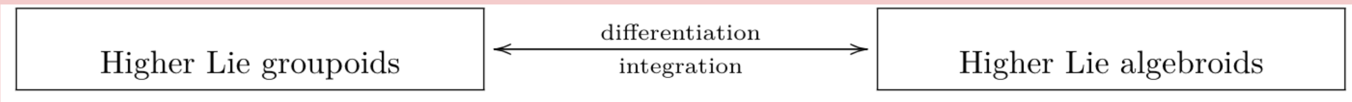
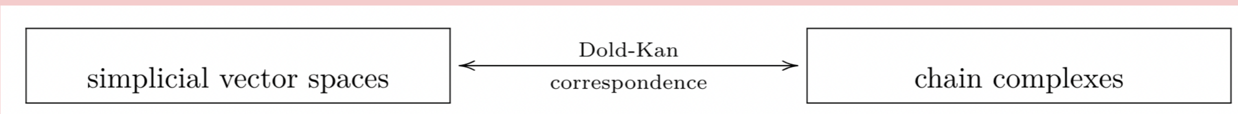
Differentiation of Lie n-groupoids (D. Li, R. Fernandes, L. Ryvkin, A. Wessel, C.Z.)

ESI, August 9th, 2022





Integration and differentiation be viewed as a non-abelian version of Dold-Kan correspondence:

Integration: (non-oid): Henriques, Getzler, Rogers-Wolfson (finite dim), (oid+local): Severa-Siran

Differentiation: A formalization of the notion Lie differentiation in higher geometry has been given by Lurie (deformation context), worked out in diff. Geom. setting by Joost Nuiten, but the result is abstract.

Pavol Ševera, L_\infty algebras as 1-jets of simplicial manifolds (and a bit beyond), idea (inspired by Konsevich) (2006)

Explicit work: Theorem 8.28 of, Du Li's thesis (arXiv:1512.04209), but it contains a mistake in Lemma 8.34 (2015) (local result)

How we found holes and how we fixed them.

- ☐ Lie n-groupoids: form a world of iCFO –simp. Localisation→ simplicial cat (<--Quillen equivalent—> quasi cat)
- ☐ Infinitesimal object (fat point) $D_\bullet$ in this world (Pavol)
- ☐ Differentiation Hom($D_\bullet$, $X_\bullet$)=H , (in this way, you are able to obtain an explicit formula of higher Lie brackets on H).
- ☐ As a graded manifold: H is the tangent complex of $X_\bullet$, more precisely:

Theorem 3.2. *The infinitesimal counterpart of a Lie n-groupoid* $X_\bullet$ *is given by* $\mathrm{Hom}(D_\bullet, X_\bullet)$ *which is a representible graded manifold, represented by*

$$\ker Tp_0^1|_{X_0}[1] \oplus \ker Tp_0^2|_{X_0}[2] \oplus \cdots \oplus \ker Tp_0^n|_{X_0}[n],$$

where $p_0^k : X_k \to X_{k,0}$ *is the horn projection.*

- ☐ A natural d.g. (or N.Q., or Lie n-algebroid) structure on H

A **simplicial manifold** $X_\bullet$: Δ —> Mfd, is a contravariant functor from Δ the category of finite ordinals

Δ: [0]={0}, [1]={0, 1}, [2]={0,1,2},..., [m]={0,1, …, m}, …

with order-preserving maps to the category of manifolds. So we have a tower of manifolds X_l and face and degeneracy

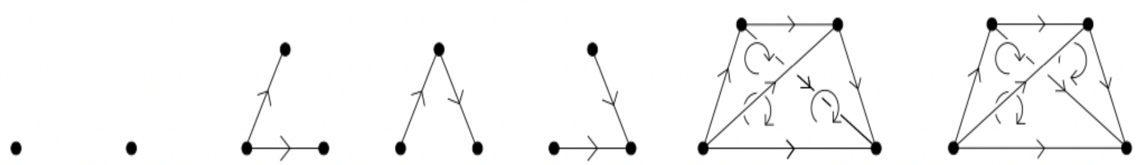
$$d^l_j : X_l \to X_{l-1}, \quad s^l_i : X_{l-1} \to X_l, \quad j = 0, \ldots, l, \; i = 0, \ldots, l-1,$$

$$\begin{aligned} d^{l-1}_i d^l_j &= d^{l-1}_{j-1} d^l_i \quad \text{if } i < j, \\ s^l_i s^{l-1}_j &= s^l_{j+1} s^{l-1}_i \quad \text{if } i \le j, \end{aligned} \qquad d^l_i s^{l-1}_j = \begin{cases} s^{l-2}_{j-1} d^{l-1}_i & \text{if } i < j, \\ \text{id} & \text{if } i = j, j+1, \\ s^{l-2}_j d^{l-1}_{i-1} & \text{if } i > j+1. \end{cases}$$

Definition 2.1. *A* ***Lie*** n***-groupoid*** *[Get09b, Hen08, Zhu09b] is a simplicial manifold* $X_\bullet$ *where the natural projections*

$$p^l_j : X_l = \hom(\Delta[l], X_\bullet) \to \hom(\Lambda[l,j], X_\bullet) =: X_{l,j} \tag{3}$$

are surjective submersions[2] *for all* $1 \le l \ge j \ge 0$ *and diffeomorphisms for all* $0 \le j \le l > n$. *It is further a* ***Lie*** n***-group*** *if* $X_0 = pt$.



Λ[1,1] Λ[1,0] Λ[2,2] Λ[2,1] Λ[2,0] Λ[3,3] Λ[3,2] ..

E.g. $X_{2,1} = X_1 \times_{d_0, X_0, d_1} X_1$.

More details, see, e.g. a minicourse online:

Lecture 1: https://youtu.be/EyOO2nagMMI

Lecture 2: https://youtu.be/qfGrK7Ndzwg

Lecture 3: https://youtu.be/vLNUDt32ILQ

To work out the explicit formula of higher Lie brackets on tangent complex, one way is to view it as Hom(fat point, -).

Fat point in gMfd: $D := spec(R[\epsilon])$, with deg $\epsilon = -1$. T a test g-mfd, (T-point), the internal hom, (a presheaf on gMfd)

Hom(D, M)(T):= hom(DxT, M)=hom(C(M), C(T)⊗ $R[\epsilon]$)

F ∈hom(C(M), C(T)⊗ $R[\epsilon]$),

$F(f) = F_0(f) + F_1(f)\epsilon$, $F_0(fg) = F_0(f)F_0(g)$, $F_1(fg) = F_0(f)F_1(g) +$

$G(f) = F_0(f)$, $G(df) = F_1(f)$, $G \in hom(\Omega^{\bullet}(M), C(T)) = hom(T, T[1]M)$

$F \Leftrightarrow G$ shows that Hom(D, M)=T[1]M is representable. Then the Lie bracket on TM (equivalent to deRham differential d), comes from the infinitesimal of the following action

$Hom(D, D) \times Hom(D, M) \to Hom(D, M)$

Which is simply the composition. Notice $Hom(D, D) = R \times D$ is group object in gMfd.

Now for the world of Lie n-gpds, an infinitesimal object can be (the nerve of) the pair groupoid

$D_\bullet = D \times D \Rightarrow D$

Hom($D_\bullet$, $X_\bullet$) $= H$, we make an approximation by H^k, as a presheaf gMfd→Set, it's defined

$$H^k(T) = \Big\{(f_0, ..., f_k) \mid f_l \in hom(T \times D_l, X_l) \text{ for } l < k, \\ f_k \in hom(T \times d^{-1}_{1,\dots,k}(\star), X_k) \text{ such that conditions } 1. - 4. \text{ hold}\Big\}$$

1. $s_i^X f_l = f_{l+1} s_i^D$ for $l \leq k-2$ and $i \in \{0, ..., l\}$.
2. $d_i^X f_{l+1} = f_l d_i^D$ for $l \leq k-2$ and $i \in \{0, ..., l+1\}$.
3. $s_i^X f_{k-1}|_{d^{-1}_{1,\dots,k-1}(\star)} = f_k s_i^D|_{d^{-1}_{1,\dots,k-1}(\star)}$ for $i \in 0, ..., k-1$.
4. $d_i^X f_k = f_{k-1} d_i^D|_{d^{-1}_{1,\dots,k}(\star)}$ for $i \in 0, ..., k$.

$$d^{-1}_{1,\dots,k}(\star) = \star \times D^{\times k} \xrightarrow{\iota_k} D^{\times k+1} = D_k,$$

For pair groupoid $F(x, y) = f(x) f(y)^{-1}$.

1. $H^0 = X_0$,
2. $H = lim\, H^k$
3. $H^k = \oplus_{i=1}^{k} kerTp^{i}_{0}[i]|_{X_0}$ → key step

3.1 pullback diagram

$$\begin{array}{ccc} H^k(T) & \xrightarrow{\mathcal{F}_k} & \hom(T \times \star \times D^k, X_k) \\ \downarrow & & \downarrow {\scriptstyle (d_1^X)_*,\ldots,(d_k^X)_*,(s_0^D)^*,\ldots,(s_{k-1}^D)^*} \\ H^{k-1}(T) & \xrightarrow{((d_1^D)^*,\ldots,(d_k^D)^*,(s_0^X)_*,\ldots,(s_{k-1}^X)_*)\circ\mathcal{F}_{k-1}} & \hom(T \times \star \times D^k, X_{k-1})^{[k]-0} \times \hom(T \times \star \times D^{k-1}, X_k)^{[k-1]} \end{array}$$

⇔

3.1.1 $(f_0, \ldots, f_k) \in H^k$ is determined by $f_k \in hom(T \times\star\times D^k, X_k)$ uniquely, and curly F_k is the forgetful map, remembering only f_k.

3.1.2 An f_k generates an element of $H^k(T)$, which restricts to $f^{(k-1)} = (f^{(k-1)}{}_0, \ldots, f^{(k-1)}{}_{k-1}) \in H^{k-1}(T)$ iff

$$\textbf{(sIk)} \quad s_i^X f_{k-1}^{(k-1)} = f_k s_i^D|_{d^{-1}_{1,\ldots,k-1}(\star)} \quad \textbf{for } i \in I = \{0, \ldots, k-1\}.$$

$$\textbf{(dJk)} \quad d_i^X f_k = f_{k-1}^{(k-1)} d_i^D|_{d^{-1}_{1,\ldots,k}(\star)} \quad \textbf{for } i \in J = \{1, \ldots, k\}.$$

Du’s Lemma 8.35 says (dJk) ⇐ (sIk) + $kerT_0^{k}[k]|_{X_0}$, Lemma 8.34 says sIk doesn’t give new information.

But one needs to prove fixing $f^{(k-1)}$, $\exists\, f_k$, namely the right vertical map is surjective. (That's where the BUG is in Du's work. Also, in Pavol's work and later in Jurco et al., (slk) was also ignored.)

To show the existence, we can try to solve for f_k by (slk), write down the components of f_k, which is a k-fold tangent vector, we see k unknown and ~k! Linear equations (Arne)

F:2 Apply S:2

$f_2^{\emptyset} + f_2^1 \epsilon_1 + f_2^2 \epsilon_1$

F:2 S: 1 Add Sx

$f_1^{\emptyset} s_1^x + f_1^1 s_1^x \epsilon_1$

F2 S: 1 Estimate Coeff.

$$f_1^{\emptyset} s_1^x = f_2^{\emptyset}$$
$$f_1^1 s_1^x = +f_2^1 + f_2^2$$

$f_3^{\emptyset} s_1^x = f_4^{\emptyset}$
$f_3^1 s_1^x = +f_4^1 + f_4^2$
$f_3^2 s_1^x = +f_4^3$
$f_3^3 s_1^x = +f_4^4$
$f_3^{12} s_1^x = +f_4^{13} + f_4^{23}$
$f_3^{13} s_1^x = +f_4^{14} + f_4^{24}$
$f_3^{23} s_1^x = +f_4^{34}$
$f_3^{123} s_1^x = +f_4^{134} + f_4^{234}$

******** S:2 ***********

F:4 Apply S:4

$f_4^{\emptyset} + f_4^1\epsilon_1 + f_4^2\epsilon_2 + f_4^3\epsilon_2 + f_4^4\epsilon_3 + f_4^{12}\epsilon_1\epsilon_2 + f_4^{13}\epsilon_1\epsilon_2 + f_4^{14}\epsilon_1\epsilon_3$
$+f_4^{24}\epsilon_2\epsilon_3 + f_4^{34}\epsilon_2\epsilon_3 + f_4^{124}\epsilon_1\epsilon_2\epsilon_3 + f_4^{134}\epsilon_1\epsilon_2\epsilon_3$

F:4 S: 2 Add Sx

$f_3^{\emptyset} s_2^x + f_3^1 s_2^x\epsilon_1 + f_3^2 s_2^x\epsilon_2 + f_3^3 s_2^x\epsilon_3 + f_3^{12} s_2^x\epsilon_1\epsilon_2 + f_3^{13} s_2^x\epsilon_1$
$\epsilon_3 + f_3^{23} s_2^x\epsilon_2\epsilon_3 + f_3^{123} s_2^x\epsilon_1\epsilon_2\epsilon_3$

F4 S: 2 Estimate Coeff.

$f_3^{\emptyset} s_2^x = f_4^{\emptyset}$
$f_3^1 s_2^x = +f_4^1$
$f_3^2 s_2^x = +f_4^2 + f_4^3$
$f_3^3 s_2^x = +f_4^4$
$f_3^{12} s_2^x = +f_4^{12} + f_4^{13}$
$f_3^{13} s_2^x = +f_4^{14}$
$f_3^{23} s_2^x = +f_4^{24} + f_4^{34}$
$f_3^{123} s_2^x = +f_4^{124} + f_4^{134}$

F6 S: 3 Estimate Coeff.

$f_5^{\emptyset} s_3^x = f_6^{\emptyset}$
$f_5^{1} s_3^x = +f_6^{1}$
$f_5^{2} s_3^x = +f_6^{2}$
$f_5^{3} s_3^x = +f_6^{3} + f_6^{4}$
$f_5^{4} s_3^x = +f_6^{5}$
$f_5^{5} s_3^x = +f_6^{6}$
$f_5^{12} s_3^x = +f_6^{12}$
$f_5^{13} s_3^x = +f_6^{13} + f_6^{14}$
$f_5^{14} s_3^x = +f_6^{15}$
$f_5^{15} s_3^x = +f_6^{16}$
$f_5^{23} s_3^x = +f_6^{23} + f_6^{24}$
$f_5^{24} s_3^x = +f_6^{25}$
$f_5^{25} s_3^x = +f_6^{26}$
$f_5^{34} s_3^x = +f_6^{35} + f_6^{45}$
$f_5^{35} s_3^x = +f_6^{36} + f_6^{46}$
$f_5^{45} s_3^x = +f_6^{56}$
$f_5^{123} s_3^x = +f_6^{123} + f_6^{124}$
$f_5^{124} s_3^x = +f_6^{125}$
$f_5^{125} s_3^x = +f_6^{126}$
$f_5^{134} s_3^x = +f_6^{135} + f_6^{145}$
$f_5^{135} s_3^x = +f_6^{136} + f_6^{146}$
$f_5^{145} s_3^x = +f_6^{156}$
$f_5^{234} s_3^x = +f_6^{235} + f_6^{245}$
$f_5^{235} s_3^x = +f_6^{236} + f_6^{246}$
$f_5^{245} s_3^x = +f_6^{256}$
$f_5^{345} s_3^x = +f_6^{356} + f_6^{456}$
$f_5^{1234} s_3^x = +f_6^{1235} + f_6^{1245}$
$f_5^{1235} s_3^x = +f_6^{1236} + f_6^{1246}$
$f_5^{1245} s_3^x = +f_6^{1256}$
$f_5^{1345} s_3^x = +f_6^{1356} + f_6^{1456}$
$f_5^{2345} s_3^x = +f_6^{2356} + f_6^{2456}$
$f_5^{12345} s_3^x = +f_6^{12356} + f_6^{12456}$

Rui: brings a trick of S_n symmetry (2019), but it didn't work out for n>1 (but maybe the model of Lie n-groupoid is not the right place for S_n, rather later on multiple vector bundles). Leonid: 2021 April, made an induction proof and solved the existence. While we polish it to an algebraic morphism, we found out another much more geometric proof, thanks much to Theorem 2.10.c of Madeleine-Malte (Multiple vector bundles: cores, splittings and decompositions, 2020), which also has a quite combinatorial proof, but much simpler, maybe multiple vector bundles has much more symmetry (we see S_n now), thus a bigger space=>a good angle to treat such combinatorial difficulty.

3.2 fix:

$$\begin{array}{ccc} H^k & \xrightarrow{\quad \mathcal{F}_k \quad} & T[1]^{(k)}X_k \\ \downarrow & & \Big\downarrow{\scriptstyle (d_J^X)_*,(s_I^D)^*} \\ H^{k-1} & \xrightarrow{((d_J^D)^*,(s_I^X)_*)\circ\mathcal{F}_{k-1}} & Q\odot P \end{array}$$

Diagram in 3.1,

$$\begin{array}{ccc} H^k(T) & \xrightarrow{\quad \mathcal{F}_k \quad} & \hom(T\times\star\times D^k, X_k) \\ \downarrow & & \Big\downarrow{\scriptstyle (d_1^X)_*,\dots,(d_k^X)_*,(s_0^D)^*,\dots,(s_{k-1}^D)^*} \\ H^{k-1}(T) & \xrightarrow{((d_1^D)^*,\dots,(d_k^D)^*,(s_0^X)_*,\dots,(s_{k-1}^X)_*)\circ\mathcal{F}_{k-1}} & \hom(T\times\star\times D^k, X_{k-1})^{[k]-0}\times\hom(T\times\star\times D^{k-1}, X_k)^{[k-1]} \end{array}$$

Thanks to representability, it reduces to

$$\begin{array}{ccc} H^k & \xrightarrow{\mathcal{F}_k} & T[1]^{(k)}X_k \\ \downarrow & & \downarrow {\scriptstyle (d_J^X)_*,(s_I^D)^*)} \\ H^{k-1} & \xrightarrow{((d_J^D)^*,(s_I^X)_*)\circ\mathcal{F}_{k-1}} & (T[1]^{(k)}X_{k-1})^{[k]-0} \times (T[1]^{(k-1)}X_k)^{[k-1]} \end{array}$$

Then it refines to the first diagram with P & Q.

$P \subset (T[1]^{(k-1)}X_k)^{[k-1]}$ is the place the multi-vector bundle $T[1]^{(k)}X_k$ projects to with $(s_I^D)^*$,

$Q \subset (T[1]^{(k)}X_{k-1})^{[k]-0}$ is the place the multi-vector bundle $T[1]^{(k)}X_k$ projects to with $(d_J^X)_*$

Surjectivity to P thanks to MM's theorem, and surjectivity to Q thanks to Kan condition. Then as we knew before, the it is not hard to see the extra information that f_k brings is $ker\, Tp^k_0[k]|_{X_0}$

$$H^k = H^{k-1} \times_{Q \odot P} T[1]^{(k)}X_k = H^{k-1} \times ker\, Tp^k_0[k]|_{X_0}$$

□

Lie n-algebroid structure on H:

0. We wish to re-have the old picture: $A \subset TX_1$ and the Lie bracket on A is induced from that on TX_1.

1. What replace TX_1 will be $T[1]^{(n)}X_n$.

Now we have a look at $T[1]^{(n)}M = Hom(D^n, M)$ is a Lie n-algebroid (or d.g. Manifold, or N.Q. manifold) with its sheaf of functions the multi-deRham sheaf (or differential worms by Pavol) $\Omega^{(n)}$, which locally on U an open of M, is a graded commutative algebra generated by

$f,\ d_i f,\ d_i d_j f,\ \ldots,\ d_1 d_2 \ldots d_n f, \qquad \forall f \in C^{\infty}(U), \quad 1 \le i \le n,\ 1 \le i,\ j \le$

And $|d_I f| = |I|$, satisfying the following relation:

$$\sum_{I' \sqcup I'' = I} \mathsf{sgn}(I', I'', I) d_{I'} f d_{I''} g = d_I(fg),$$

$sgn(2,\ 13,\ 123) = -1,\ sgn(1, 23, 123) = 1.$

On this algebra, we define an R-linear operator d_i by

$$d_i(d_I f d_J g) = d_i(d_I f) d_J g + (-1)^{|I|} \mathsf{sgn}(i, J, iJ) d_I f d_{iJ} g, \quad d_i(d_I f) = \mathsf{sgn}(i, I, iI) d_{iI} f.$$

$\Rightarrow\ d_i$ is a graded derivation on $\Omega^{(n)}(U)$, $d_i d_j = -\ d_j d_i$, $d_i^2 = 0$. Thus

$(\Omega^{(n)}(U),\ d := \sum_{i=1}^{n} d_i)$ is a d.g. Algebra. This gives $T[1]^{(n)}M$ a d.g. Manifold structure. Can be viewed as an infinitesimal multi-Artin-Mazur codiagnoal construction.

2. Δ: $R \times D = Hom(D,\ D) \rightarrow Hom(D_{\bullet},\ D_{\bullet})$, c: $Hom(D_{\bullet},\ D_{\bullet}) \times Hom(D_{\bullet},\ X_{\bullet}) \rightarrow Hom($

, gives an infinitesimal D action on H, which in turn gives a homological vector field Q with degree 1, thus H is a Lie n-algebroid.

3. $F_n : H \to T[1]^{(n)}X_n$ is a d.g. Manifold embedding. With this embedding, one can write down the higher Lie brackets on H explicitly once F_n is explicit and the higher brackets on $T[1]^{(n)}X_n$ are explicit. But we think these higher brackets will depend on the choice of connections, except for n=1.

Example with explicit formula (when n=2)

There is a Lie 2-algebroid structure on

$T[1]^{(2)}M \simeq (T[1]M \oplus T[1]M) \oplus T[1]M$ once we fix a torsion free TM

For $T[1]T[1]M = T[1]^{(2)}M$, *we have the following short exact sequence of vector bundles*

$$0 \to T[2]M \to T[1]^{(2)}M \to T[1]M \oplus T[1]M \to 0, \tag{35}$$

connection ∇ on TM, with ρ, l_1, l_2, l_3

$$\rho(X_1, X_2) = X_1 + X_2, \quad l_1(Y) = (Y, -Y), \quad l_2((X_1, X_2), Y) = \nabla_{X_1}Y + \nabla_{X_2}Y,$$
$$l_2((X_1, X_2), (Y_1, Y_2)) = (\nabla_{X_2}Y_1 - \nabla_{Y_2}X_1 + [X_1, Y_1], \nabla_{X_1}Y_2 - \nabla_{Y_1}X_2 + [X_2, Y_2]),$$
$$l_3((X_1, X_2), (Y_1, Y_2), (Z_1, Z_2)) = -(R(X_2, Y_2)Z_1 + c.p.) + R(X_1, Y_1)Z_2 + c.p.$$

The formulas are highly inspired by Madeleine's article Lie 2-algebroids and matched pairs of 2-representations. We think it is isomorphic to the d.g. Manifold structure in general on $T[1]^{(n)}M$, when $n = 2$. By now we have local isomorphism.

The embedding $F: H \to T[1]^{(2)}X_2$ can be construct rather explicitly when n=2:

$$\mathcal{F}(v_1, v_2) = ((d_1^D)^*(s_0^X)_*v_1, (d_2^D)^*(s_1^X)_*v_1, v_2) \in T[1]^{(2)}X_2 \cong (T[1]X_1 \oplus T[1]X_1) \oplus T[2]X_2.$$

Thus, one should be able to read a Lie 2-algebroid structure explicitly from a Lie 2-groupoid. For higher case, we will need much more computational power, and Leonid is currently developing some computer assisting tools. It can certainly be helpful for this problem. Moreover, there is also a work in progress of Madeleine and Malte on n-fold multi-vector bundles with Sn symmetry and n-graded manifolds, which is probably the first step to take for the explicit formula.

Other examples, e.g. Lie group(oid), strict Lie 2-groups, see Du’s thesis.