Geometry of quantum dynamics in infinite dimensions

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Implicit dynamics

Definition

An implicit first-order ordinary differential equation (implicit dynamics) on a manifold N is a submanifold (subset) $\mathcal{D}$ of the tangent bundle $\mathsf{T}N$.

We say that a smooth curve $\gamma : \mathbb{R} \to N$ is a solution of $\mathcal{D}$) if its tangent prolongation $\mathsf{t}\gamma = (\gamma, \dot{\gamma}) : \mathbb{R} \to \mathsf{T}N$ takes values in $\mathcal{D}$.

A curve $\widetilde{\gamma}$ in $\mathsf{T}N$ we call admissible, if it is the tangent prolongation of its projection $\widetilde{\gamma}_N$ on N, $\widetilde{\gamma} = \mathsf{t}\widetilde{\gamma}_N$.
Solutions of an implicit dynamics $\mathcal{D} \subset \mathsf{T}N$ are projections $\widetilde{\gamma}_N$ of admissible curves $\widetilde{\gamma}$ lying in $\mathcal{D}$.

Example

An explicit differential equation is the range $\mathcal{D} = X(N) \subset \mathsf{T}N$ of a vector field $X : N \to \mathsf{T}N$ on N. Solutions in this case are called trajectories of X.

Integrability conditions

Indeed, if $X = f^i(x)\partial_{x^i}$, then

$$X(N) = \left\{ \left(x^i, \dot{x}^j = f^j(x)\right) : x \in N \right\} \subset \mathsf{T}N$$

and $\widetilde{\gamma}(t) \in X(N)$ means $\dot{\gamma}^j(t) = f^j\left(\gamma(t)\right)$.

Note, however, that different implicit differential equations may have the same set of solutions.

First of all, if $\mathcal{D}$ is supported on a subset N_0, $\tau_N(\mathcal{D}) = N_0$, only vectors from $\mathcal{D} \cap \mathsf{T}N_0$ do matter if solutions are concerned. Hence, the first integrability extract

$$\mathcal{D}^1 = \mathcal{D} \cap \mathsf{T}N_0$$

has the same solutions as $\mathcal{D}$, and $\mathcal{D} \subset \mathsf{T}N_0$ is the first integrability condition. Explicit differential equations are automatically integrable.

Of course, replacing $\mathcal{D}$ with $\mathcal{D}^1$, then $\mathcal{D}^1$ by $\mathcal{D}^2$, etc., may turn out to be an infinite procedure, but this will not happen in examples considered during this talk.

Lagrangian submanifolds

Any 2-form ω on N induces a VB-morphism

$$\omega^\flat : \mathsf{T}N \to \mathsf{T}^*N, \quad \omega^\flat(X) = i_X\omega.$$

ω is called symplectic if it is closed and $\omega^\flat$ is an isomorphism.
T^*Q possess a canonical symplectic form (Darboux coordinates)

$$\omega_Q = \mathrm{d}q^k \wedge \mathrm{d}p_k.$$

A Lagrangian submanifold $\mathcal{L}$ of a symplectic manifold (N,ω) of dimension $2n$ is a submanifold of dimension n on which the symplectic form vanishes, $\omega\big|_{\mathcal{L}} = 0$. For $(N,\omega) = (\mathsf{T}^*Q, \omega_Q)$:

Proposition

The range $\mathcal{L} = \eta(Q)$ of a 1-form $\eta : Q \to \mathsf{T}^*Q$ is a Lagrangian submanifold in T^*Q if and only if η is a closed form.

For $\eta = \mathrm{d}f$, $\mathcal{L} = \{(q^k, p_j = \partial f/\partial q^j)\}$,

$$\omega_Q\big|_{\mathcal{L}} = \mathrm{d}q^k \wedge \mathrm{d}\Big(\frac{\partial f}{\partial q^k}\Big) = \Big(\frac{\partial^2 f}{\partial q^k \partial q^j}\Big)\mathrm{d}q^k \wedge \mathrm{d}q^j = 0.$$

Tangent lifts and Hamiltonian vector fields

Hamiltonian Mechanics: phase space N is a symplectic manifold (N,ω) and the dynamics $\mathcal{D}$ is determined by a Hamiltonian function H on N,

$$\mathcal{D} = X_{\mathsf{H}}(N) \subset \mathsf{T}N, \quad \omega^\flat(X_{\mathsf{H}}) = \mathrm{d}\mathsf{H}.$$

(Locally) Hamiltonian vector fields correspond, *via* $\omega^\flat$ to (closed) exact one-forms. Any symplectic form ω on N lifts canonically to a symplectic form $\mathrm{d}_{\mathsf{T}}\omega$ on $\mathsf{T}N$.
In Darboux coordinates, the tangent lift takes the form

$$\mathrm{d}_{\mathsf{T}}\big(\mathrm{d}q^k \wedge \mathrm{d}p_k\big) = \mathrm{d}\dot{q}^k \wedge \mathrm{d}p_k + \mathrm{d}q^k \wedge \mathrm{d}\dot{p}_k.$$

Proposition

A vector field $X\colon N \to \mathsf{T}N$ is locally Hamiltonian if and only if its image $X(N)$ is a Lagrangian submanifold of $(\mathsf{T}N, \mathrm{d}_{\mathsf{T}}\omega)$.

Generalized Hamiltonian systems on (N,ω) can be defined as Lagrangian submanifolds of $(\mathsf{T}N, \mathrm{d}_{\mathsf{T}}\omega)$.

Relativistic particle

The dynamics of a relativistic particle is an example of such a system.

Example

The (implicit) phase-space dynamics of a free relativistic massless particle in a space-time Q is described by equations

$$\begin{aligned} 0 &= g^{\kappa\lambda} p_\kappa p_\lambda \\ \dot{q}^\kappa &= v \cdot g^{\kappa\lambda} p_\lambda \\ \dot{p}_\kappa &= -\frac{v}{2} \cdot g_\kappa^{\mu\nu} p_\mu p_\nu \,, \end{aligned}$$

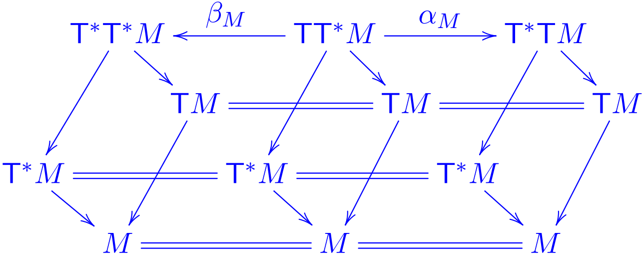
where $g_{\kappa\lambda}$ is the Minkowski metric and $v > 0$. The equations describe a Lagrangian submanifold $\mathcal{D}$ in TT^*Q which is not the range of any vector field on T^*Q due to the constraint $g^{\kappa\lambda} p_\kappa p_\lambda = 0$. However, following Tulczyjew, it is possible to obtain the above dynamics from a constrained Lagrangian.

$$\beta_M = (\omega_M)^\flat : \mathsf{T}\mathsf{T}^* M \to \mathsf{T}^*\mathsf{T}^* M\,,$$

composed with $\mathcal{R}_{\mathsf{T}M} : \mathsf{T}^*\mathsf{T}^* M \to \mathsf{T}^*\mathsf{T} M$, yields an isomorphism

$$\alpha_M : \mathsf{T}\mathsf{T}^* M \to \mathsf{T}^*\mathsf{T} M\,.$$

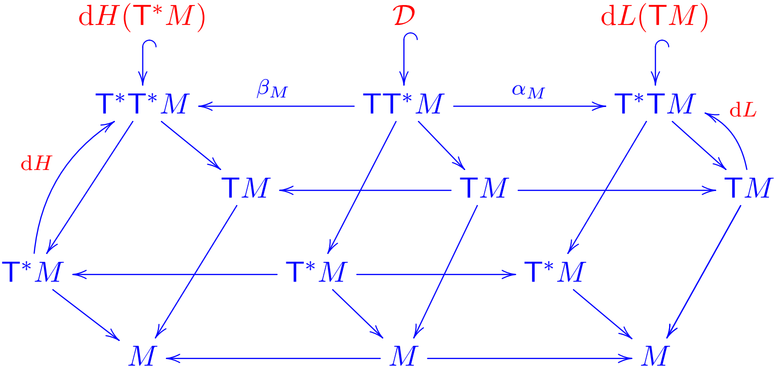
In the adapted coordinates, $\alpha_M(x, p, \dot{x}, \dot{p}) = (x, \dot{x}, \dot{p}, p)$. Hence, we have the commutative diagram of double vector bundle isomorphisms being simultaneously symplectomorphisms:



. (1)

Geometric Mechanics on one picture

Starting with a Lagrangian $L : \mathsf{T}M \to \mathbb{R}$ or a Hamiltonian $H : \mathsf{T}^*M \to \mathbb{R}$, we get the diagram for the phase dynamics $\mathcal{D}$.



- The right-hand side is Lagrangian, $\mathcal{D} = \alpha_M^{-1}\big(\mathrm{d}L(\mathsf{T}M)\big)$,
- the left-hand side is Hamiltonian, $\mathcal{D} = \beta_M^{-1}\big(\mathrm{d}H(\mathsf{T}^*M)\big)$.
- In both cases, $\mathcal{D}$ is a Lagrangian submanifold in $\mathsf{T}\mathsf{T}^*M$.

Both sides give the same $\mathcal{D}$ only for regular Lagrangians.

Classical mechanical systems

Consider the standard mechanical system with the Lagrangian

$$L(x,\dot{x}) = \frac{m}{2}\sum_i (\dot{x}^i)^2 - W(x).$$

It generates the Lagrangian submanifold $\mathrm{d}L(\mathsf{T}M)$ in $\mathsf{T}^*\mathsf{T}M$ which in adapted coordinates reads

$$\left\{\left(x^i, \dot{x}^j, -\partial W/\partial_{x^k}, m\dot{x}^l\right)\right\},$$

and induces the implicit dynamics $\mathcal{D} = \alpha_M^{-1}\left(\mathrm{d}L(\mathsf{T}M)\right)$,

$$\mathcal{D} = \left\{\left(x^i, m\dot{x}^j, \dot{x}^k, -\partial W/\partial_{x^l}\right)\right\} \subset \mathsf{T}\mathsf{T}^*M.$$

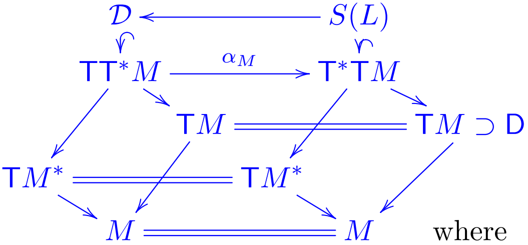
The corresponding implicit differential equation is

$$m\ddot{x}^i + \frac{\partial W}{\partial x^i}(x) = 0.$$

The dynamics $\mathcal{D}$ can also be obtained from the Hamiltonian

$$H(x,p) = \frac{1}{2m}\sum_i p_i^2 + W(x).$$

Starting with a constrained Lagrangian $L : \mathsf{T}M \supset \mathsf{D} \to \mathbb{R}$, we get



where

$S(L) = \left\{\theta_e \in \mathsf{T}^*_e\mathsf{T}M : e \in \mathsf{D} \text{ and } \langle \theta_e, v_e \rangle = \mathrm{d}L(v_e) \text{ for } v_e \in \mathsf{T}_e\mathsf{D}\right\}.$

The constrained phase dynamics is just $\mathcal{D} = \alpha_M^{-1}\big(S(L)\big)$.
Analogously for a constrained Hamiltonian $\mathsf{H} : \mathsf{T}^*M \supset \mathsf{D} \to \mathbb{R}$.

The implicit dynamics $\mathcal{D}$ of a free relativistic particle is of this form for the trivial Hamiltonian $\mathsf{H} = 0$ defined on the constraint $\mathsf{D} \subset \mathsf{T}^*Q$ being the 'future part' of the cone $g^{\kappa\lambda}p_\kappa p_\lambda = 0$.

Hilbert spaces - notation

- $\mathcal{H}$ - a separable Hilbert space equipped with a Hermitian inner product $\langle\cdot,\cdot\rangle$ (anti-linear in the second argument) and the corresponding norm
$$\|x\| := \sqrt{\langle x, x\rangle}\,.$$
- $\mathrm{gl}(\mathcal{H})$ - the C^*-algebra of all continuous complex linear maps $A:\mathcal{H}\to\mathcal{H}$, with the operator norm
$$\|A\| = \sup\{\|Ax\|\ : x\in\mathcal{H}\,,\quad \|x\|\le 1\}$$
and the $*$-operation being the Hermitian conjugation, $A\mapsto A^\dagger$, where $\langle A^\dagger x, y\rangle = \langle x, Ay\rangle$; operators satisfying $A^\dagger = A$ we call Hermitian; those with $A^\dagger = -A$ anti-Hermitian.
The operator $A^\dagger$ makes sense even for a densely defined operator $A:\mathcal{H}\supset \mathsf{D}\to\mathcal{H}$. The domain of $A^\dagger$ is
$$\mathsf{D}^\dagger = \left\{x\in\mathcal{H}\,\middle|\, y\mapsto\langle x, Ay\rangle\in\mathbb{C}\text{ is continuous}\right\}$$
and defines $A^\dagger$ on this domain. If $\mathsf{D}=\mathsf{D}^\dagger$ and $A = A^\dagger$ ($A=-A^\dagger$) on D, we call A selfadjoint (anti-selfadjoint).

Unitary group

- $\mathrm{u}(\mathcal{H})$ denotes the (real) Banach subspace of $\mathrm{gl}(\mathcal{H})$ of anti-Hermitian operators;
- $\mathrm{GL}(\mathcal{H})$ - the group of invertible elements in $\mathrm{gl}(\mathcal{H})$;
- $\mathrm{U}(\mathcal{H})$ - the unitary group, the subgroup in $\mathrm{GL}(\mathcal{H})$ of elements $UU^\dagger = \mathrm{Id}$;

The group $\mathrm{GL}(\mathcal{H})$ is a (complex) Banach-Lie group modelled on $\mathrm{gl}(\mathcal{H})$, and $U(\mathcal{H})$ is its (real) Lie subgroup corresponding to the (real) Lie subalgebra $\mathrm{u}(\mathcal{H})$ of the Banach-Lie algebra $\mathrm{gl}(\mathcal{H})$ with the commutator bracket.

The unitary group carries also the strong topology,

$$\lim_{k\to+\infty} U_k = U \quad \Leftrightarrow \quad \forall\, x \in \mathcal{H}\, [\, \lim_{k\to+\infty} U_k(x) = U(x)\,],$$

in which it is also a topological group.

Strong one-parameter subgroups are generated by (generally unbounded) anti-selfadjoint operators, i.e., operators $-iA$, where A is selfadjoint (Stone theorem),

$$\mathbb{R} \ni t \to e^{-itA} \in \mathrm{U}(\mathcal{H}).$$

Hilbert space is a Kähler manifold

The Hilbert space $\mathcal{H}$ is an infinite-dimensional Kähler manifold with the standard complex structure J, and the Riemannian and symplectic structures $g_{\mathcal{H}}$ and $\omega_{\mathcal{H}}$,

$$J(x) = i \cdot x, \qquad g_{\mathcal{H}}(x,y) + i \cdot \omega_{\mathcal{H}}(x,y) = \langle x, y \rangle .$$

The group $\mathrm{U}(\mathcal{H})$ acts by isometric symplectomorphisms.

We will view the (real) symplectic manifold $(\mathcal{H}, \omega_{\mathcal{H}})$ as $\mathsf{T}^*H = H \oplus H^*$, where H is a real part of $\mathcal{H}$, $\mathcal{H} = H \oplus_{\mathbb{R}} iH$.

We can span the real vector space H by an orthonormal basis (e_k) in $\mathcal{H}$ and real coefficients (coordinates) $q = (q^k)$, making H into a real Hilbert space with the scalar product (metric)

$$g_H = \sum_{k=1}^{\infty} \mathrm{d}q^k \otimes \mathrm{d}q^k.$$

On $\mathcal{H}$ we have the coordinates (q, p), so that $x \in \mathcal{H}$ reads

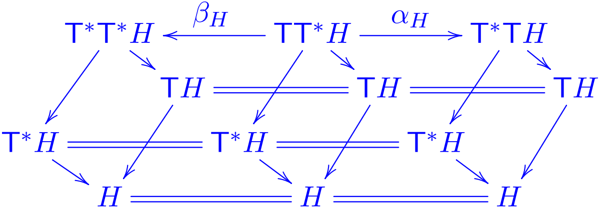
$$x = \sum_{k=1}^{\infty} (q^k + i \cdot p_k) e_k, \quad \sum_{k=1}^{\infty} \left(|q^k|^2 + |p_k|^2\right) < +\infty.$$

Quantum Tulczyjew triple

In these coordinates,

$$g_{\mathcal{H}} = \sum_{k=1}^{\infty} \left(\mathrm{d}p_k \otimes \mathrm{d}p_k + \mathrm{d}q^k \otimes \mathrm{d}q^k \right), \quad \omega_{\mathcal{H}} = \sum_{k=1}^{\infty} \mathrm{d}q^k \wedge \mathrm{d}p_k \,.$$

Having chosen H as the configuration manifold, we can write the corresponding Tulczyjew triple:



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Isomorphisms α_H and β_H identify coordinates:

$$(q, p, \dot{q}, \dot{p}) \quad \text{on} \quad (H \oplus H^*) \oplus (H \oplus H^*) = \mathsf{T}\mathsf{T}^* H \,,$$

$$(q, \dot{q}, \dot{p}, p) \quad \text{on} \quad (H \oplus H) \oplus (H^* \oplus H^*) = \mathsf{T}^*\mathsf{T} H \,,$$

$$(q, p, \dot{p}, -\dot{q}) \quad \text{on} \quad (H \oplus H^*) \oplus (H^* \oplus H) = \mathsf{T}^*\mathsf{T}^* H \,.$$

Canonical symplectic forms

Note that H and iH are real Hilbert spaces with the real scalar products induced from $g_{\mathcal{H}}$, and that one can view iH *via* $\omega_{\mathcal{H}}$ as the dual space H^* of H. We have a canonical isomorphism $H \simeq H^* = iH$, associated with the metric on H, and the identification

$$\mathcal{H} = H \oplus iH = H \oplus H^* = \mathsf{T}^*H,$$

so $\omega_{\mathcal{H}}$ coincides with ω_{T^*H}. Identifying TT^*H with $\mathcal{H} \oplus_{\mathbb{C}} \mathcal{H}$ *via*

$$(q, p, \dot{q}, \dot{p}) \mapsto (x, \dot{x}) = (q + i \cdot p, \dot{q} + i \cdot \dot{p}),$$

we can write the canonical symplectic form ω_0 on TT^*H as

$$\omega_0 = \omega_{\mathsf{TT}^*H} = \mathrm{d}\dot{q}^k \wedge \mathrm{d}p_k + \mathrm{d}q^k \wedge \mathrm{d}\dot{p}_k.$$

It is the imaginary part of the lifted pseudo-Hermitian form

$$\langle (x, \dot{x}), (y, \dot{y}) \rangle_0 = \langle \dot{x}, y \rangle + \langle x, \dot{y} \rangle.$$

The canonical symplectic forms $\omega_{\mathsf{T}^*\mathsf{T}H}$ and $\omega_{\mathsf{T}^*\mathsf{T}^*H}$ read as above if we use the indicated identification of coordinates, corresponding to canonical Tulczyjew isomorphisms

$$\mathsf{T}^*\mathsf{T}^*H \simeq \mathsf{TT}^*H \simeq \mathsf{T}^*\mathsf{T}H.$$

Quantum dynamics

In the Tulczyjew approach, an implicit dynamics is a Lagrangian submanifold in TT^*M.

In the quantum case, we will consider only complex linear Lagrangian submanifolds, i.e., those complex linear subspaces $V \subset \mathsf{TT}^*H = \mathcal{H} \oplus_{\mathbb{C}} \mathcal{H}$ which are maximally isotropic for the symplectic form ω_0,

$$\omega_0(v, v') = 0 \quad \text{for all} \quad v, v' \in V.$$

From the maximality condition, V must be closed.

In particular, if $A : \mathcal{H} \supset \mathsf{D} \to \mathcal{H}$ is a complex linear operator in the domain D, its graph,

$$\mathfrak{G}(A) = \{(x, Ax), x \in \mathsf{D}\},$$

is a linear relation in $\mathcal{H} \oplus_{\mathbb{C}} \mathcal{H} \simeq \mathcal{H} \times \mathcal{H}$. The operator A is called closed if $\mathfrak{G}(A)$ is closed in $\mathcal{H} \times \mathcal{H}$.

Complex linear Lagrangian submanifolds in $\mathcal{H} \oplus_{\mathbb{C}} \mathcal{H}$ we will call anti-selfadjoint relations.
Of course, there is an analogous selfadjoint picture.

Anti-selfadjoint operators

Theorem

The graph $\mathfrak{G}(A)$ of a complex linear operator $A : \mathcal{H} \supset \mathsf{D} \to \mathcal{H}$ is an isotropic submanifold for the symplectic structure ω_0 if and only if the operator is anti-symmetric:

$$\langle Ax, y\rangle + \langle x, Ay\rangle = 0 \quad \textit{for} \quad x, y \in \mathsf{D}\,.$$

Moreover, $\mathfrak{G}(A)$ is a Lagrangian submanifold if and only if D is dense in $\mathcal{H}$ and A is anti-selfadjoint, $A^\dagger = -A$.

In our framework, anti-selfadjoint relations play the rôle of implicit quantum dynamics.

For a complex linear relation $V \subset \mathcal{H} \oplus_{\mathbb{C}} \mathcal{H}$, its domain and kernel are

$$\mathsf{D}(V) = \{x \in \mathcal{H} \,:\, (x, y) \in V \text{ for some } y\}\ ,$$

$$\ker(V) = \{x \in \mathcal{H} \,:\, (x, 0) \in V\}\ .$$

The inverse relation is defined as

$$V^{-1} = \{(y, x) \in \mathcal{H} \oplus_{\mathbb{C}} \mathcal{H} \,:\, (x, y) \in V\}\,.$$

A characterization of anti-selfadjoint relations

Theorem

If a complex linear relation V is anti-selfadjoint, then

$$\mathsf{D}(V)^{\perp} = \ker(V^{-1})$$

and

$$V_A = \left\{(x, Ax + v) \,|\, x \in \mathsf{D}\,, \quad v \in \mathsf{D}^{\perp}\right\},$$

where $A : \mathcal{H} \supset \mathsf{D} \to \mathcal{H}$ is an anti-selfadjoint operator, densely defined in the Hilbert space $\overline{\mathsf{D}} \subset \mathcal{H}$.
In particular, V is a graph, $V = \mathfrak{G}(A)$, if and only if V is densely defined. In this case, A is anti-selfadjoint and A is a bounded operator if and only if $\mathsf{D}(V) = \mathcal{H}$.

Example

Consider $\mathcal{H} = \mathsf{L}^2(\mathbb{R})$ with H being the (real) subspace of real functions. The natural domain of the momentum operator $A(f) = f'$ is $\mathsf{D} = W^{1,2}(\mathbb{R})$ and $\mathfrak{G}(A) \subset \mathcal{H} \times \mathcal{H}$ is an anti-selfadjoint relation, due to $(gf' + g'f) = (gf)'$. Usually, the quantum momentum operator is understood as $\widehat{\mathbf{p}} = -i\hbar\partial_t$.

Quadratic Lagrangians

Let g_0 be the canonical scalar product on $\mathsf{T}H = H \oplus H$,

$$g_0(Q, Q') = g_H(q, q') + g_H(\dot{q}, \dot{q}'),$$

where we write Q for $(q, \dot{q})$. Since we work only with linear relations, we consider only Lagrangians $L : \mathsf{D}_0 \to \mathbb{R}$, quadratic in $Q = (q, \dot{q})$ in domains of differentiability D_0 being real subspaces of $\mathsf{T}H = H \oplus H$.

More precisely, there is a (real) linear operator $B : \mathsf{D}_0 \to \overline{\mathsf{D}_0}$ such that B is g_0-symmetric,

$$g_0(Q', BQ) = g_0(BQ', Q) \quad \text{for all} \quad Q, Q' \in \mathsf{D}_0,$$

and

$$L(Q) = \frac{1}{2} g_0(Q, BQ), \quad Q \in \mathsf{D}_0.$$

Hence, $\mathrm{d}L(Q)(Q') = g_0(BQ, Q')$ for $Q \in \mathsf{D}_0$, $Q' \in \overline{\mathsf{D}_0}$.

We regard D_0 as a constraint and generate the Lagrangian submanifold $S(L)$ in $\mathsf{T}^*\mathsf{T}H$.

Quantum dynamics in the Tulczyjew picture

Actually, $S(L)$ is the closure of the linear subspace

$$S(L)^0 = \left\{(Q,P) \in \mathsf{T}^*\mathsf{T}H \;:\; Q \in \mathsf{D}_0\,,\ \mathrm{pr}(P) = \mathrm{d}L(Q)\right\},$$

where $\mathrm{pr} : H^* \oplus H^* \to \overline{\mathsf{D}_0}^*$ is the canonical projection, dual to the embedding $\overline{\mathsf{D}_0} \hookrightarrow H \oplus H$.

Now, *via* the symplectomorphism α_H, we view $S(L)$ as a Lagrangian submanifold $V(L)$ in $(\mathsf{T}\mathsf{T}^*H, \omega_0)$. If we assume that $V(L)$ is a complex relation (J-invariant), it is anti-selfadjoint and represents the implicit quantum dynamics.

The first integrability extract,

$$V(L)^1 = V(L) \bigcap \left(\mathcal{D} \oplus \overline{\mathcal{D}}\right)\,,$$

is now the graph of an anti-selfadjoint operator $-iA$ defined on the domain $\mathcal{D}$ which is dense in the closed subspace $\overline{\mathcal{D}}$ of $\mathcal{H}$ representing the Hamiltonian constraint, $V(L)^1 = \mathfrak{G}(-iA)$. The operator A is the Schrödinger operator in the Schrödinger picture.

Example

Consider the real Hilbert space $H = \mathsf{L}^2_{\mathbb{R}}(\mathbb{R})$ and a quadratic Lagrangian on $\mathsf{T}H$,

$$L(x,\dot{x}) = \frac{1}{2}\int_{\mathbb{R}} \Big(\dot{x}^2(t)/t - tx^2(t)\Big)\mathrm{d}t.$$

The Lagrangian is densely defined and its domain of differentiability is

$$\mathsf{D}_0 = \Big\{(x,\dot{x}) \in \mathsf{T}H \,:\, \int_{\mathbb{R}} \big(\dot{x}(t)/t\big)^2 \mathrm{d}t < \infty\,,\ \int_{\mathbb{R}} \big(x(t)t\big)^2 \mathrm{d}t < \infty\Big\}\,.$$

The generated Lagrangian submanifold in $\mathsf{T}^*\mathsf{T}H$ is

$$S(L) = \Big\{\Big(x,\dot{x},-tx,\dot{x}/t\Big) \,:\, (x,\dot{x}) \in \mathsf{D}_0\Big\},$$

so that the Lagrangian submanifold $V(L) = \alpha_H^{-1}\big(S(L)\big)$ reads

$$V(L) = \big\{\,(x,\dot{x}/t,\dot{x},-tx) \,:\, (x,\dot{x}) \in \mathsf{D}_0\big\} \subset \mathsf{T}\mathsf{T}^*H \simeq \mathcal{H} \oplus \mathcal{H}\,.$$

The Legendre map is ‘regular’,

$$\mathsf{T}H \ni \big(x,\dot{x}\big) \mapsto \big(x, p = \dot{x}/t\big) \in \mathsf{T}^*H.$$

The quantum position operator

It is easy to see that $V(L)$ is the graph of the complex linear operator $-iA$, where

$$(Ay)(t) = ty(t)\,, \quad y(t) = x(t) + ip(t)$$

is selfadjoint in the domain

$$\mathsf{D} = \left\{ y \in \mathcal{H} = \mathsf{L}^2(\mathbb{R}) \,:\, \int_{\mathbb{R}} |ty(t)|^2 \mathrm{d}t < \infty \right\}.$$

There is also a Hamiltonian description of $-iA$ with the Hamiltonian $\mathsf{H} : \mathcal{H} \to \mathbb{R}$,

$$\mathsf{H}(y) = \frac{1}{2} \int_{\mathbb{R}} t|y(t)|^2 \mathrm{d}t.$$

Traditionally, $\widehat{\mathbf{q}}(y) = ty$ is called the quantum position operator and $-i\widehat{\mathbf{q}}$ induces s 1-parametr group of unitary transformations of $\mathcal{H}$ (quantum dynamics),

$$e^{-i\widehat{\mathbf{q}}}(y) = e^{-it} y.$$

Final Message: Quantum position and momentum operators have classical Lagrangian and Hamiltonian description.

THANK YOU FOR YOUR ATTENTION!

(Sokolica- Polish Carpathians)