Lecture 4 Birational Geom, Singularities, Min Model

Chp 11, 15 of
Cox, Little, Schenck.

§ Singularities:

X normal is Gorenstein if K_X Cartier

(Recall: for toric, Cartier $\Leftrightarrow \forall$ max σ, $\exists\, m_\sigma$ s.t. $\langle m_\sigma, v_i \rangle = -a_i$ $\forall\, v_i \in \sigma$.

$D = \sum a_\rho D_\rho$

$K_{X_\Sigma} = -\sum_{\rho \in \Sigma(1)} D_\rho$ So K_{X_Σ} Cartier $\Leftrightarrow \langle m_\sigma, v_i \rangle = 1\ \forall v_i$ in σ.

X is $\mathbb{Q}$-Gorenstein if K_X $\mathbb{Q}$-Cartier (nK_X is Cartier, some $n \in \mathbb{Z}_+$. $X' \xrightarrow{f} X$ (sm) $f^*(nK_X)$ is Cartier, $f^*(K_X) = \frac{1}{m} f^*(mK_X)$ and is Crepant if $K_{X'} = f^*(K_X)$

(Crepant are rare, $X'_{sm} \xrightarrow[\text{proper birational}]{f} X$ $K_X' = f^*(K_X) + \sum a_i E_i$

with E_i exceptional divisors (irr divisor in exceptional locus/f)

Sings terminal if $a_i > 0$, canonical if $a_i \geq 0$

Toric case Ψ = support fn. of $\mathbb{Q}$-Cartier K_X

$\phi: X_{\Sigma'} \to X_\Sigma$ has Σ' refinement of Σ. Then

$$K_{X_{\Sigma'}} = \phi^* K_{X_\Sigma} + \sum_{\rho \in \Sigma'(1) \setminus \Sigma(1)} (\Psi(u_\rho) - 1) D_\rho \qquad \left[\Psi: X_{\Sigma'} \to X_\Sigma \text{ refinement}\right]$$

Since $\Psi(u_\rho) \geq 1$, normal Gorenstein toric has canonical sings

Def: Fano-Gorenstein if $-K$ ample,

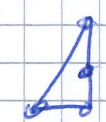
Kreuzer-Skarke:

dim	reflexive	sm	terminal
2	16	5	5
3	4319	18	100
4	473,800,776	~~12~~ 124	634

(Some) toric facts

- X_Σ canonical sings has simplicial refinement $X_{\Sigma'} \to X_\Sigma$ projective, crepant
- X_Σ terminal sings $\Rightarrow \operatorname{codim}(X_\Sigma)_{sing} \geq 3$, if also simplicial Gorenstein ≥ 4
- U_σ $\mathbb{Q}$ Gorenstein $\Leftrightarrow \exists\, m \,/\, \langle m, u_\rho\rangle = 1\ \forall \rho \in \Sigma$ ~~$\Leftrightarrow$ unique facet~~
  - $\rightarrow$ term sings $\Leftrightarrow \operatorname{conv}(0, u_\rho (\rho \in \Sigma(1)))$ has only lattice pts as vertices
  - $\rightarrow$ canonical sings $\Leftrightarrow$ only nonzero lattice pts on facet not conting 0

Surface
smooth
only rational double pts



term.

can, not term.

Example of rational double pts Δ lattice so rays are

$$\{e_2,\ de_1 - ke_2\}$$

$[d>0,\ 0 \leq k \leq d-1,\ \gcd(d,k)=1]$ $\rightsquigarrow$ lattice pts $\langle m, u\rangle \leq 1$ w/

$$m = \frac{k+1}{d} e_1 + e_2$$

$\Rightarrow k < d-1$ then $\langle m, e_1\rangle = \frac{k+1}{d} < 1 \Rightarrow e_1$ interior pt, impossible because canonical hence $k = d-1$, which is (definition) rational double pt.

[10.1.6 CLO's]

lowest term defg eq of deg 2

$XY - Z^{k+1} \Leftrightarrow A_k$

§ MMP

Def: A minimal model is $\mathbb{Q}$-Gorenstein normal projective toric w/ terminal sings + K nef. ($\mathbb{Q}$GNPT).

MMP: GOAL Birational classificatn of proj. varieties

If X $\mathbb{Q}$GNPT, then $\exists$ a sequence of divisor blowdowns + flips $X \to X_1 \to X_2 \cdots \to Y$ s.t. either

- Y is a minimal model
- $\exists\, f : Y \to Z^{\dim < Y}$ s.t. $C \to p \Rightarrow K_Y \cdot C < 0$ (Mori Fibratn)

Rmk If K not nef, $\exists$ C with $K \cdot C < 0$, so C is an edge of $\overline{NE}(X)$ "extremal ray", giving an extremal contractn $X \to X'$

- $\to$ fibering
- $\to$ divisorial
- $\to$ flipping.

(of course, all torics are rational, but good for illustrating)

Thm 15.4.1 ~~15.4.5~~ $\Sigma \subseteq N_{\mathbb{R}}$ simplicial, ~~X_Σ semiprojective~~

R an extremal ray (($V(\tau)$ gen Mori, [simplicial → prim relns gen, projective])

Then $\exists$ (possibly not SC) fan $\Sigma_0 \subseteq N_{\mathbb{R}}$ s.t.

- Σ refines Σ_0 (so toric morphism)
- (★) $X_\Sigma \xrightarrow{\phi} X_{\Sigma_0}$ toric, $\phi[V[\tau]] = pt \iff V[\tau] \in R$

"Extremal contraction"

15.4.2 **Thm** If R (extremal ray) gen'd by $\sum b_\rho v_\rho = 0$ (prim reln), Let $J_- = \{i \mid b_i < 0\}$

Then $\Sigma_R = \{\sigma_J \mid J \subsetneq J_+\}$ is a complete simplicial fan (in a certain quotient lattice) with $|J_+| \geq 2$

s.t X_R is $\mathbb{Q}$-factorial, $\mathbb{Q}$-Fano, of dim $|J_+|-1$, $\operatorname{Pic}(X_R)=1$ and a finite abelian quotient of weighted proj. space.

[~~omit if time tight~~]

15.4.5 Situation as above. The map ϕ at (★) is

- fibering $\iff J_- = \phi$, X_{Σ_0} simplicial, fibers $\phi \cong X_R$
- divisorial $\iff |J_-| = 1$, exceptional locus a divisor with fibers of ϕ on $\cong X_R$
- flipping $\iff |J_-| > 1$, exceptional locus of codim $|J_-|$, fibers

SKIP IF time issue

④

§ Generalities on birational map impact on divisors.

Rmk: birational map not functorial; blow up X at sm. point of Y, $X \xrightarrow{f} Y$, E = exceptional locus.

f^{-1} is birational, but $f_*^{-1} f_* E = f_*^{-1}(0) = 0$, while

$(f^{-1} \circ f)_* E = I_* E = E$.

If $X \xrightarrow{f} X'$ is a birational map on normal vars, defined on U, then $f_* D = \overline{f(D \cap U)}$ if codim $=1$, zero else.

Lemma: in this situation, if f is an iso in codim 1, then $D_1 \sim D_2$ on $X \Rightarrow f_* D_1 \sim f_* D_2$ on X'.

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§ Thm: X_Σ simplicial, R extremal ray and ϕ a flipping contraction to X_{Σ_0}, then $\exists$ diagram

$$\begin{tikzcd} X_\Sigma \arrow[rr, "\psi"] \arrow[dr, "\phi"'] & & X_{\Sigma'} \arrow[dl, "\phi'"] \\ & X_{\Sigma_0} & \end{tikzcd}$$

ψ ← elementary flip, equivalently iso in codim 1

flipping contraction for R → ϕ

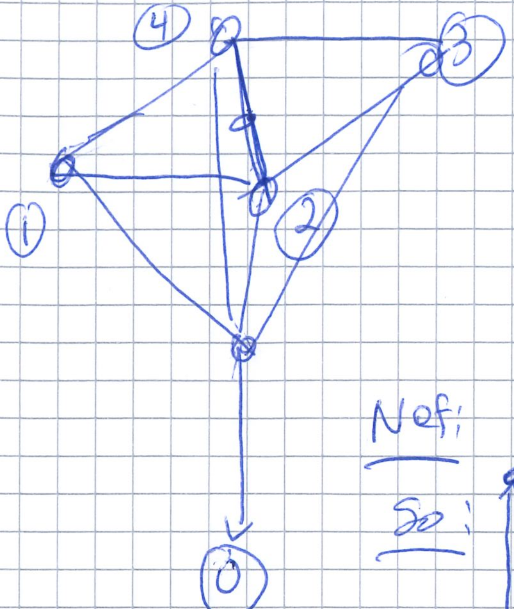
ϕ': flipping contraction for $R' = -R$

Example next page

- (Recall: elementary flip is birational map of "flipping type" from end of last lecture.)

- We had nice Corollary: birat equiv map iso codim 1 of torics is a composition of elem flips + toric iso.

Example



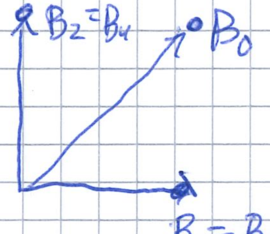
$$\mathbb{Z}^3 \longrightarrow \mathbb{Z}^5 \longrightarrow \mathbb{Z}^2 \longrightarrow 0$$

$$\begin{array}{c} 0 \\ \\ \\ \\ 4 \end{array}\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

simplicial, check $\mathrm{Pic} \neq 0$ so projective.

Nef: Use that $D_1 \sim D_3, D_2 \sim D_4, D_0 \sim D_1 + D_2 + D_3 + D_4 \Rightarrow \underline{D = xD_1 + yD_2}$

So:



Prim Coll: $\{1,3\}, \{0,2,4\}$

$\Longleftarrow$ 1^{st} qual $= \mathrm{eff}(X_\Sigma)$

$\varphi_D(1+3) \geq \varphi_D(1) + \varphi_D(3)$

$\overset{\shortparallel}{\varphi_D(2+4)} \qquad \overset{\shortparallel}{-x} \qquad \overset{\shortparallel}{0}$

in a cone where PL applies $\varphi_D\begin{pmatrix}0\\0\\2\end{pmatrix} = -y$

so $\varphi_D(2+4) = \varphi_D(2) + \varphi_D(4)$

$\Longrightarrow \boxed{x \geq y}$

For $\varphi_D(0+2+4) \geq \varphi_D(0) + \varphi_D(2) + \varphi_D(4)$

$\boxed{-\tfrac{1}{2}y} = \tfrac{1}{2}\varphi_D(002) \overset{\shortparallel}{\begin{pmatrix}0\\0\\1\end{pmatrix}} \quad \geq \quad \overset{\shortparallel}{0} \quad \overset{\shortparallel}{-y} \quad \overset{\shortparallel}{0}$

$y \geq \tfrac{1}{2}y \qquad \tfrac{1}{2}y \geq 0 \Rightarrow \boxed{y \geq 0}$

$\Longrightarrow \mathrm{Nef}(X_\Sigma) =$

MORI

$u_1 + u_3 = u_2 + u_4$

$\Rightarrow b_1 = \langle 0, 1, -1, 1, -1 \rangle$

$u_0 + u_2 + u_4 = \tfrac{1}{2}(u_2 + u_4)$

$u_0 + \tfrac{1}{2}u_2 + \tfrac{1}{2}u_4 = 0$

$\Rightarrow b_2 = \langle 1, 0, \tfrac{1}{2}, 0, \tfrac{1}{2} \rangle$

$\overline{NE}$ has 2 extremal rays (dual to Nef)

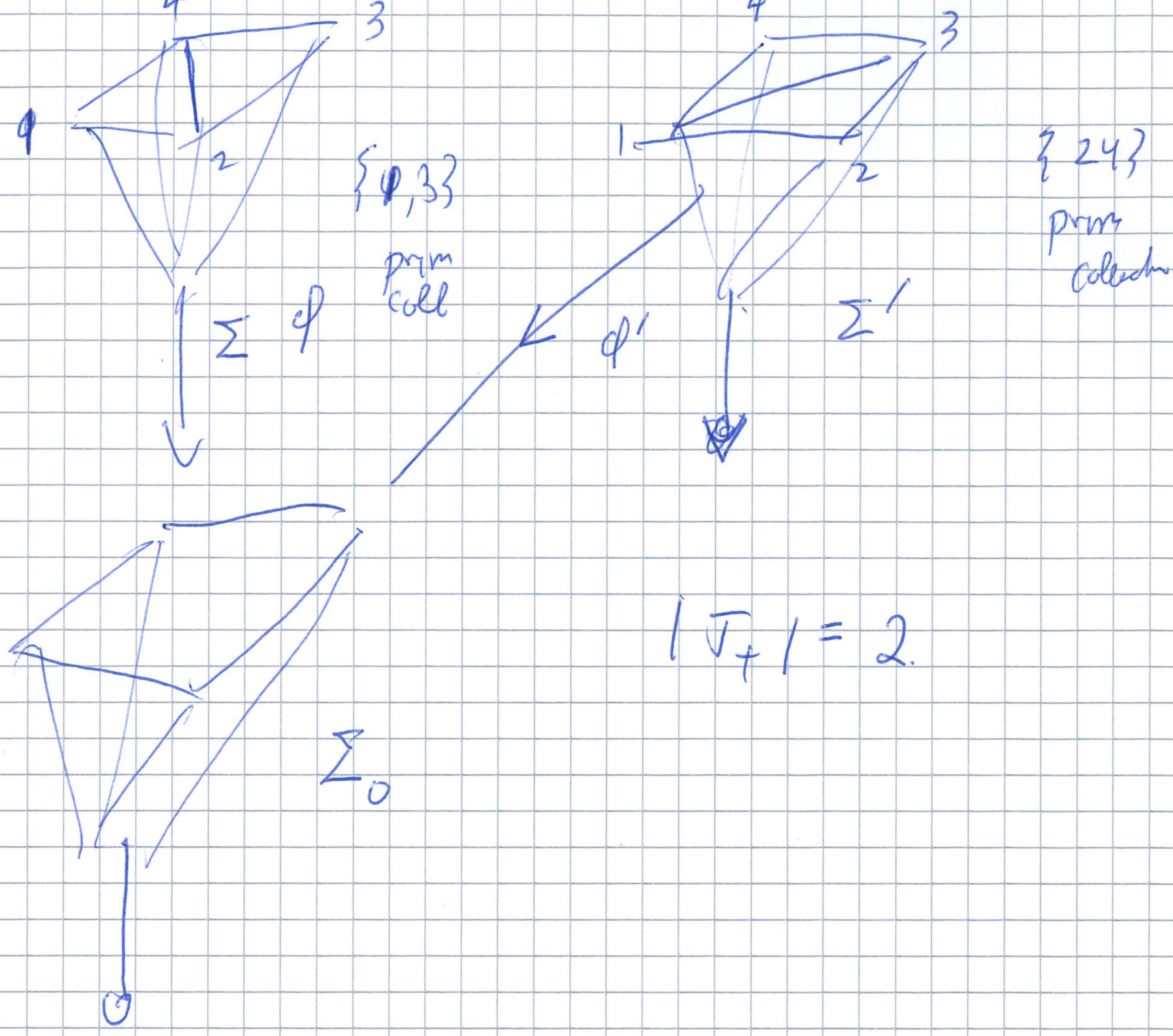
<u>Exercise 1</u> Determine the Mori cone pairing.

<u>Exercise 2</u>: for F_3, find the nef + Mori cones. (Example 6.3.23 of C-L-O's)

Mori

Nef

<u>Example, continued</u>. Thm. gives us



$|J_f| = 2.$