Tropical field theory

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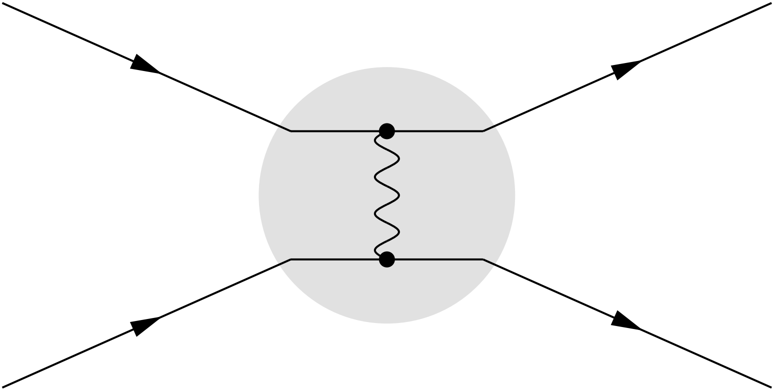
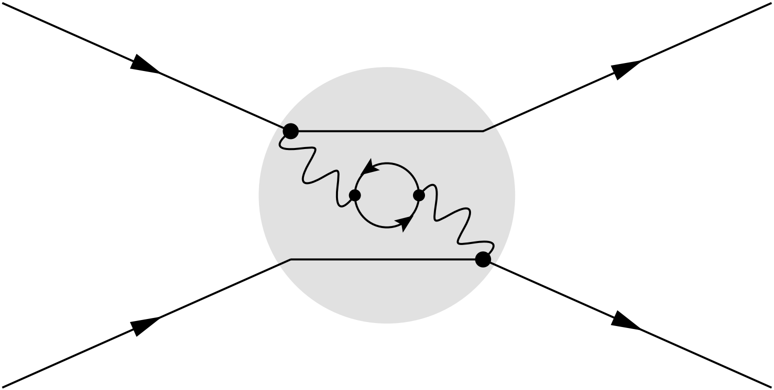
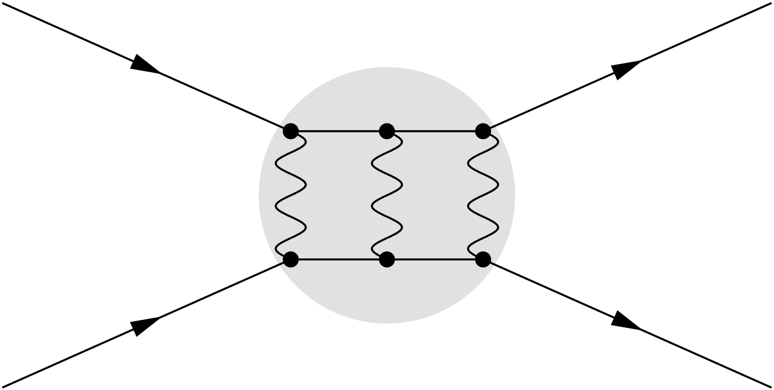
Perturbative Quantum Field Theory



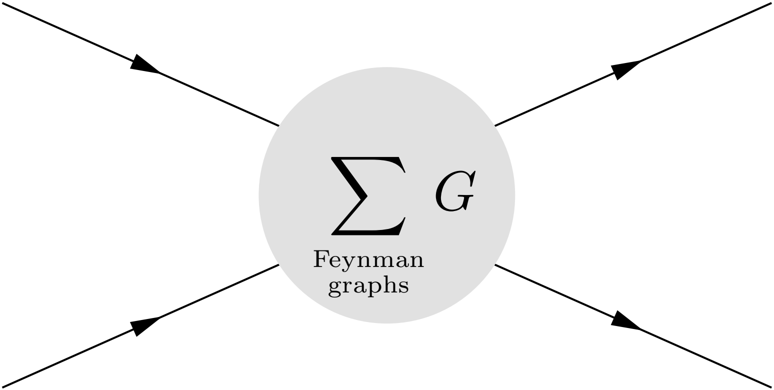
Perturbative Quantum Field Theory



Perturbative Quantum Field Theory



Perturbative Quantum Field Theory



- Feynman graph $G \mapsto$ Feynman integral $\Phi(G, \{m_i^2, \vec{p}_i \cdot \vec{p}_j\})$
- compute more graphs $\sum_G \Phi(G) \Rightarrow$ higher precision

Problems:

1. $\Phi(G)$ extremely complicated

   ➡ *polylogarithms, iterated elliptic integrals, modular forms, $K3$ surfaces, Calabi-Yau manifolds, ...*

2. $\sum_G \Phi(G) = \infty$

   ➡ *factorial growth $A \cdot n! \cdot c^n \cdot n^\alpha$, resummation, Borel transformation, resurgence, ...*

perturbation series are very poorly understood in QCD, QED, ϕ^4

Simplifications:

1. integrable models

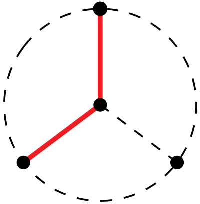
   ➡ $\Phi(G)$ *simplify somewhat, full* $\sum_G \Phi(G)$, *convergent expansion*

2. truncated Dyson-Schwinger equations

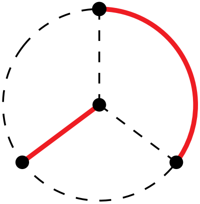
   ➡ *factorial growth possible, sums very restricted class of diagrams*

3. tropical limit

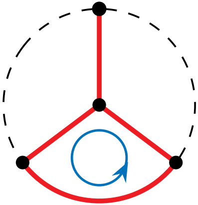
   ➡ *all $\Phi(G)$ simplify drastically, asymptotics unchanged*



not spanning



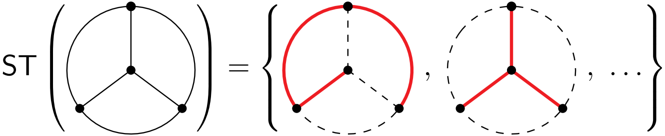
not connected



has a loop

Definition

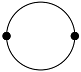
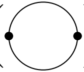
A spanning tree $T \subset G$ is a spanning, simply connected subgraph.



Definition

The graph polynomial $\mathcal{U}$ and Feynman period of G are

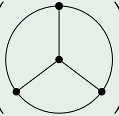
$$\mathcal{U} = \sum_{T \in \mathrm{ST}(G)} \prod_{e \notin T} x_e \qquad \text{and} \qquad \mathcal{P}(G) = \left(\prod_{e>1} \int_0^\infty \mathrm{d}x_e \right) \frac{1}{\mathcal{U}^2|_{x_1=1}}$$

$G =$  $\Rightarrow$ $\mathcal{U} = x_1 + x_2$ and $\mathcal{P}\Big($  $\Big) = \int_0^\infty \frac{\mathrm{d}x_2}{(1+x_2)^2} = 1$

- contribute to the β-function

  ➔ *renormalization constants, running coupling, critical exponents*
- very hard to compute, even numerically

Example

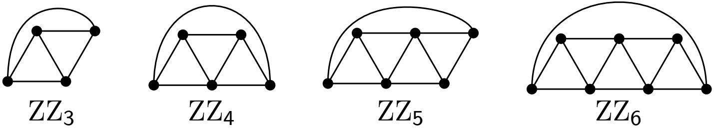
$\mathcal{P}\Big($  $\Big) = \int_{\mathbb{R}_+^5} \frac{\mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 \mathrm{d}x_5 \mathrm{d}x_6}{(x_1 x_2 x_3 + 15 \text{ more terms})^2|_{x_1=1}} = 6\zeta(3) = 6 \sum_{n=1}^\infty \frac{1}{n^3}$

Only one *infinite* family of periods is known in ϕ^4:

Theorem (Brown & Schnetz 2012)

$$\mathcal{P}(\mathrm{ZZ}_n) = 4\frac{(2n-2)!}{n!(n-1)!}\left(1-\frac{1-(-1)^n}{2^{2n-3}}\right)\zeta(2n-3)$$

conjectured by Broadhurst & Kreimer

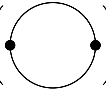


- >1000 periods are known in ϕ^4 [Broadhurst, Kreimer, Schnetz, Panzer]
- complete only up to 7 loops

Hepp bound

$$\mathcal{H}(G) = \left(\prod_{e>1} \int_0^\infty \mathrm{d}x_e \right) \frac{1}{\mathcal{U}^2_{\max}|_{x_1=1}} \quad \text{where} \quad \mathcal{U}_{\max} = \max_{T \in \mathrm{ST}} \prod_{e \notin T} x_e$$

Example:

$\mathcal{H}\Bigg($  $\Bigg)$

$$= \int_0^\infty \frac{\mathrm{d}x_2}{(\max\{1, x_2\})^2} = \int_0^1 \mathrm{d}x_2 + \int_1^\infty \frac{\mathrm{d}x_2}{x_2^2} = 2$$

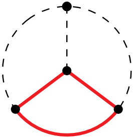
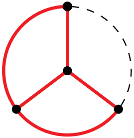
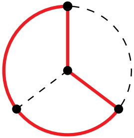
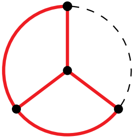
Properties:

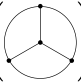
- $\mathcal{H}(G) > \mathcal{P}(G) > \mathcal{H}(G)/\left|\mathrm{ST}(G)\right|^2$ ⇒ same asymptotics up to $\mathcal{O}(c^\ell)$
- $\mathcal{H}(G) \in \mathbb{Q}_{>0}$
- computable for all G
- correlates with $\mathcal{P}(G)$
- respects symmetries of $\mathcal{P}(G)$
- generalizes to $\Phi(G, m_e^2, p_i \cdot p_j)$

$$\omega(\gamma_\ell) = |\gamma| - 2\ell$$

Theorem

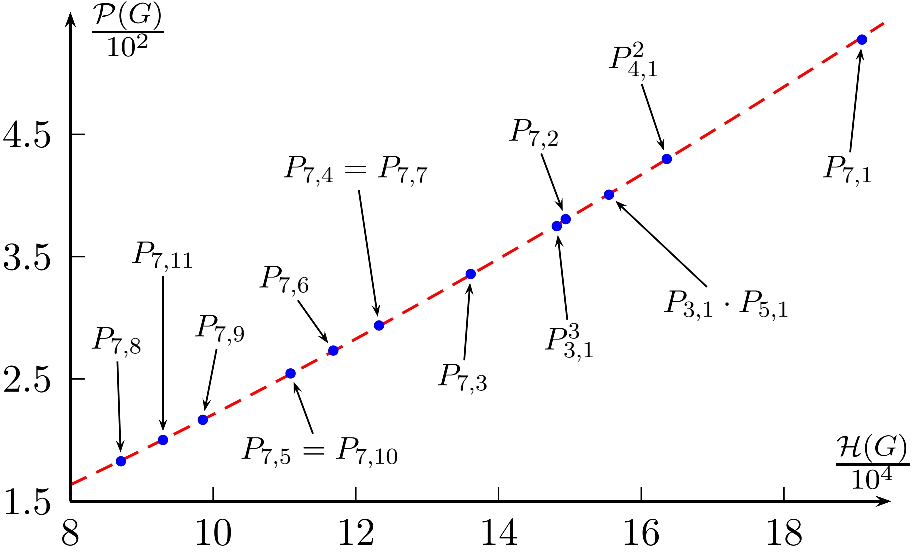
$$\mathcal{H}(G) = \sum_{\substack{\gamma_1 \subset \gamma_2 \subset \cdots \subset \gamma_\ell = G \\ \text{each } \gamma_i \text{ is 1PI}}} \frac{|\gamma_1| \cdot |\gamma_2 \setminus \gamma_1| \cdots |G \setminus \gamma_{\ell-1}|}{\omega(\gamma_1) \cdots \omega(\gamma_{\ell-1})}$$

γ_1	$\subset$	γ_2	summand	#	$\sum$
	$\subset$		$\frac{3\cdot 2\cdot 1}{1\cdot 1} = 6$	12	72
	$\subset$		$\frac{4\cdot 1\cdot 1}{2\cdot 1} = 2$	6	12

$\Rightarrow \mathcal{H}\Big($  $\Big) = 84$

7 loops in ϕ^4:

G	$\mathcal{P}(G \setminus v)$	$\mathcal{H}(G \setminus v)$
$P_{7,1}$	527.7	190952
$P_{4,1} \cdot P_{4,1}$	430.1	163592
$P_{3,1} \cdot P_{5,1}$	400.9	155484
$P_{7,2}$	380.9	149426
$P_{3,1} \cdot P_{3,1} \cdot P_{3,1}$	375.2	148176
$P_{7,3}$	336.1	136114
$\{P_{7,4}, P_{7,7}\}$	294.0	123260
$P_{7,6}$	273.5	116860
$\{P_{7,5}, P_{7,10}\}$	254.8	110864
$P_{7,9}$	216.9	98568
$P_{7,11}$	200.4	92984
$P_{7,8}$	183.0	87088



$\frac{\mathcal{P}(G)}{10^2}$
4.5
3.5
2.5
1.5
8
10
12
14
16
18
$\frac{\mathcal{H}(G)}{10^4}$
$P_{7,8}$
$P_{7,11}$
$P_{7,9}$
$P_{7,5} = P_{7,10}$
$P_{7,6}$
$P_{7,4} = P_{7,7}$
$P_{7,3}$
$P_{7,2}$
$P_{3,1}^3$
$P_{3,1} \cdot P_{5,1}$
$P_{4,1}^2$
$P_{7,1}$

Michael Borinsky:

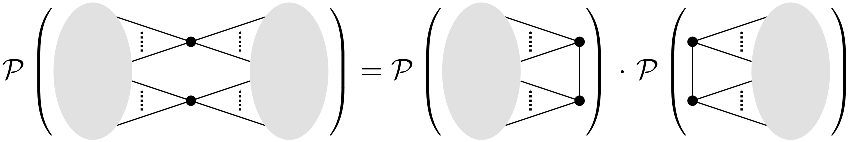
Tropical Monte Carlo quadrature for Feynman integrals

➡ numeric evaluation at large loop orders

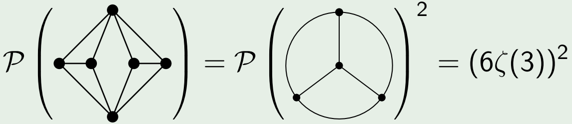
Symmetries

When is $\mathcal{P}(G_1) = \mathcal{P}(G_2)$?

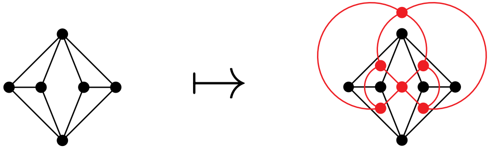
1. Product:



Example

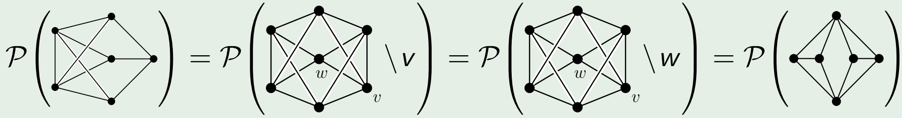


2. Planar duality: $\mathcal{P}(G) = \mathcal{P}(G^{\mathrm{dual}})$

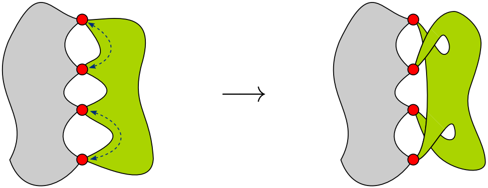


3. Completion: $\mathcal{P}(G \setminus v) = \mathcal{P}(G \setminus w)$

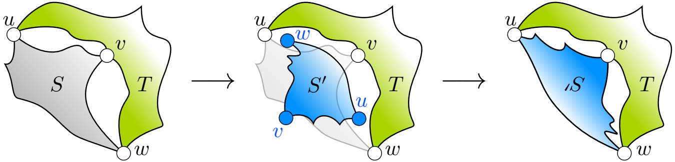
Example



4. Twist:



5. Fourier split:



Goal:

Construct simpler graph invariants with those symmetries.

1. point count [Schnetz]

$$c_2(p) = \frac{1}{p^2} \left| \left\{ \vec{x} \in (\mathbb{Z}/p\mathbb{Z})^N \colon\ \mathcal{U}(\vec{x}) = 0 \right\} \right| \quad \mod p$$

2. extended graph permanent [Crump]

$P_{7,11}$

p	2	3	5	7	11	13	17	19	23
$c_2(p)$	1	0	1	-1	1	-1	1	-1	1
$\mathrm{Perm}(p)$		0	1	1	1	11	5	0	22

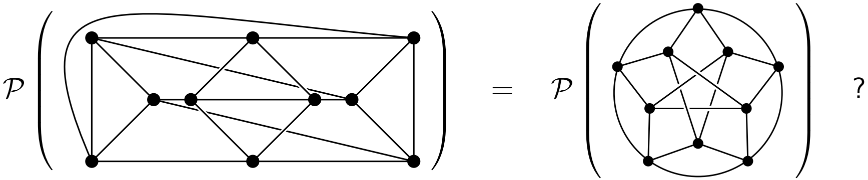
3. $\mathcal{H}$ $(\in \mathbb{Q})$
4. $\#\{$minimal 6-cuts$\}$ $(\in \mathbb{Z})$
5. $O(-2)$ symmetry factor $(\in \mathbb{Z})$

Conjecture (ϕ^4)

$$\mathcal{H}(G_1) = \mathcal{H}(G_2) \quad \Leftrightarrow \quad \mathcal{P}(G_1) = \mathcal{P}(G_2)$$

Example:

- $\mathcal{H}(P_{8,30}) = \frac{1724488}{3} = \mathcal{H}(P_{8,36})$
- $\mathcal{P}(P_{8,30}) \approx 505.5 \approx \mathcal{P}(P_{8,36})$ (exact period unknown)



there are further symmetries of Feynman integrals