Duality Hierarchies

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- with Bergshoeff et. al. 0901.2054
- with Samtleben, 1805.03220, 1903.02821
- with Bonezzi, 1904.11036, 1910.10399
- closely related work by Strobl & friends

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Overview

Central Question:

Can physics be encoded in first order duality relations?

- Part I: Duality Relations & Tensor Hierarchy
- Part II: Duality Hierarchy & Differential Graded Lie Algebra
- Part III: Embedding Tensor of ExFT

Part I: Duality Relations & Tensor Hierarchy

First Order Duality Relations

Free scalar in $D = 3$:

$$\Box\phi = \partial_\mu \partial^\mu \phi = 0$$

integrability condition of duality relation defining dual vector

$$\partial^\mu \phi = \frac{1}{2}\varepsilon^{\mu\nu\rho} F_{\nu\rho} = \varepsilon^{\mu\nu\rho}\partial_\nu A_\rho$$

Linearized Gravity ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$):

$$G^{(1)}_{\mu\nu}(h) = \partial^\rho Y_{\rho\mu,\nu} = 0$$

where

$$Y_{\mu\nu,\rho} = -\partial_{[\mu}\bar{h}_{\nu]\rho} + \eta_{\rho[\mu}\partial^\lambda \bar{h}_{\nu]\lambda}\,, \qquad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}$$

Duality relation for $(D-3, 1)$ dual graviton:

$$Y_{\mu\nu,\rho} = \varepsilon_{\mu\nu}{}^{\lambda_1 \dots \lambda_{D-2}}\partial_{\lambda_1} D_{\lambda_2 \dots \lambda_{D-2},\rho}$$

Obstacles to Duality Relations

In presence of scalar potential V,

$$\Box\phi = \frac{\partial V}{\partial \phi}$$

we can no longer “pull out a derivative” $\Rightarrow$ no first order duality relation

Similarly, for gravity with sources,

$$G^{(1)}_{\mu\nu}(h) = \kappa(T_{\mu\nu} - G^{(2)}_{\mu\nu}(h) + \cdots)$$

one cannot “pull out a derivative” on the RHS, because otherwise “improvement term” and hence trivial

[Bergshoeff, de Roo, Kerstan, Kleinschmidt, Riccioni (2008)]

Hierarchy of Duality Relations

In gauged supergravity and ExFT we encounter field strengths

$$\mathcal{F}_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} + \cdots + \mathfrak{D}B_{\mu\nu}$$

with Bianchi identity (“Tensor Hierarchy”)

$$D_{[\mu}\mathcal{F}_{\nu\rho]} = \mathfrak{D}\mathcal{H}_{\mu\nu\rho}\,, \qquad \mathcal{H}_{\mu\nu\rho} = \partial_{[\mu}B_{\nu\rho]} + \cdots$$

→ integrability condition of *two duality relations*

$$D^\mu\phi = \varepsilon^{\mu\nu\rho}\mathcal{F}_{\nu\rho}$$
$$\varepsilon^{\mu\nu\rho}\mathcal{H}_{\mu\nu\rho} = f(\phi)$$

where $\mathfrak{D}f(\phi) = V'(\phi)$ then

$$D_\mu D^\mu\phi = \varepsilon^{\mu\nu\rho}D_\mu\mathcal{F}_{\nu\rho} = \varepsilon^{\mu\nu\rho}\mathfrak{D}\mathcal{H}_{\mu\nu\rho} = \mathfrak{D}f(\phi) = V'(\phi)$$

→ non-linear scalar equation of gauged supergravity

(Note: in Gauged Sugra/ExFT gauging and scalar potential correlated)

Part II: Duality Hierarchy & Differential Graded Lie Algebra

Leibniz Algebras

Gauge algebra of gauged sugra/DFT/ExFT governed by Leibniz algebra, a vector space X_0 with bilinear map $\circ$ obeying

[Strobl (2013), Kotov & Strobl (2018)]

$$V \circ (W \circ U) \; = \; (V \circ W) \circ U + W \circ (V \circ U)$$

Minimal structure needed to have consistent infinitesimal variations

$$\delta_V W \;\equiv\; \mathcal{L}_V W \;\equiv\; V \circ W$$

Closure implied by Leibniz relation: $[\mathcal{L}_V, \mathcal{L}_W]U \;=\; \mathcal{L}_{V \circ W} U$.

Symmetric part

$$\{V, W\} \equiv \tfrac{1}{2}(V \circ W + W \circ V) \;\equiv\; \tfrac{1}{2}\mathfrak{D}(V \bullet W)$$

Differential $\mathfrak{D}$ and $\bullet : X_0 \otimes X_0 \to X_1$. If trivial Leibniz reduces to Lie.

In general, higher algebra on chain complex (in particular L_∞ algebra)

$$\cdots \to X_1 \;\stackrel{\mathfrak{D}}{\longrightarrow}\; X_0$$

[O.H. & Samtleben (2018), Kotov & Strobl (2018), O.H. & Bonezzi (2019)]

Embedding Tensor

Derive Leibniz algebra from *Lie* algebra $\mathfrak{g}$ and embedding tensor map. Given Lie brackets $[\cdot, \cdot]$ we have *adjoint* and *coadjoint* reps

$$\delta_\zeta a \equiv \mathsf{ad}_\zeta a \equiv [\zeta, a] \,, \qquad \delta_\zeta \mathcal{A} \equiv \mathsf{ad}^*_\zeta \mathcal{A}$$

leaving pairing $\mathcal{A}(a) \in \mathbb{R}$ invariant. Given representation R, $\delta_\zeta V = \rho_\zeta V$, embedding tensor is map

$$\vartheta \ : \ R \ \to \ \mathfrak{g}$$

Transport Lie algebra to higher algebra on R:

$$V \circ W \equiv \rho_{\vartheta(V)} W$$

defines Leibniz algebra provided *quadratic constraint* obeyed

$$\vartheta(V \circ W) \ = \ [\vartheta(V), \vartheta(W)]$$

Equivalently, embedding tensor $\Theta \, : \, R \otimes \mathfrak{g}^* \to \mathbb{R}$ defined by pairing

$$\Theta(V, \mathcal{A}) \equiv -\mathcal{A}(\vartheta(V))$$

is Leibniz invariant.

Derived from differential graded Lie algebra

Upon suspension (overall shift of degree) and addition of vector spaces the graded symmetric $\bullet$ can be interpreted as dgLa (bracket $[\,,]$ satisfying graded Jacobi and differential $\mathfrak{D}$ with $\mathfrak{D}^2$) on

$$\cdots \longrightarrow \quad X_2 \xrightarrow{\mathfrak{D}} X_1 \xrightarrow{\mathfrak{D}=\vartheta} X_0 = \mathfrak{g} \xrightarrow{\mathfrak{D}} X_{-1} \longrightarrow \quad \cdots$$

Leibniz algebra on X_1 then "derived" from dgLa bracket:

[Lavau & Palmkvist (2019), O.H. & Bonezzi (2019)]

$$x \circ y \;\equiv\; -\mathfrak{D}x \bullet y \;=\; \pm[\mathfrak{D}x, y]$$

Leibniz relations follow from dgLa axioms.

Entire chain complex forms representation space of $\mathfrak{g}$.

Gauge theory or tensor hierarchy

Given dgLa X take dgLa $Z \equiv X \otimes \Omega(M)$ of forms on M valued in X

$$
\begin{array}{ccccccc}
\cdots & \xrightarrow{\mathfrak{D}} & Z^3_{[0]} & \xrightarrow{\mathfrak{D}} & Z^2_{[0]}(\chi_0) & \xrightarrow{\mathfrak{D}} & Z^1_{[0]}(\lambda_0) \\
\downarrow \mathrm{d} & & \downarrow \mathrm{d} & & \downarrow \mathrm{d} & & \downarrow \mathrm{d} \\
\cdots & \xrightarrow{\mathfrak{D}} & Z^3_{[1]}(\chi_1) & \xrightarrow{\mathfrak{D}} & Z^2_{[1]}(\lambda_1) & \xrightarrow{\mathfrak{D}} & Z^1_{[1]}(A_1) \\
\downarrow \mathrm{d} & & \downarrow \mathrm{d} & & \downarrow \mathrm{d} & & \downarrow \mathrm{d} \\
\cdots & \xrightarrow{\mathfrak{D}} & Z^3_{[2]}(\lambda_2) & \xrightarrow{\mathfrak{D}} & Z^2_{[2]}(A_2) & \xrightarrow{\mathfrak{D}} & Z^1_{[2]}(F_2) \\
\downarrow \mathrm{d} & & \downarrow \mathrm{d} & & \downarrow \mathrm{d} & & \downarrow \mathrm{d} \\
\cdots & \xrightarrow{\mathfrak{D}} & Z^3_{[3]}(A_3) & \xrightarrow{\mathfrak{D}} & Z^2_{[3]}(F_3) & \xrightarrow{\mathfrak{D}} & Z^1_{[3]}(\mathrm{d}F_2)
\end{array}
$$

dgLa structure w.r.t. diagonal grading $|Z^k_{[p]}| = k - p$ and differential

$$\partial \equiv d + \mathfrak{D}$$

Combing forms into formal sums, remembering only diagonal degree,

$$\cdots \longrightarrow Z_1 \xrightarrow{\partial} Z_0 \xrightarrow{\partial} Z_{-1} \longrightarrow \cdots ,$$
$$\Lambda \qquad \mathcal{A} \qquad \mathcal{F}$$

where Λ gauge parameters, $\mathcal{A}$ gauge fields, and $\mathcal{F}$ field strengths.

Maurer-Cartan forms

$$\Omega \equiv e^{-\mathcal{A}}\,\partial\, e^{\mathcal{A}}$$

identically satisfy Maurer-Cartan equations
[Greitz, Howe & Palmkvist (2014)]

$$\partial\Omega + \tfrac{1}{2}[\Omega, \Omega] = 0$$

In terms of $\Omega = \mathcal{F} + \mathfrak{D} A_1$ yields *Bianchi identity* of tensor hierarchy

$$DF_p + \frac{1}{2}\sum_{k=2}^{p-1}[F_k, F_{p+1-k}] + \mathfrak{D}F_{p+1} = 0$$

Scalars and generalization of G/H

Include zero-forms (scalars) $\phi \in \mathfrak{g} = X_0$:

$$\Omega \equiv e^{-\phi} e^{-\mathcal{A}} \partial \left(e^{\mathcal{A}} e^{\phi} \right)$$

satisfies Maurer-Cartan $\Leftrightarrow$ $\partial_\Omega = \partial + \Omega$ squares to zero: $\partial_\Omega^2 = 0$

in components:

$$\partial_\Omega = D_Q + P + T + \sum_{p=2}^{\infty} \mathcal{V}^{-1} F_p \mathcal{V} \ ,$$

"coset representative" $\mathcal{V} \equiv e^{\phi} \in G$, $\mathcal{V}^{-1} D \mathcal{V} = P + Q$, $Q \in \mathfrak{h} \subset \mathfrak{g}$

and embedding tensor $\Theta \in X_{-1}$ and T-tensor

$$\mathcal{V}^{-1} \mathfrak{D} \mathcal{V} = [\Theta, \phi] + \cdots \qquad T \equiv \mathcal{V}^{-1} \Theta \mathcal{V}$$

Duality Hierarchy & Dynamics

Goal: complete gauged supergravity/ExFT as tower of duality relations
[Bergshoeff, Hartong, O. H., Huebscher & Ortin (2009)]

In n external dimensions following structures needed:

- G covariant isomorphisms $I_p : X_p^* \to X_{n-p-2}$
- H invariant metric $\Delta_p : X_p \to X_p^*$

$\to$ 'generalized metric' $\mathcal{M}_1 \equiv \mathcal{V}\Delta_1\mathcal{V}^\top$ extended to map $\mathcal{M}$ on entire X

Duality relations: $\qquad \mathcal{F} = \star I \mathcal{M} \mathcal{F}$

Integrability conditions imply non-linear equations, including scalars with

$$V = \frac{1}{2}(T, \Delta_{-1}T)$$

Part III: Embedding Tensor of ExFT

Underlying Lie Algebra

$\mathfrak{g}_0$ Lie algebra of U-duality group (e.g. $\mathrm{E}_{7(7)}$),

$$[t_\alpha, t_\beta] \;=\; f_{\alpha\beta}{}^{\gamma} t_\gamma$$

Pick representation R_0 with matrices $(t_\alpha)_M{}^N$, $M, N = 1, \ldots, \dim(R_0)$.

Infinite-dimensional Lie algebra $\mathfrak{G}$ given by pairs of functions $\zeta \equiv (\lambda^M, \sigma^\alpha)$ of coordinates Y^M of $\mathbb{R}^{\dim(R_0)}$,

$$\begin{aligned} [\zeta_1, \zeta_2] \;=\; \big(2\,\lambda_{[1}{}^N \partial_N \lambda_{2]}{}^M, \\ 2\,\lambda_{[1}{}^N \partial_N \sigma_{2]}{}^\alpha + f_{\beta\gamma}{}^\alpha \sigma_1{}^\beta \sigma_2{}^\gamma\big) \end{aligned}$$

Coadjoint action on $\mathcal{A} = (A_\alpha, B_M) \in \mathfrak{G}^*$ so that invariant pairing with $a = (p^M, q^\alpha) \in \mathfrak{G}$ given by the integral:

$$\mathcal{A}(a) \;=\; \int \mathrm{d}Y \big(p^M B_M \;+\; q^\alpha A_\alpha\big)$$

Subtlety: eventually need to impose section constraint

$\Rightarrow$ non-vanishing vectors λ^M so that $\lambda^M \partial_M = 0$

$\Rightarrow$ subalgebra $\mathfrak{I}$ defined as

$$\mathfrak{G} \supset \mathfrak{I} = \left\{ \zeta = (\lambda^M, 0) \in \mathfrak{G} \middle| \lambda^M \partial_M = 0 \right\}$$

is generally non-empty, forming an abelian ideal of $\mathfrak{G}$.

The Lie algebra $\mathfrak{g}$ is then

$$\mathfrak{g} = \mathfrak{G} / \mathfrak{I}$$

Its dual $\mathfrak{g}^*$ consists of functions $\mathcal{A} = (A_\alpha, B_M)$ with a pairing for which the non-trivial denominator $\mathfrak{I}$ requires the B_M to satisfy

$$\forall \lambda^M : \quad \lambda^M \partial_M = 0 \quad \Longrightarrow \quad \lambda^M B_M = 0$$

$\Rightarrow$ "covariantly constrained" objects B_M

Embedding Tensor

Representation R of $\mathfrak{g}$ given by functions $V^M(Y)$:

$$\rho_\zeta V^M \equiv \lambda^N \partial_N V^M - \sigma^\alpha (t_\alpha)_N{}^M V^N$$

Define embedding tensor map

$$\vartheta(V) \ = \ \big[\big(V^M\,,\ -\kappa(t^\alpha)_M{}^N \partial_N V^M\big)\big] \ \in \ \mathfrak{g}$$

where $[\,]$ indicates equivalence class.

Derived Leibniz structure defines *generalized Lie derivative*

$$\begin{aligned}(\Lambda \circ V)^M \equiv \mathcal{L}_\Lambda V^M \equiv \rho_{\vartheta(\Lambda)} V^M \\ = \Lambda^N \partial_N V^M + \ \kappa\,(t^\alpha)_N{}^M (t_\alpha)_L{}^K\, \partial_K \Lambda^L\, V^N\end{aligned}$$

closure/quadratic constraint $\Rightarrow$ strong section constraint on ∂_M.

$O(d,d)$ or Double Field Theory

$\mathfrak{g}_0 = \mathfrak{o}(d,d)$ with representation matrices and invariant metric

$$(t^{IJ})_M{}^N \;=\; 2\,\delta^{[I}{}_M\,\eta^{J]N}\,, \qquad \eta_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

∞-dimensional Lie algebra $\mathfrak{g}$ consists of functions $\zeta = (\lambda^M, \sigma^{IJ})$, with embedding tensor map $\vartheta : R \to \mathfrak{g}$

$$\vartheta(V) \;=\; \big(V^M\,,\; -\kappa(t^{IJ})_M{}^N \partial_N V^M\big) \;=\; \big(V^M\,,\; 2\,\partial^{[I} V^{J]}\big)$$

Derived Leibniz algebra

$$(V \circ W)^M \;=\; V^N \partial_N W^M + \partial^M V_N W^N - \partial_N V^M W^N$$

The symmetric part obeys $\{V,W\} = \frac{1}{2}\mathfrak{D}(V \bullet W)$ for

$$\mathfrak{D} \;:\; X_1 \to X_0\,, \qquad (\mathfrak{D} f)^M \;\equiv\; \partial^M f$$

$$\bullet \;:\; X_0 \otimes X_0 \to X_1\,, \qquad V \bullet W \;\equiv\; \eta_{MN} V^M W^N$$

X_0: $O(d,d)$ vectors, X_1: $O(d,d)$ scalars $\rightarrow$ Courant algebroid

$\mathsf{E}_{7(7)}$ Exceptional Field Theory

Generators t_α, $\alpha = 1, \ldots, 133$, $\quad \dim R_0 = 56$, $\quad M, N = 1, \ldots, 56$,
symplectic embedding $\mathsf{E}_{7(7)} \subset \mathsf{Sp}(56) \;\Rightarrow\;$ invariant form Ω_{MN}
Embedding tensor yields Leibniz algebra

$$\begin{aligned}(V \circ W)^M \; &= V^N \partial_N W^M - W^N \partial_N V^M - \tfrac{1}{2} \partial^M V_N \, W^N \\ &\quad - 12 \, (t_\alpha)^{MN} (t^\alpha)_{KL} \, \partial_N V^K W^L\end{aligned}$$

Symmetric part obeys $\{V, W\} = \frac{1}{2} \mathfrak{D}(V \bullet W)$ for $\bullet : X_0 \otimes X_0 \to X_1 \cong \mathfrak{g}^*$,
$\mathcal{A} = (A_\alpha, B_M) \in \mathfrak{g}^*$. The bullet operation is defined by

$$V \bullet W \;\equiv\; \Big((t_\alpha)_{KL} V^K W^L ,\; \tfrac{1}{2} \big(V_N \, \partial_M W^N + W_N \, \partial_M V^N \big) \Big) \;\in\; \mathfrak{g}^*$$

and the differential $\mathfrak{D} : \mathfrak{g}^* \to R$ by

$$\big(\mathfrak{D} \mathcal{A} \big)^M \;\equiv\; -12 \Big((t^\alpha)^{MN} \partial_N A_\alpha \;-\; \frac{1}{12} \Omega^{MN} B_N \Big) \in R$$

Quadratic constraints require

$$(t_\alpha)^{MN} \, \partial_M f \, \partial_N g \;=\; 0 \,, \quad \Omega^{MN} \, \partial_M f \, \partial_N g \;=\; 0$$

$E_{8(8)}$ Exceptional Field Theory

$\mathfrak{e}_{8(8)}$ representation given by (co)adjoint

$\to$ ∞-dimensional Lie algebra given by pairs $\zeta = (\lambda^M, \sigma_M)$, $M = 1, \ldots, 248$

$\to$ co-adjoint vectors are pairs $\mathcal{A} = (A^M, B_M)$

Embedding tensor is map $\vartheta : \mathfrak{g}^* \to \mathfrak{g}$ for $\mathfrak{g}^* \ni \Upsilon = (\Lambda^M, \Sigma_M)$ given by

$$\vartheta(\Upsilon) = \big(\Lambda^M,\ f_M{}^N{}_K \partial_N \Lambda^K + \Sigma_M\big) \equiv \big(\Lambda^M, R_M(\Lambda, \Sigma)\big)$$

Leibniz algebra defined on $\mathfrak{g}^*$

$$\Upsilon_1 \circ \Upsilon_2 \ \equiv \ \Big(\mathcal{L}^{[1]}_{\Upsilon_1} \Lambda_2{}^M\ ,\ \mathcal{L}^{[0]}_{\Upsilon_1} \Sigma_{2M}\ +\ \Lambda_2{}^N \partial_M R_N(\Upsilon_1)\Big)$$

$\to$ tensor hierarchy of ExFT necessarily employs ‘doubled’ vectors

$$\mathcal{A}_\mu \ \equiv \ \big(A_\mu{}^M,\ B_{\mu M}\big)$$

Bilinear form

$$\Theta(\mathcal{A}_1, \mathcal{A}_2) = -\int \mathrm{d}Y \big(2A_{(1}{}^M B_{2)M} - f^M{}_{NK} A_1{}^N \partial_M A_2{}^K\big)$$

gives invariant 3D Chern-Simons theory.

Outlook

- Inclusion of external metric $g_{\mu\nu}$ as part of tensor hierarchy or dgLa?
- Detailed formulation of ExFT in terms of these structures, e.g. ExFT potential as $V = \frac{1}{2}(T, \Delta_{-1}T)$?
- (Non-covariant) actions for duality relations? c.f. exotic $D = 6$ theories in Henning's talk
- Universal dgLa unifying all $E_{n(n)}$, $n = 2, \ldots, 9$, without split into internal/external?