaly in Yang-Mills gauge theory.
  $\hookrightarrow$ that the type-A anomaly should arise via some *descent equations* à la Stora-Zumino was therefore conjectured.

In the BRST formalism

- Apart from $g_{\mu\nu}$, the other fields of the problem are the Weyl ghost ω and the diffeomorphisms ghosts ξ^{μ} , $gh(\xi^{\mu}) = gh(\omega) = 1$.

- The BRST transformations on the fields $\Phi^A = \{g_{\mu\nu}, \omega, \xi^{\mu}\}$ read

$$s_{_D} g_{\mu\nu} = \xi^{\rho}\partial_{\rho}g_{\mu\nu} + \partial_{\mu}\xi^{\rho}g_{\rho\nu} + \partial_{\nu}\xi^{\rho}g_{\mu\rho} \,, \; s_{_W} g_{\mu\nu} = 2\omega g_{\mu\nu}$$

$$s_{_D}\xi^{\mu} = \xi^{\rho}\partial_{\rho}\xi^{\mu} \,, \quad s_{_D}\omega = \xi^{\rho}\partial_{\rho}\omega \,, \quad s_{_W}\xi^{\mu} = 0 = s_{_W}\omega \;,$$

where the BRST differentials $s_{_W}$ and $s_{_D}$ implement the Weyl transformations and the diffeomorphisms transformations, respectively.

WZ conditions for WA

Upon quantization one always chooses to preserve diffeomorphism invariance. With $s = s_{_W} + s_{_D}$, decomposing $s\,a + d\,b = 0$, $a \sim a + s\,c + d\,f$ w.r.t. the Weyl ghost degree gives the WZ consistency conditions for the Weyl anomalies in terms of local forms :

$$(*)\quad \begin{cases} s_{_W} a^{1,n} + d\,b^{2,n-1} = \; 0\,, & a^{1,n} \neq s_{_W} p^{0,n} + d\,f^{1,n-1}\,, \\ s_{_D} a^{1,n} + d\,c^{2,n-1} = \; 0\,, & \forall\, p^{0,n} \quad s.t. \quad s_{_D} p^{0,n} + d\,h^{1,n-1} = 0\,. \end{cases}$$

Stora's trick

- Denoting $\boxed{\tilde{s}_W = s_W + d}$ and similarly for s_D, the problem $(*)$ consists in determining the $\tilde{s}_D$-invariant $(n+1)$-local total forms $\alpha(\mathscr{W})$ satisfying

$$\tilde{s}_W\,\alpha(\mathscr{W}) = 0\,, \quad \alpha(\mathscr{W}) \neq \tilde{s}_W\,\zeta(\mathscr{W}) + constant\,, \tag{3}$$

$$\boxed{totdeg = formdeg + gh}$$

  where $\zeta(\mathscr{W})$ must be $\tilde{s}_D$-invariant.

- Using very general results obtained in [Friedemann Brandt, CMP 1996], we know that the solution of (3) will take the form

$$\alpha(\mathscr{W}) = 2\omega\,\tilde{C}^{N_1}\ldots\tilde{C}^{N_n}\,a_{N_1\ldots N_n}(\mathscr{T})\,.$$

Elimination of trivial pairs in Jet space

Lemma

Suppose there is a set of local jet coordinates

$$\mathscr{B} = \{\mathscr{U}^{\ell}, \mathscr{V}^{\ell}, \mathscr{W}^{\Lambda}\}$$

such that the change of coordinates from $\mathscr{J} = \{[\Phi^A], x^\mu, dx^\mu\}$ to $\mathscr{B}$ is local and locally invertible and

$$\begin{aligned} \tilde{s}\,\mathscr{U}^{\ell} &= \mathscr{V}^{\ell} \quad \forall \ell\,, \\ \tilde{s}\,\mathscr{W}^{\Lambda} &= \mathscr{R}^{\Lambda}(\mathscr{W}) \quad \forall\,\Lambda\,. \end{aligned}$$

Then, locally the $\mathscr{U}$'s and $\mathscr{V}$'s can be eliminated from the $\tilde{s}$-cohomology, i.e., the latter reduces locally to the $\tilde{s}$-cohomology on total local forms depending only on the $\mathscr{W}$'s.

In the case at hand

- Jet space $\mathscr{J} = \{[g_{\mu\nu}], [\omega], [\xi^\mu], x^\mu, dx^\mu\}$ and $\tilde{s} = s_{_D} + s_{_W} + d$ the differential acting on $\mathscr{J}$

- The $\{\mathscr{U}, \mathscr{V}, \mathscr{W}\}$-decomposition of $\mathscr{J}$ corresponding to $\tilde{s}$:

$$\begin{aligned}
\{\mathscr{U}^\ell\} &= \{x^\mu, \partial_{(\mu_1 \ldots \mu_k}\Gamma_{\mu_{k+1}\mu_{k+2})}{}^\nu, \nabla_{(\mu_1 \ldots \mu_k} P_{\mu_{k+1}\mu_{k+2})}, \ k \in \mathbb{N}\}, \\
\{\mathscr{V}^\ell\} &= \{\tilde{s}\mathscr{U}^\ell\}, \quad \{\mathscr{W}^\Lambda\} = \{\mathscr{T}^i, \tilde{C}^N\}, \\
\{\mathscr{W}^\Lambda\} &= \{\mathscr{T}^i\} \cup \{\tilde{C}^N\}, \\
\{\mathscr{T}^i\} &= \{g_{\mu\nu}, \mathscr{D}_{(\alpha_k} \ldots \mathscr{D}_{\alpha_1} W^\beta{}_{\gamma\delta)\epsilon}, \ k \in \mathbb{N}\}, \\
\{\tilde{C}^N\} &= \{2\omega, \tilde{\xi}^\nu, \tilde{C}_\nu{}^\rho, \tilde{\omega}_\alpha\}, \\
& \quad \tilde{\xi}^\nu := \xi^\nu + dx^\nu, \ \tilde{C}_\nu{}^\rho := \partial_\nu \xi^\rho + \tilde{\xi}^\alpha \Gamma_{\alpha\nu}{}^\rho, \ \tilde{\omega}_\alpha := \omega_\alpha - \tilde{\xi}^\beta P_{\alpha\beta}.
\end{aligned}$$

- Assignment of degrees :

$$totdeg(\mathscr{T}^i) = 0\,, \quad totdeg(\tilde{C}^N) = 1\,, \quad \tilde{C}^N = \hat{C}^N + \mathscr{A}^N\,,$$
$$gh(\hat{C}^N) = 1 = formdeg(\mathscr{A}^N)\,, \quad gh(\mathscr{A}^N) = 0 = formdeg(\hat{C}^N)\,,$$

- *covariant ghosts* and *connection 1-forms* :

$$\begin{aligned} \{\hat{C}^N\} &= \{2\omega, \xi^\nu, \hat{C}_\nu{}^\rho := \partial_\nu \xi^\rho + \xi^\alpha \Gamma_{\alpha\nu}{}^\rho, \hat{\omega}_\alpha := \omega_\alpha - \xi^\mu P_{\mu\alpha}\}\,, \\ \{\mathscr{A}^N\} &= \{0\,, dx^\mu \delta^\nu_\mu\,, dx^\mu \Gamma_{\mu\nu}{}^\rho\,, -dx^\mu P_{\mu\alpha}\}\,. \end{aligned}$$

- The differential $\tilde{s}$ raises the total degree by 1 unit, so

$$\tilde{s}\mathscr{T}^i = \tilde{C}^N \Delta_N \mathscr{T}^i \quad \Leftrightarrow \quad \left\{ \begin{array}{l} s\mathscr{T}^i = \hat{C}^N \Delta_N \mathscr{T}^i \\ d\mathscr{T}^i = \mathscr{A}^N \Delta_N \mathscr{T}^i \end{array} \right. ,$$
$$\{\Delta_N\} = \{\Delta^{ex}_g\,, \mathscr{D}_\nu\,, \Delta_\rho{}^\nu\,, \mathbf{\Gamma}^\alpha\}\,.$$

The W-tensors, by iteration :

$$W_{\Omega_k} = \mathscr{D}_{\alpha_k} W_{\Omega_{k-1}} = (\nabla_{\alpha_k} + P_{\beta\alpha_k}\mathbf{\Gamma}^{\beta}) W_{\Omega_{k-1}} \ .$$

They transform as

$$s_W W_{\Omega_i} = \omega_\alpha \mathbf{\Gamma}^\alpha W_{\Omega_i}$$

$\hookrightarrow$ *only the first derivative* $\omega_\alpha = \partial_\alpha \omega$ of the Weyl parameter *appears* and

$$\mathbf{\Gamma}^\alpha W_{\Omega_j} = [T^\alpha]_{\Omega_j}^{\Omega_{j-1}} W_{\Omega_{j-1}} \ .$$

The $[T^\alpha]_{\Omega_j}^{\Omega_{j-1}}$'s are built iteratively starting with $[T^\alpha]_{\Omega_j}^{\Omega_{j-1}} = 0 \ \ \forall\, j \leqslant 0$.
The operators acting on the space $\mathscr{T}$ of tensors and connections :

$$\{\Delta_N\} = \{\Delta_g^{ex}\,, \mathscr{D}_\nu\,, \Delta_\rho{}^\nu\,, \mathbf{\Gamma}^\alpha\}\ ,$$

$$\mathscr{D}_\mu := \partial_\mu - \Gamma_{\mu\nu}{}^\rho \Delta_\rho{}^\nu + P_{\mu\alpha}\mathbf{\Gamma}^\alpha\ ,$$

Δ_g^{ex} counts the number of metric tensors appearing in a given expression.

Gauge covariant algebra

With $\Delta^{\mu}{}_{\nu}$ the generators of $GL(n)$-transformations of world indices acting on a type-$(1,1)$ tensor T_{α}^{β} as $\Delta^{\mu}{}_{\nu}T_{\alpha}^{\beta} = \delta_{\alpha}^{\mu}T_{\nu}^{\beta} - \delta_{\nu}^{\beta}T_{\alpha}^{\mu}$, the gauge covariant algebra $\mathscr{G}$ generated by $\{\Delta_N\} = \{\Delta_g^{ex}\,, \mathscr{D}_\nu\,, \Delta^{\mu}{}_{\nu}\,, \mathbf{\Gamma}^\alpha\}$ reads

$$
\begin{aligned}
[\Delta^{\mu}{}_{\nu}, \mathbf{\Gamma}^\alpha] &= -\delta_\nu^\alpha \mathbf{\Gamma}^\mu\,, \quad [\Delta^{\mu}{}_{\nu}, \mathscr{D}_\alpha] = \delta_\alpha^\mu \mathscr{D}_\nu\,, \\
[\Delta^{\rho}{}_{\mu}, \Delta^{\sigma}{}_{\nu}] &= \delta_\nu^\rho \Delta^{\sigma}{}_{\mu} - \delta_\mu^\sigma \Delta^{\rho}{}_{\nu}\,, \quad [\mathbf{\Gamma}^\alpha, \mathbf{\Gamma}^\beta] = 0\,, \\
[\mathscr{D}_\beta, \mathbf{\Gamma}^\alpha] &= \mathscr{P}_{\beta\mu}^{\nu\alpha} \Delta^{\mu}{}_{\nu} - \delta_\beta^\alpha \Delta_g^{ex}\,, \\
[\mathscr{D}_\rho, \mathscr{D}_\sigma] &= -W^{\mu}{}_{\nu\rho\sigma} \Delta^{\nu}{}_{\mu} - C_{\alpha\rho\sigma}\, \mathbf{\Gamma}^\alpha\,,
\end{aligned}
$$

where $C_{\alpha\mu\nu} := 2\,\nabla_{[\nu}P_{\mu]\alpha}$ is the Cotton tensor and $\mathscr{P}_{\beta\mu}^{\nu\alpha} := (-g^{\nu\alpha}g_{\beta\mu} + \delta_\beta^\nu\delta_\mu^\alpha + \delta_\beta^\alpha\delta_\mu^\nu)$. The operator Δ_g^{ex} commutes with all the other generators.

- With $\{\Delta_N\} = \{\Delta_g^{ex}\,, \mathscr{D}_\nu\,, \Delta^\mu{}_\nu\,, \mathbf{\Gamma}^\alpha\}$, the action of $\tilde{s}_W$ on the tensor fields $\{\mathscr{T}^i\}$ and generalized connections $\{\tilde{C}^N\}$ can be written as

$$\tilde{s}_W\,\mathscr{T}^i = \tilde{C}^N \Delta_N \mathscr{T}^i\,, \quad \tilde{s}_W\,\tilde{C}^N = \tfrac{1}{2}\,\tilde{C}^L \tilde{C}^K \mathscr{F}_{KL}{}^N(\mathscr{T})\,,$$

where $\mathscr{F}_{KL}{}^N(\mathscr{T})$ denote the structure functions of the gauge covariant algebra $\mathscr{G}$:

$$[\Delta_M, \Delta_N] = \mathscr{F}_{MN}{}^L(\mathscr{T})\Delta_L\,.$$

- The relation $\tilde{s}_W\,\tilde{C}^N = \frac{1}{2}\,\tilde{C}^L \tilde{C}^K \mathscr{F}_{KL}{}^N(\mathscr{T})$ generalizes what Stora coined the Russian formula.

Jacobi identities for gauge covariant algebra

- From $\tilde{s}^2\tilde{C}^N = 0$, get the following set of Bianchi identities

  - $\tilde{s}^2\omega = 0 \;\Rightarrow\; C_{[\mu\rho\sigma]} = 0$
  - $\tilde{s}^2\tilde{C}_\mu{}^\nu = 0 \;\Rightarrow\; \nabla_{[\gamma}W_{\delta\epsilon]\alpha\beta} - C_{\alpha[\gamma\delta}g_{\epsilon]\beta} + C_{\beta[\gamma\delta}g_{\epsilon]\alpha} = 0$
  - $\tilde{s}^2\tilde{\xi}^\mu = 0 \;\Rightarrow\; \begin{cases} \mathscr{P}^{\alpha\mu}_{[\rho\nu]} = 0 \\ W^\mu{}_{[\nu\rho\sigma]} = 0 \end{cases}$
  - $\tilde{s}^2\tilde{\omega}_\alpha = 0 \;\Rightarrow\; \begin{cases} \mathbf{\Gamma}^\alpha C_{\beta\rho\sigma} + W^\alpha{}_{\beta\rho\sigma} = 0 \\ \mathscr{D}_{[\beta}C_{\rho\sigma]\alpha} = 0 \end{cases}$

Relation with conformal algebra

Introducing the new set of generators $\{ P_\mu \,,\; K_\nu \,,\; M_{\mu\nu} \,,\; D \}$ via

$$\{ \Delta_{\mu\nu} \,,\; \mathbf{\Gamma}_\alpha \,,\; D \} = \{ g_{\mu\rho}\Delta^\rho{}_\nu \,,\; g_{\alpha\beta}\mathbf{\Gamma}^\beta \,,\; \delta^\mu_\nu \Delta^\nu{}_\mu - \Delta^{ex}_g \} \,,$$

$$\{ P_\mu \,,\; K_\nu \,,\; M_{\mu\nu} \} = \{ \tfrac{1}{4}\,\mathscr{D}_\mu \,,\; 2\,\mathbf{\Gamma}_\nu \,,\; -2\,\Delta_{[\mu\nu]} \} \,,$$

one gets

$$\begin{aligned}
[P_\alpha, M_{\mu\nu}] &= 2\,g_{\alpha[\mu}P_{\nu]} \,, \quad [K_\alpha, M_{\mu\nu}] = 2\,g_{\alpha[\mu}K_{\nu]} \,, \\
[D, P_\mu] &= P_\mu \,, \quad [D, K_\mu] = -K_\mu \,, \\
[M_{\alpha\mu}, M_{\beta\nu}] &= 2\,g_{\alpha[\beta}M_{\nu]\mu} - 2\,g_{\mu[\beta}M_{\nu]\alpha} \,, \\
[P_\mu, K_\nu] &= 2\,(g_{\mu\nu}D + M_{\mu\nu}) \,, \quad [K_\mu, K_\nu] = 0 \,, \\
[P_\mu, P_\nu] &= -\frac{1}{2}\,W^{\rho\sigma}{}_{\mu\nu}\,M_{\rho\sigma} - \frac{1}{2}\,C_{\alpha\mu\nu}\,K^\alpha
\end{aligned}$$

which is isomorphic to the conformal algebra $\mathfrak{so}(2,n)$ when $g_{\mu\nu} = \eta_{\mu\nu}$.

1 Conformal Anomalies

- Introduction
- Wess-Zumino consistency conditions
- Solution of the WZ conditions for the anomaly
- Results

2 Conformal Invariants

- Another cohomological group
- Type-A invariants
- Type-B invariants
- Action and field equations for pure Lorentz-Chern-Simons theories

Lemma : Let $\psi_{\mu_1 \dots \mu_{2p}}$ be the local total form

$$\begin{aligned}\psi_{\mu_1 \dots \mu_{2p}} &:= \frac{\omega}{\sqrt{-g}}\,\varepsilon^{\alpha_1 \dots \alpha_r}{}_{\nu_1 \dots \nu_r \mu_1 \dots \mu_{2p}} \tilde{\omega}_{\alpha_1} \dots \tilde{\omega}_{\alpha_r}\, dx^{\nu_1} \dots dx^{\nu_r}\,, \\ p &= m - r\,, \quad m = n/2\,, \quad 0 \leqslant r \leqslant m\end{aligned}$$

and let $W^{\mu\nu}$ denote the tensor-valued two-form $W^{\mu\nu} = W^{\mu}{}_{\rho}\, g^{\rho\nu}$, then the local total forms $\Phi_r^{[n-r]}$ $(0 \leqslant r \leqslant m)$

$$\Phi_r^{[n-r]} := \frac{(-1)^p}{2^p}\,\frac{m!}{r!\,p!}\,\psi_{\mu_1 \dots \mu_{2p}}\, W^{\mu_1 \mu_2} \dots W^{\mu_{2p-1}\mu_{2p}}$$

obey a descent equations so that the following relations hold :

$$\tilde{s}_W\, \alpha = 0 = \tilde{s}_W\, \beta$$

with

$$\alpha := \sum_{r=1}^{m} \Phi_r^{[n-r]}\,, \quad \beta := \Phi_0^{[n]}\,.$$

Theorem (A)

The top form-degree component $a^{1,n}$ *of* α *satisfies the WZ consistency conditions for the Weyl anomalies. The WZ conditions for* $a^{1,n}$ *give rise to a non-trivial descent and* $a^{1,n}$ *is the unique anomaly with such a property, up to the addition of trivial terms and anomalies satisfying a trivial descent.*

Theorem (B)

The top form-degree component $e^{1,n}$ *of* $(\alpha+\beta)$ *is proportional to the Euler density of the manifold* $\mathscr{M}_n$ *:*

$$e_1^n \quad = \quad \frac{(-1)^m}{2^m}\,\sqrt{-g}\,\omega\,(R^{\mu_1\nu_1}\dots R^{\mu_m\nu_m})\,\varepsilon_{\mu_1\nu_1\dots\mu_m\nu_m}\ .$$

The anomaly $\beta=\Phi_0^{[n]}$ *— a contraction of a product of Weyl tensors — satisfies a trivial descent. It is a type-B anomaly.*

Example for $n = 6$

- From the definitions above, one gets for $n = 6$

$$\begin{aligned}
\beta = \Phi_0^{[6]} &= \frac{-\omega}{8}\sqrt{-g}\,\varepsilon_{\mu_1\dots\mu_6}\,W^{\mu_1\mu_2}W^{\mu_3\mu_4}W^{\mu_5\mu_6}\,, \\
\Phi_1^{[5]} &= \frac{3\,\omega}{4}\sqrt{-g}\,\varepsilon^{\alpha}{}_{\nu\mu_1\dots\mu_4}\,\tilde{\omega}_\alpha\,dx^\nu\,W^{\mu_1\mu_2}W^{\mu_3\mu_4}\,, \\
\Phi_2^{[4]} &= \frac{-3\,\omega}{2}\sqrt{-g}\,\varepsilon^{\alpha\beta}{}_{\mu\nu\rho\sigma}\,\tilde{\omega}_\alpha\tilde{\omega}_\beta\,dx^\mu dx^\nu\,W^{\rho\sigma}\,, \\
\Phi_3^{[3]} &= \omega\sqrt{-g}\,\varepsilon^{\alpha\beta\gamma}{}_{\mu\nu\rho}\,\tilde{\omega}_\alpha\tilde{\omega}_\beta\tilde{\omega}_\gamma\,dx^\mu dx^\nu dx^\rho\,.
\end{aligned}$$

- Extracting from $\alpha = \Phi_1^{[5]} + \Phi_2^{[4]} + \Phi_3^{[3]}$ its top form-degree component amounts to selecting everywhere the contribution $\mathscr{A}_\mu$ of $\tilde{\omega}_\mu = \omega_\mu + \mathscr{A}_\mu$. As a consequence, the top form-degree component of $(\alpha + \beta)$ reproduces the expression $e_1^6 = -\frac{\omega}{8}\sqrt{-g}\,\varepsilon_{\mu_1\dots\mu_6}R^{\mu_1\mu_2}R^{\mu_3\mu_4}R^{\mu_5\mu_6}$ making use of $R^{\mu\nu} = W^{\mu\nu} - 2\,\mathscr{A}^{[\mu}dx^{\nu]}$ and $\mathscr{A}^\mu = -g^{\mu\nu}P_{\nu\rho}dx^\rho$.

A regularization-free understanding

- Universal structure of Weyl anomalies established in a purely algebraic manner, independently of any regularization scheme and in *arbitrary* dimensions n . In particular, we do not resort to dimensional analysis. That the anomalies exist in even dimension $n = 2$, only is *not* an assumption but arises in the cohomological analysis. The type-A Weyl anomaly is the *unique* (up to trivial terms) Weyl anomaly satisfying a non-trivial descent of equations.
- the Weyl anomalies satisfying a trivial descent equations are all (integral) of product of the Weyl parameter times a strictly Weyl-invariant scalar density. They are called type-B Weyl anomalies.

1 Conformal Anomalies

- Introduction
- Wess-Zumino consistency conditions
- Solution of the WZ conditions for the anomaly
- Results

2 Conformal Invariants

- Another cohomological group
- Type-A invariants
- Type-B invariants
- Action and field equations for pure Lorentz-Chern-Simons theories

From anomalies to invariants

- Conformal anomalies are related to *global conformal invariant*. The Deser-Schwimmer paper triggered the interest of some conformal geometers.
- Global conformal invariants are given by the integral over a n-dimensional (pseudo) Riemannian manifold $\mathscr{M}_n(g)$ of linear combinations of strictly Weyl-invariant scalar densities and scalar densities that are invariant under Weyl rescalings *only up to* a total derivative.
- What is the general structure of the latter ?
  $\hookrightarrow$ relevant for (quasi-)Weyl-invariant Lagrangians densities.

- By the assumption of *locality*, a global invariant is a ghost-zero scalar density whose Hodge dual $a^{0,n}$ obeys the cocycle equation

$$sa^{0,n} + db^{1,n-1} = 0 \; .$$

- The local conformal invariants are (the integral of) scalar densities that are strictly Weyl invariant. They can be built using various techniques, be them algebraic or geometric [tractor calculus].

- The global invariants are scalar densities that are Weyl invariant only up to a total derivative $\Rightarrow$ Produce a non-trivial descent equations.

- Non-trivial *descent equations* :

$$\left.\begin{aligned} s\,a^{0,n} + d\,a^{1,n-1} &= 0 \\ s\,a^{1,n-1} + d\,a^{2,n-2} &= 0 \\ &\vdots \\ s\,a^{p-1,n-p+1} + d\,a^{p,n-p} &= 0 \\ s\,a^{p,n-p} &= 0 \end{aligned}\right\}$$

It stops either because $p = n$ or because one encounters an s-cocycle $a^{p,n-p}$.

- Decomposing the first equation wrt Weyl-ghost degree :

$$\left\{\begin{array}{rcl} s_{_D}\,a^{0,n} + d\,f^{1,n-1} &=& 0\,, \\ s_{_W}\,a^{0,n} + d\,g^{1,n-1} &=& 0\,, \end{array}\right. \quad a^{0,n} \neq d\,b^{0,n-1}\,.$$

- The classification of global conformal invariants is also given by the cohomology of the associated BRST differential in top form degree n, but this time, at ghost number *zero*, i.e., $H^{0,n}(s|d)$. The two cohomological groups $H^{1,n}(s|d)$ (anomalies) and $H^{0,n}(s|d)$ present some similarities but also important differences. The latter group is the larger!
- The conjecture of Deser and Schwimmer on the structure of Weyl anomalies led the geometer Spyros Alexakis to study the problem of the *classification of global conformal invariants*.
  $\hookrightarrow$ Gave rise to several publications culminating with the monograph "The Decomposition of Global Conformal Invariants" in the Annals of Mathematics Studies series at Princeton U. Press, 2012.

Pursuing the cohomological analysis

- From
$$\begin{cases} s_{_D}\, a^{0,n} + d\, f^{1,n-1} = & 0\,, \\ s_{_W}\, a^{0,n} + d\, g^{1,n-1} = & 0\,, \end{cases} \quad a^{0,n} \neq d\, b^{0,n-1}\;,$$
$\hookrightarrow$ Find the cocycles of the differential $s_{_W}$ modulo d, in the cohomology of the diffeomorphism-invariant local n-forms.

- The latter cohomology class already been worked out in [Brandt-Dragon-Kreuzer89] and [Barnich-Brandt-Henneaux95].

- Denote by $f_K := \mathrm{Tr}(R^{m(K)})$, $K \in \{1, \ldots, r = [n/2]\}$, the invariant polynomials of the Lorentz algebra $so(1, n-1)$ and q^0_K the corresponding Chern-Simons $(2m(K)-1)$-forms obeying $dq^0_K = f_K$. The general solution of the first equation above decomposes into two main classes :

- Two main classes :

$$a^{0,n} = \underbrace{\sqrt{-g}\, L(\nabla, R, g) d^n x}_{class\ I} + \underbrace{\sum_m \sum_{K:m(K)=m} q^0_K \frac{\partial}{\partial f_K} P_m(f_1, \ldots, f_r)}_{class\ II} \ .$$

- The second class only contributes for spacetimes of dimensions $n = 4p - 1$, $p \in \mathbb{N}^*$. Taking $n = 7$ as a definite example, the second class gives two structures

$$\mathrm{Tr}(\Gamma d\Gamma + \tfrac{2}{3}\,\Gamma^3)\mathrm{Tr}(R^2) \equiv L^3_{CS}\,\mathrm{Tr}(R^2) \text{ and } L^7_{CS} = \mathrm{Tr}(I_7)\ ,$$
$$I_7 = \Gamma(d\Gamma)^3 + \tfrac{8}{5}(d\Gamma)^2\Gamma^3 + \tfrac{4}{5}\Gamma(\Gamma d\Gamma)^2 + 2\,\Gamma^5 d\Gamma + \tfrac{4}{7}\Gamma^7,$$

where Γ denotes the matrix-valued 1-form $dx^\mu\, \Gamma^\alpha{}_{\beta\mu}$ whose components $\Gamma^\alpha{}_{\beta\mu}$ are the Christoffel symbols and $\mathrm{Tr}(\cdot)$ denotes the matrix trace. $\mathrm{Tr} R^2 \equiv R^\alpha{}_\beta R^\beta{}_\alpha$ for $R^\alpha{}_\beta = \frac{1}{2}\, dx^\mu dx^\nu\, R^\alpha{}_{\beta\mu\nu}$ the curvature 2-form.

1. Conformal Anomalies
   - Introduction
   - Wess-Zumino consistency conditions
   - Solution of the WZ conditions for the anomaly
   - Results
2. Conformal Invariants
   - Another cohomological group
   - Type-A invariants
   - Type-B invariants
   - Action and field equations for pure Lorentz-Chern-Simons theories

Lemma 1 :

Let $\psi_{\mu_1\dots\mu_{2p}}$ be the local total form

$$\psi_{\mu_1\dots\mu_{2p}} \;=\; \frac{1}{\sqrt{-g}}\,\varepsilon^{\alpha_1\dots\alpha_r}{}_{\nu_1\dots\nu_r\mu_1\dots\mu_{2p}}\tilde{\omega}_{\alpha_1}\dots\tilde{\omega}_{\alpha_r}\,dx^{\nu_1}\dots dx^{\nu_r}\,,$$
$$p \;=\; m-r\,,\quad m=n/2\,,\quad r\in\{0,\dots,m\}\,.$$

Then, the local total forms

$$\Phi^{[n-r]}_r = \frac{(-1)^p}{2^p}\,\frac{m!}{r!\,p!}\,\psi_{\mu_1\dots\mu_{2p}}\,W^{\mu_1\mu_2}\dots W^{\mu_{2p-1}\mu_{2p}}$$

satisfy non-trivial descent equations and give solutions

$$\tilde{s}_W\,\alpha \;=\; 0 \;=\; \tilde{s}_W\,\beta \quad \text{for}$$
$$\alpha \;=\; \sum_{r=1}^{m}\Phi^{[n-r]}_r \quad \text{and} \quad \beta=\Phi^{[n]}_0\,.$$

[Lemma 2 Invariants of class I]

The top form-degree component $a^{0,n}$ of α in Lemma 1 satisfies the cocycle condition for the conformal invariants. It gives rise to a non-trivial descent in $H(s_W|d)$. The invariant $\beta = \Phi_0^{[n]}$ satisfies a trivial descent and is obtained by taking contractions of products of Weyl tensors (m of them in dimension $n = 2m$). The top form-degree component $e^{0,n}$ of $\alpha + \beta$ is proportional to the Euler density of the manifold $\mathscr{M}_n$:

$$e^{0,n} = \frac{(-1)^m}{2^m} \sqrt{-g}\, \varepsilon_{\alpha_1 \beta_1 \ldots \alpha_m \beta_m} \left(R^{\alpha_1 \beta_1} \wedge \ldots \wedge R^{\alpha_m \beta_m}\right)$$

It is the *only* conformal invariant of the class I that satisfies a non-trivial descent in $H(s_W|d)$.

Lemma 3 [Invariants of class II]

Let $\alpha^{4p-1}_{[2m-1]}$ be the total $(4p-1)$-form of degree $2m-1$ in the connection 1-form Γ, defined by

$$\alpha^{4p-1}_{[2m-1]} := -\tfrac{1}{2m-1}\,\mathrm{Tr}\left([\omega dx - R]^{2p-m}\Gamma^{2m-1}\right)\ , \quad m = 1, 2, \ldots 2p\ ,$$
$$\alpha^{4p-1}_{[0]} := 2\omega(d\omega)^{2p-1}\ ,$$

where $[\omega dx - R]$ stands for the matrix-valued total 2-form with components $\omega^{\alpha} dx_{\beta} - R^{\alpha}{}_{\beta}$ and Γ denotes the matrix-valued 1-form with $\Gamma^{\alpha}{}_{\beta}$ for components. Then, the total form

$$\tilde{\alpha}^{4p-1} := \alpha^{4p-1}_{[0]} + \sum_{m=1}^{2p} \alpha^{4p-1}_{[2m-1]}$$

obeys the equation

$$\tilde{s}_W \tilde{\alpha}^{4p-1} = \mathrm{Tr}R^{2p}\ .$$

By decomposing the equation $\tilde{s}_W \tilde{\alpha}^{4p-1} = \mathrm{Tr} R^{2p}$ with respect to the form degree, we obtain, in dimension $n = 4p - 1$, the descent equations

$$\begin{aligned} \mathrm{Tr} R^{2p} &= dL^n_{CS} \;, \\ s_W L^n_{CS} + da^{1,n-1} &= 0 \;, \\ s_W a^{1,n-1} + da^{2,n-2} &= 0 \;, \\ &\vdots \\ s_W a^{2p-1,2p} + da^{2p,2p-1} &= 0 \;, \\ s_W a^{2p,2p-1} &= 0 \;, \quad a^{2p,2p-1} \equiv \alpha^{4p-1}_{[0]} \;. \end{aligned}$$

- Finally, descent equations associated with a product of the type $L_{CS}^{4p-1} f_{K_1} \dots f_{K_m}$ will be exactly the same as the descent associated with L_{CS}^{4p-1}, where each element $a^{q,n-q}$ is obtained from the corresponding one in the descent for L_{CS}^{4p-1} upon taking the wedge product with $f_{K_1} \dots f_{K_m}$. In other words, the products of the type $f_{K_1} \dots f_{K_m}$ are completely spectators in a descent of s_W modulo d. That the f_K's are s_W-closed is trivial once one realizes the identity $\mathrm{Tr}(R^{m(K)}) \equiv \mathrm{Tr}(W^{m(K)})$ that is obtained from the relation $R^{ab} = W^{ab} + 2e^{[a}P^{b]}$ where e^a are the vielbein 1-forms and P^a is the Schouten 1-form.

1. Conformal Anomalies
   - Introduction
   - Wess-Zumino consistency conditions
   - Solution of the WZ conditions for the anomaly
   - Results
2. Conformal Invariants
   - Another cohomological group
   - Type-A invariants
   - Type-B invariants
   - Action and field equations for pure Lorentz-Chern-Simons theories

Type B global conformal invariants

- The W-tensors $\{W_{\Omega_i}\}_{i\in\mathbb{N}}$ are the building blocks for the construction of Weyl invariants. They had been constructed earlier by Gerlach, Günther and Wünsch circa 1985 [R. Gerlach and V. Wünsch (1999)]. The Bach tensor is the double trace of W_{Ω_2} :

$$B_{\mu\nu} := \nabla^\alpha C_{\mu\nu\alpha} - P^{\alpha\beta} W_{\alpha\mu\nu\beta} \equiv \tfrac{1}{(3-n)}\, g^{\alpha\rho} \mathscr{D}_\alpha \mathscr{D}_\beta W^\beta{}_{\mu\nu\rho}\,.$$

- In $n = 6$, the invariant found in [T. Parker and S. Rosenberg, *J. Diff. Geometry* 25 (1987) 199] writes as

$$\mathscr{I}_6 = \sqrt{-g}\,\Big(W^{\alpha\beta\mu\nu} \mathscr{D}_\lambda \mathscr{D}^\lambda W_{\alpha\beta\mu\nu} + \tfrac{1}{2}\, \mathscr{D}^\lambda W^{\alpha\beta\mu\nu} \mathscr{D}_\lambda W_{\alpha\beta\mu\nu} + \tfrac{8}{9}\, \mathscr{D}^\lambda W_{\lambda\rho\mu\nu} \mathscr{D}_\sigma W^{\sigma\rho\mu\nu} \Big)\,.$$

Type-B invariants in 8D

The strictly Weyl-invariant scalar densities in $8D$ in 18-dimensional basis :

$$
\begin{aligned}
I_8 \quad = \quad & a_1\, W_{\rho\gamma\mu\sigma}\mathscr{D}^{\alpha}\mathscr{D}_{\alpha}\mathscr{D}^{\beta}\mathscr{D}_{\beta}W^{\rho\gamma\mu\sigma} + \\
& b_1\, \mathscr{D}_{\beta}W^{\beta}{}_{\gamma\mu\alpha}\mathscr{D}_{\nu}\mathscr{D}^{\nu}\mathscr{D}_{\rho}W^{\rho\gamma\mu\alpha} + b_2\, \mathscr{D}_{\alpha}W_{\mu\beta\gamma\nu}\mathscr{D}_{\rho}\mathscr{D}^{\rho}\mathscr{D}^{\alpha}W^{\mu\beta\gamma\nu} + \\
& c_1\, \mathscr{D}^{\alpha}\mathscr{D}^{\beta}W_{\gamma\alpha\beta\mu}\mathscr{D}_{\nu}\mathscr{D}_{\rho}W^{\gamma\nu\rho\mu} + c_2\, \mathscr{D}^{\gamma}\mathscr{D}_{\gamma}W_{\alpha\mu\nu\beta}\mathscr{D}^{\rho}\mathscr{D}_{\rho}W^{\alpha\mu\nu\beta} + \\
& c_3\, \mathscr{D}_{\alpha}\mathscr{D}_{\beta}W_{\nu\gamma\mu\rho}\mathscr{D}^{\alpha}\mathscr{D}^{\beta}W^{\nu\gamma\mu\rho} + c_4\, \mathscr{D}_{\alpha}\mathscr{D}_{\gamma}W^{\gamma}{}_{\mu\nu\beta}\mathscr{D}^{\alpha}\mathscr{D}_{\rho}W^{\rho\mu\nu\beta} + \\
& d_1\, \mathscr{D}^{\rho}\mathscr{D}^{\sigma}W^{\beta}{}_{\rho\sigma}{}^{\nu}W_{\beta}{}^{\alpha\gamma\mu}W_{\nu\alpha\gamma\mu} + d_2\, \mathscr{D}^{\beta}\mathscr{D}^{\gamma}W^{\mu}{}_{\nu\rho}{}^{\sigma}W_{\mu\beta\gamma\alpha}W_{\sigma}{}^{\nu\rho\alpha} + \\
& d_3\, \mathscr{D}_{\sigma}\mathscr{D}^{\sigma}W^{\nu}{}_{\gamma}{}^{\rho}{}_{\mu}W_{\alpha}{}^{\gamma}{}_{\beta}{}^{\mu}W^{\alpha}{}_{\nu}{}^{\beta}{}_{\rho} + d_4\, \mathscr{D}^{\sigma}\mathscr{D}_{\sigma}W^{\beta\nu\mu\rho}W_{\beta\nu}{}^{\alpha\gamma}W_{\alpha\gamma\mu\rho} + \\
& e_1\, W^{\gamma}{}_{\alpha}{}^{\mu}{}_{\beta}\mathscr{D}^{\rho}W_{\rho\gamma}{}^{\nu}{}_{\mu}\mathscr{D}^{\sigma}W_{\sigma}{}^{\alpha}{}_{\nu}{}^{\beta} + e_2\, W_{\alpha}{}^{\gamma}{}_{\beta}{}^{\mu}\mathscr{D}^{\rho}W_{\rho\gamma\sigma\nu}\mathscr{D}^{\alpha}W^{\beta}{}_{\mu}{}^{\sigma\nu} + \\
& e_3\, W_{\alpha\beta\gamma\mu}\mathscr{D}_{\nu}W^{\alpha\beta\rho\sigma}\mathscr{D}^{\nu}W^{\gamma\mu}{}_{\rho\sigma} + e_4\, W_{\alpha}{}^{\gamma}{}_{\beta}{}^{\mu}\mathscr{D}^{\rho}W_{\rho}{}^{\nu\sigma}{}_{\mu}\mathscr{D}^{\alpha}W^{\beta}{}_{\nu\sigma\gamma} + \\
& e_5\, W_{\alpha\beta\gamma\mu}\mathscr{D}^{\rho}W_{\rho\nu}{}^{\alpha\beta}\mathscr{D}^{\sigma}W_{\sigma}{}^{\nu\gamma\mu} + e_6\, W^{\alpha}{}_{\mu\gamma\beta}\mathscr{D}_{\alpha}W^{\gamma}{}_{\nu\rho\sigma}\mathscr{D}^{\beta}W^{\mu\nu\rho\sigma} + \\
& e_7\, W^{\alpha}{}_{\mu}{}^{\gamma}{}_{\beta}\mathscr{D}^{\nu}W^{\rho}{}_{\alpha}{}^{\sigma}{}_{\gamma}\mathscr{D}_{\nu}W^{\mu}{}_{\rho}{}^{\beta}{}_{\sigma}\,.
\end{aligned}
$$

[Classification of Weyl-invariant scalar densities in $8D$] [N.B. and J. Erdmenger, 2004] Besides the seven Weyl invariants of the type $\sqrt{-g}\, W\, W\, W\, W$ given in [Fulling et al.], there are

$$\mathscr{I}_j = \sqrt{-g}\, I_j\,, \quad j = 1, \ldots, 5\,.$$

The first one starts with the quadratic term $I_1 = W^{\mu\nu\rho\sigma}\Box^2 W_{\mu\nu\rho\sigma} + \cdots$ whereas the other four are at least cubic in the Riemann tensor.

$\hookrightarrow$ The coefficients in the 18-dimensional basis given above :

$$\begin{aligned}
\mathbf{I_1} &= (1, 48/25, 2, 42/125, 9/10, 3/5, 96/125, 74/25, 208/5, \\
&\qquad -8, 16/5, -144/25, -104/5, 0, 0, -88/25, 0, 0)\,, \\
\mathbf{I_2} &= (0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 5, 0, 0, 5, 0, 12/5, 0, 0)\,, \\
\mathbf{I_3} &= (0, 0, 0, 0, 0, 0, 0, 1, 0, -20, 0, -48/5, 0, 0, 0, 0, 0, -20)\,, \\
\mathbf{I_4} &= (0, 0, 0, 0, 0, 0, 0, 1, 12, -5/6, -5/24, 4/5, -28/5, -13/8, -12/5, -63/50, -1, 1/2)\,, \\
\mathbf{I_5} &= (0, 0, 0, 0, 0, 0, 0, 1, 8, -2/3, 5/6, 24/25, -16/5, 0, -16/5, -12/25, 0, 0)\,.
\end{aligned}$$

1. Conformal Anomalies
   - Introduction
   - Wess-Zumino consistency conditions
   - Solution of the WZ conditions for the anomaly
   - Results
2. Conformal Invariants
   - Another cohomological group
   - Type-A invariants
   - Type-B invariants
   - Action and field equations for pure Lorentz-Chern-Simons theories

Action and field equations for L_{CS}

- Given a pseudo-Riemannian spacetime $\mathscr{M}_{4p-1}$ of dimension $n = 4p - 1$ with an orientation, consider the functional

$$I[g_{\mu\nu}] = \frac{1}{2p} \int_{\mathscr{M}_{4p-1}} L_{CS}^{4p-1} \; .$$

- The Euler-Lagrange derivative (wrt the metric) of the functional is

$$\mathscr{E}^{\mu\nu} := \frac{\delta I}{\delta g_{\mu\nu}} \equiv \frac{1}{2^{2p-1}} \nabla^{\lambda} \mathscr{A}^{(\mu|\nu)}{}_{\lambda} \; ,$$

where

$$\mathscr{A}^{\mu|\nu}{}_{\lambda} := \varepsilon^{\mu\nu_2\nu_3\dots\nu_{4p-1}} \left[R_{\nu_2\nu_3} \dots R_{\nu_{4p-2}\nu_{4p-1}}\right]^{\nu}{}_{\lambda} \; .$$

and $[R_{\nu_2\nu_3} \dots R_{\nu_{4p-2}\nu_{4p-1}}]^{\nu}{}_{\lambda}$ denotes the $(2p-1)$-fold product of the 2-form valued matrix $[R_{\nu_2\nu_3}]^{\alpha}{}_{\beta} \equiv R^{\alpha}{}_{\beta\nu_2\nu_3}$.

- Weyl and diffeomorphism invariances of the action $I[g_{\mu\nu}]$ get translated into the Noether identites

$$g_{\mu\nu}\mathscr{E}^{\mu\nu} \equiv 0 \,, \qquad \text{and} \qquad \nabla_{\mu}\mathscr{E}^{\mu\nu} \equiv 0 \,.$$

- For the second identity, one must use

$$\varepsilon^{\nu_1 \dots \nu_{4p-1}} \operatorname{Tr}[R_{\nu_1\nu_2} \dots R_{\nu_{4p-3}\nu_{4p-2}} R_{\nu_{4p-1}\nu}] \equiv 0 \,,$$

(Schouten identity and cyclicity of the trace)

- Finally, one has the strict invariance under Weyl transformations :

$$s_W \mathscr{E}^{\mu\nu} = -2\,\omega\,\mathscr{E}^{\mu\nu} \;\Leftrightarrow\; s_W \mathscr{E}^{\mu}{}_{\nu} = 0 \,.$$

that can be seen by expressing

$$\mathscr{A}^{\mu|\nu}{}_{\lambda} = \varepsilon^{\mu\nu_2\nu_3\dots\nu_{4p-1}} \, [W_{\nu_2\nu_3} \dots W_{\nu_{4p-2}\nu_{4p-1}}]^{\nu}{}_{\lambda} \,.$$

Conclusions

- As a consequence of our decomposition, global conformal invariants are *not* in one-to-one correspondence with the conformal anomalies. Indeed, multiplying the Lorentz Chern-Simons densities by the Weyl parameter $\sigma(x)$ does *not* produce any consistent conformal anomaly.

- Our work generalises the analyses devoted to the three-dimensional case $p = 1$ [Deser-Jackiw-Templeton, van Nieuwenhuizen] and completes the classification of Alexakis.