What do dg categories form?
(traces and higher structures)

Objects: $A, B, \ldots$ algebras

1-morphisms: bimods { perfect as B-mods

for us: just graph(f)

$f: A \to B$ morphism

$$\mathrm{graph}(f) = {}_f B = B \qquad A \xrightarrow[\text{via } f]{} B_B$$

Two-morphisms T:

$$\begin{tikzcd} A \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] \arrow[r, phantom, "\Downarrow\varphi"] & B \end{tikzcd}$$

$$C^\bullet(A, \underbrace{{}_{f_1}B_{f_2}}_{B\ \ A\text{-bimod}})$$

$$R\mathrm{Hom}_{A\text{-}B}({}_{f_2}B, {}_{f_1}B)$$

$$\begin{tikzcd} A \arrow[r, "f_1"'] & B & A \arrow[l, "f_2"] \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] \arrow[r, phantom, "\xi"] & B \end{tikzcd}$$

$$C_\bullet(A, {}_{f_1}B_{f_2})$$

Hochschild (co)chains

$$C^\bullet(A, M) = \mathrm{Hom}_k(A[1]^{\otimes \bullet}, M)$$

$$C_\bullet(A, M) = A[1]^{\otimes \bullet} \otimes M$$

differentials...

What kind of alg structure on those?

what do dg cats form?

① a category in <u>cocategories</u>

<u>stricts</u>

$\left(= \mathrm{Bar}\, C^{\bullet}(A,B) \right)$

$\underbrace{A, B}_{\text{algs}} \rightsquigarrow \mathbb{B}(A,B)$

dg cocategory

$m_{ABC} : \mathbb{B}(A,B) \otimes \mathbb{B}(B,C) \to \mathbb{B}(A,C)$

morphism of dg cocats

associative

$\mathbb{B}(A,A)(\mathrm{id}_A, \mathrm{id}_A)$ a <u>bialgebra</u>

Getz.-J. Ger.-Voronov

Hopf

<u>What else</u> ?

2. Categ in cocategs with a trace functor.

0) $\mathbb{B}(A,B)$ $\forall A, B$

1) T_A a dg mod / $\mathbb{B}(A,A)$ $\forall A$

2)
$$\begin{tikzcd}
 & \mathbb{B}(A,B)\otimes\mathbb{B}(B,A) \arrow[rr, "\text{flip}"] \arrow[d, "m_{ABA}"] & & \mathbb{B}(B,A)\otimes\mathbb{B}(A,B) \arrow[d, "m_{BAB}"] \\
T_A & \mathbb{B}(A,A) \arrow[l, bend right] & T_B & \mathbb{B}(B,B) \arrow[l, bend right]
\end{tikzcd}$$

$\tau_{AB} : m^*_{ABA} T_A \xrightarrow{\simeq} m^*_{BAB} T_B$

2)

$$\begin{tikzcd} B(A,B)\otimes B(B,C)\otimes B(C,A) \arrow[r, "\sim"] & B\ C\ A \arrow[d, "\wr"] \\ & C\ A\ B \arrow[ul, "\sim"] \end{tikzcd}$$

$$\left.\begin{array}{c} \mathrm{id} \quad m^*_{ABCA}(T_A) \\ \wr\ \sigma_{ABC} \\ T_{AB}T_{BC}T_{CA} \end{array}\right\}$$

2 dg endoms of $m^*_{ABCA}(TA)$

homotopy; CENTRAL

$\pi \in \Lambda^{\circ} T$

$\alpha \in \Omega^{\bullet}$

0) $A, B \rightsquigarrow$ DG coalg $\mathbb{B}(A,B)$

NC version of $\pi_1 \wedge \pi_2$

1) $A, B, C \rightsquigarrow \mathbb{B}(A,B) \otimes \mathbb{B}(\ ) \xrightarrow{m} \mathbb{B}(A,C)$

NC (higher) vers. of $[\,,\,]_{sch}$

2) $\mathbb{B}(A,A) \circlearrowleft J_A$ 3) T_{AB}

NC $\iota_\pi \alpha$ L_π

4) σ_{ABC} — the only homotopy needed

NC dDR (a.k.a. B)

Category in cocategories with a trace functor

Obj's: $A, B, \ldots$ I. Any A, B: dg cocat $\mathbb{B}(A,B)$

II. Any A, B, C: m_{ABC} : $\mathbb{B}(A,B) \otimes \mathbb{B}(B,C) \to \mathbb{B}(A,C)$

associative. $m_{A_1,\ldots,A_{n+1}}$: ———

III. Any A: dg comod T_A over $\mathbb{B}(A,A)$

IV Any A, B: τ_{AB} : $m^*_{ABA} T_A \xrightarrow{\sim} m^*_{BAB} T_B$

V. Any A, B, C: σ_{ABC} : $\mathrm{id} \sim \tau_{CA} \tau_{BC} \tau_{AB}$ $m^*_{ABCA}(T_A)$

CENTRAL

$$
\begin{tikzcd}
\mathbb{B}(A,B)\otimes\mathbb{B}(B,A) \arrow[r, "\mathrm{flip}"] \arrow[d, "m_{ABA}"] & \mathbb{B}(B,A)\otimes\mathbb{B}(A,B) \arrow[d, "m_{BAB}"] \\
\mathbb{B}(A,A) \arrow[d, "\circlearrowright"] & \mathbb{B}(B,B) \arrow[d, "\circlearrowright"] \\
T_A & T_B
\end{tikzcd}
$$

$$\mathbb{B}(A,B)\otimes\mathbb{B}(B,C)\otimes\mathbb{B}(C,A) \rightarrow \qquad \rightarrow$$

$\mathbb{B}(A,A) \circlearrowleft T_A$:

NC analog of contraction

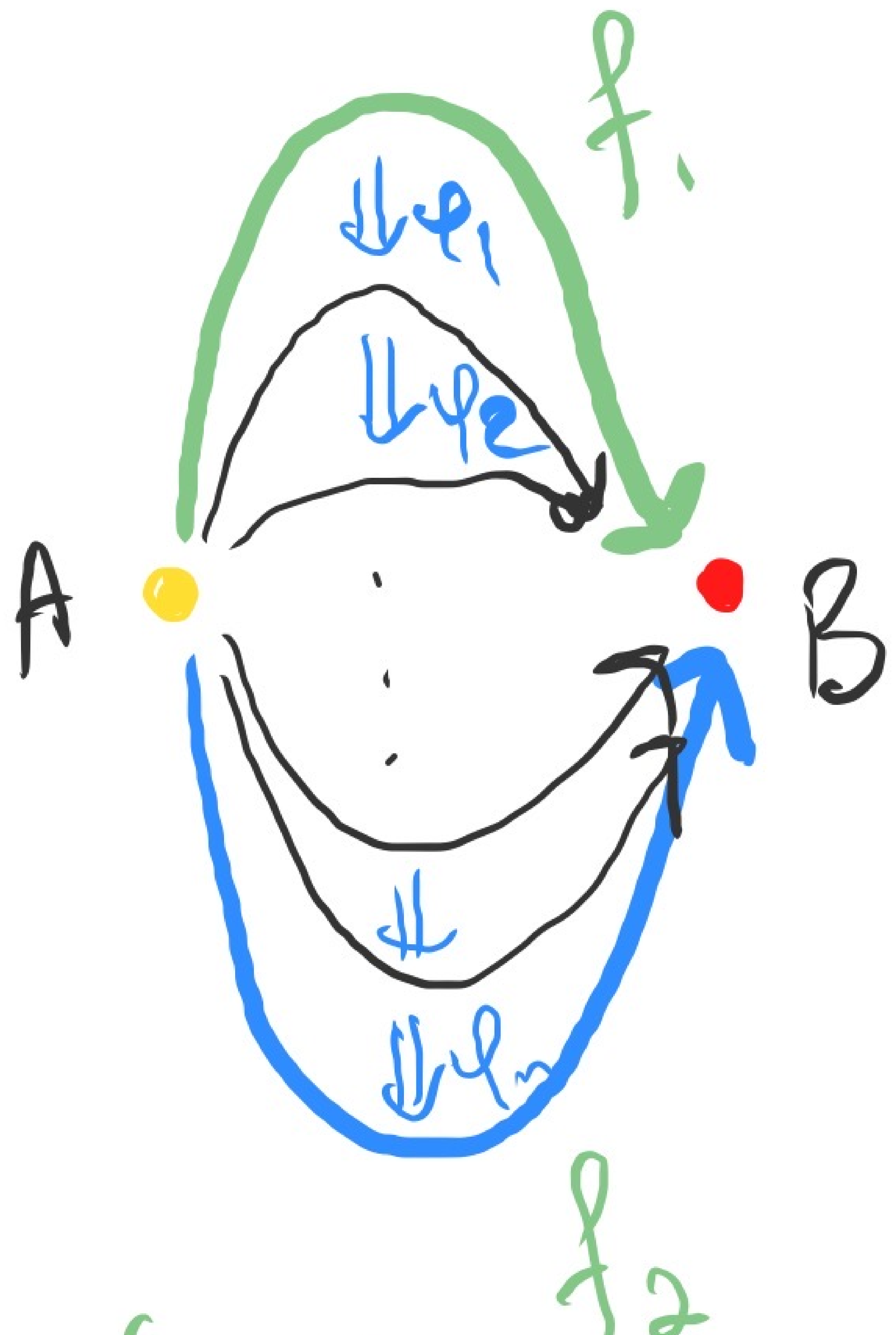
$\iota_\pi \alpha$ $\quad \pi \in \Lambda^1 T \quad \alpha \in \Omega$

τ_{AB} : $m^*_{ABA} T_A \sim m^*_{BAB} T_B$

NC analog of Lie derivative

$L_\pi \alpha$

σ_{ABC} : $\mathrm{id} \sim \tau^3$: NC analog of d_{DR} (a.k.a. B)

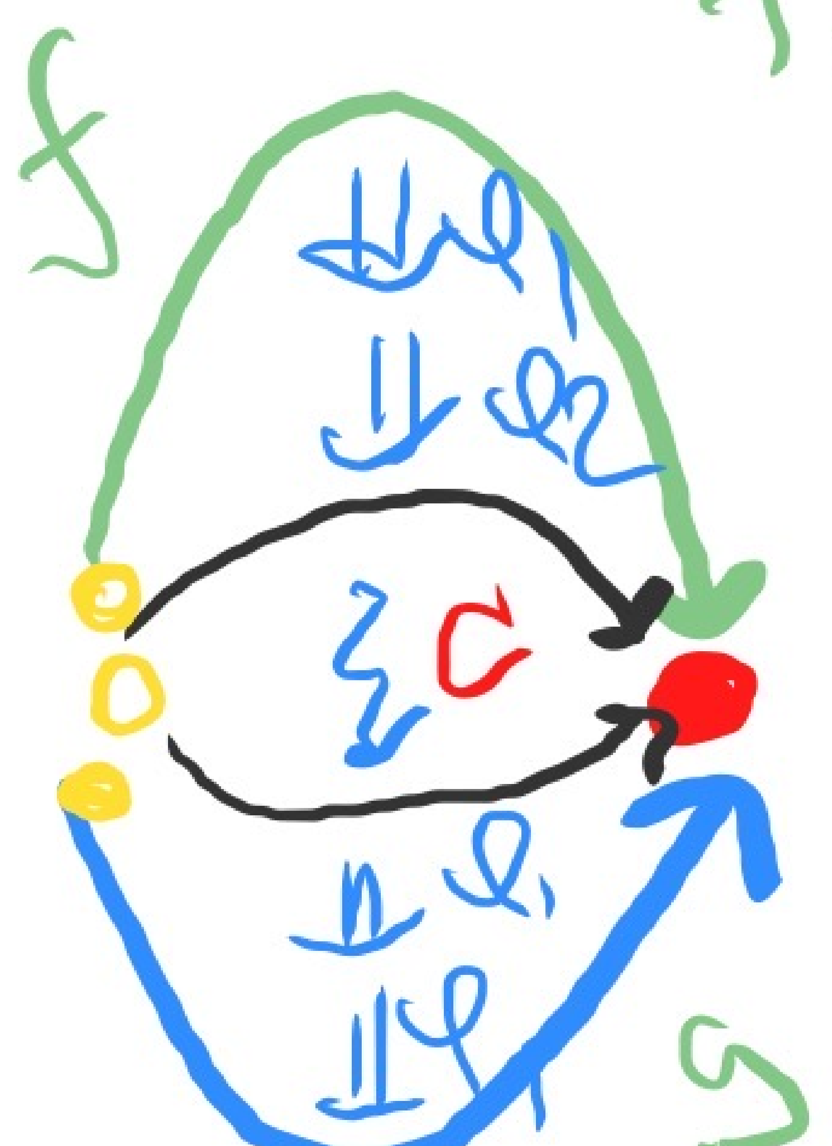


$\in \mathbb{B}(A,B)$ — dg cocategory

objects:

$f : A \to B$

Bar $\underbrace{C^\bullet(A,B)}_{\text{dg cat}}$

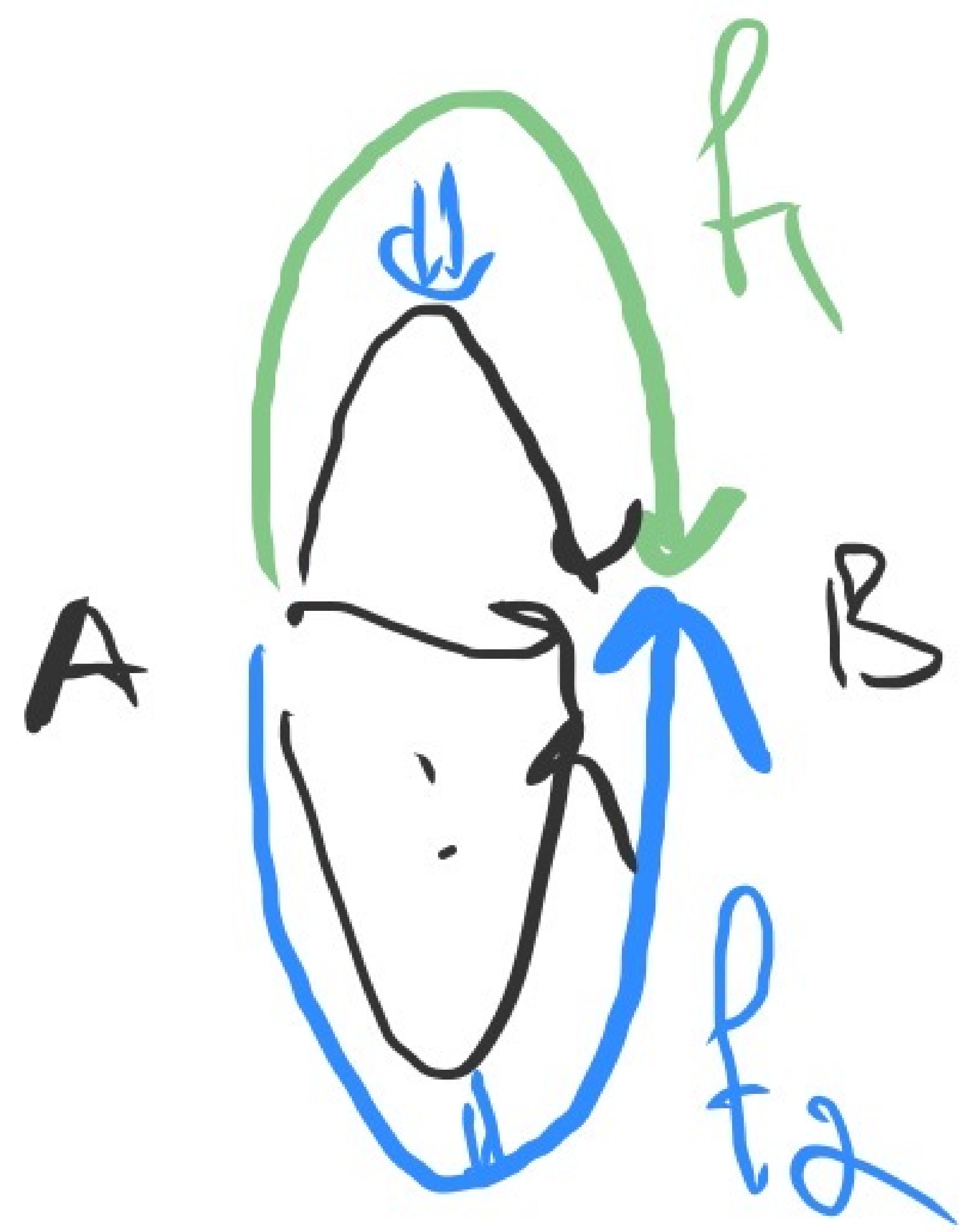


$\in M(A,B)$ — dg cobimodule

(bar of —)

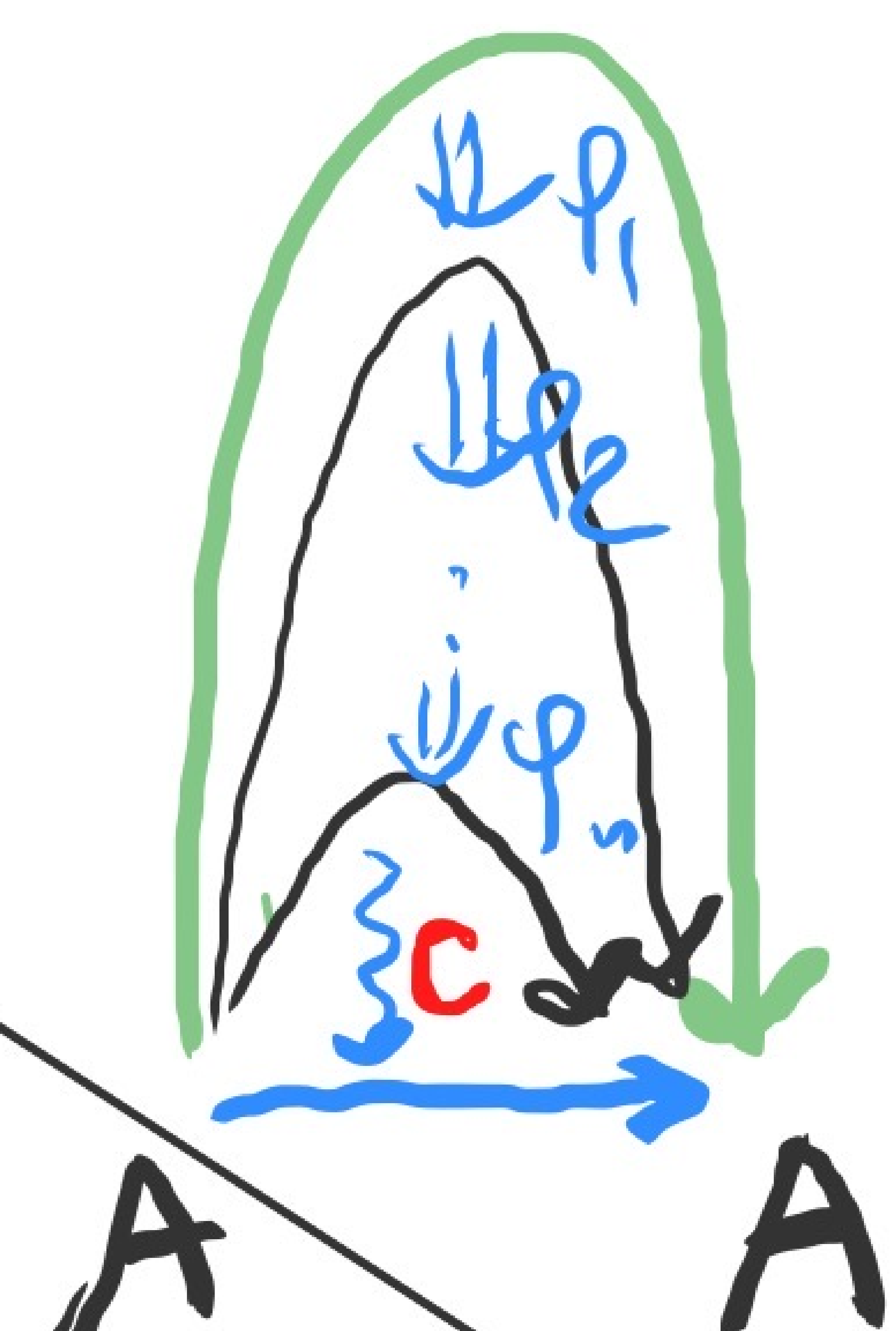
The trace functor:

still $\mathbb{B}(A,B)$



dg cocategory

$\mathbb{T}(A)$



dg comodule

Applying $\underbrace{\text{cobar}}$, get some version of a (homotopy)

Categ in Categs +

+ a TRACE functor

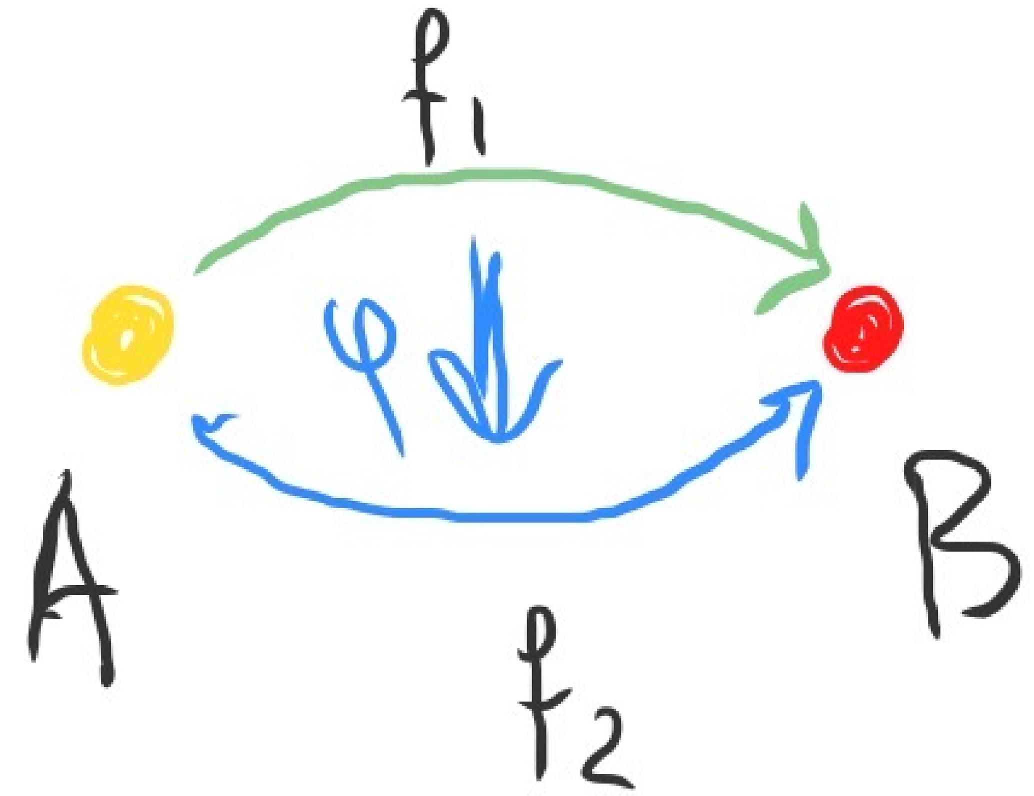
dfn due to Kaledin

example: NC Witt vectors

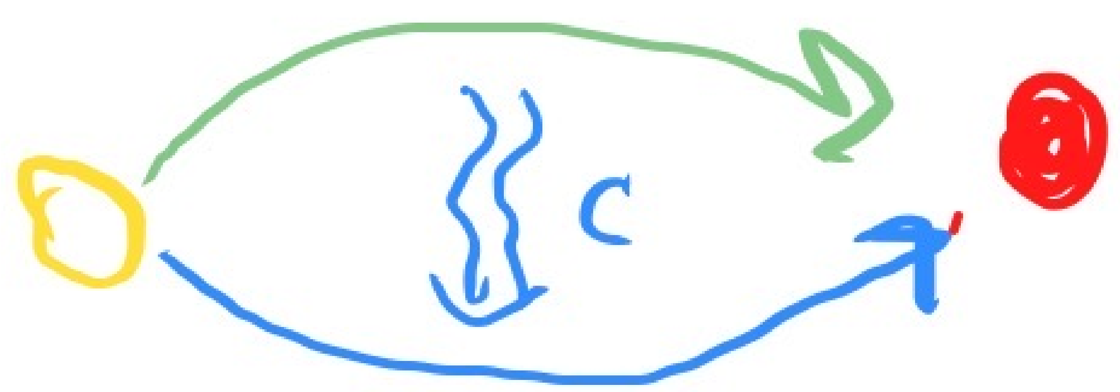
What about all
$$\begin{tikzcd} \circ \arrow[r, bend left] \arrow[r, bend right] & \circ \end{tikzcd}$$
$$C_\bullet(A, {}_f B_g)$$

not just

$$C_\bullet(A, {}_f A)\,?$$

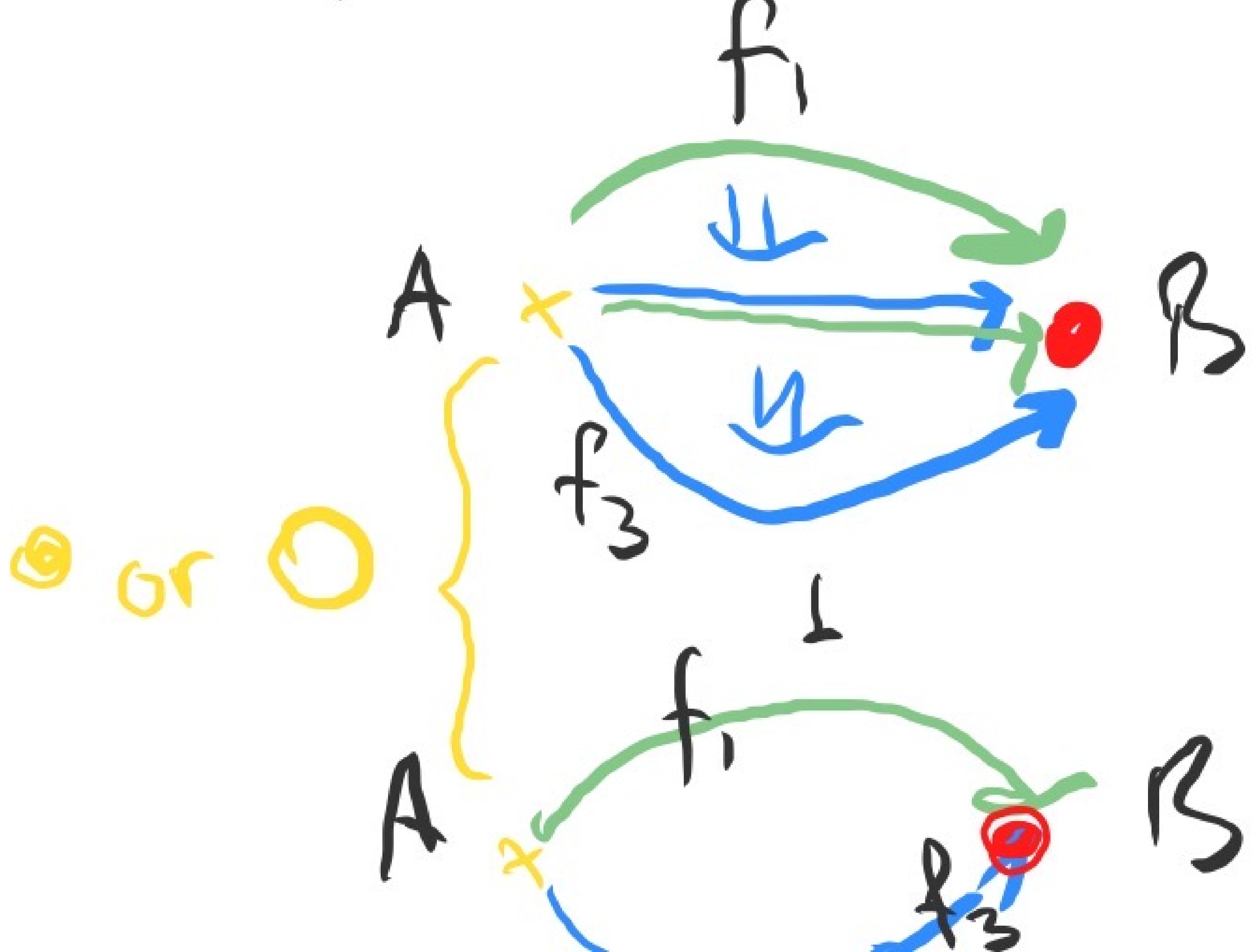


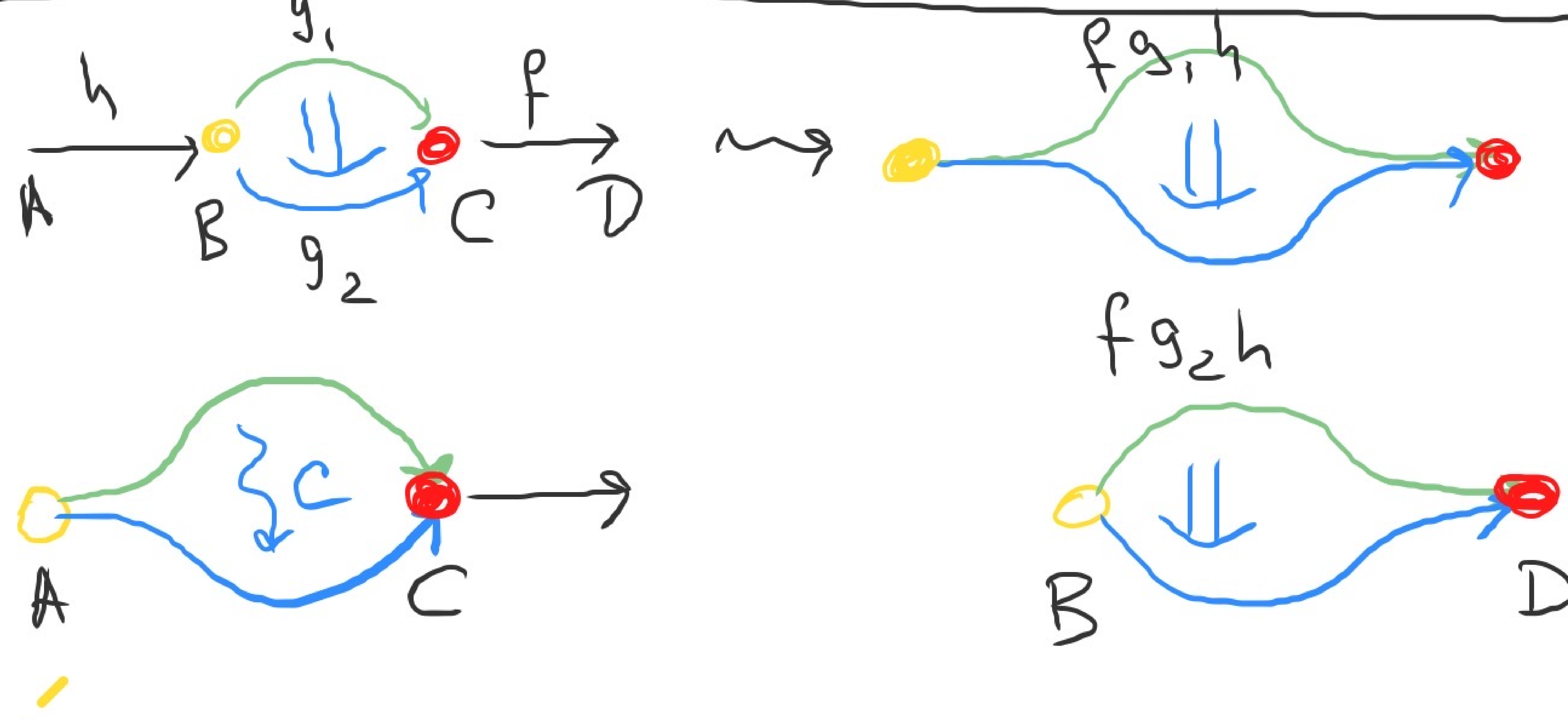
$$\varphi \in C^{\circ}(A, {}_{f_1}B_{f_2})$$



$$c \in C^{\cdot}(A, {}_{f_1}B_{f_2})$$

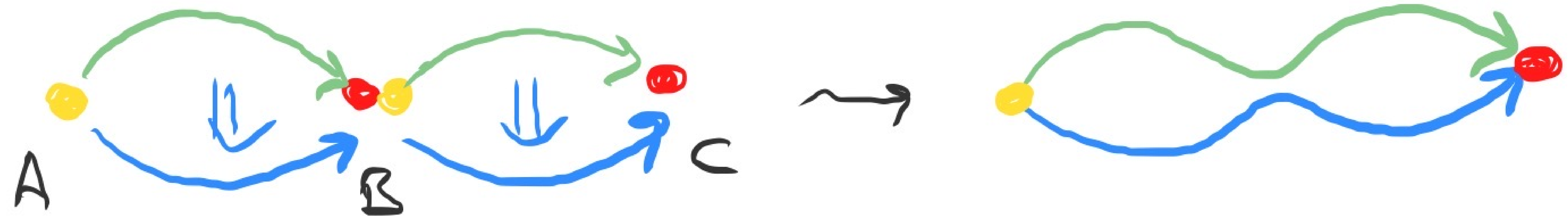
Operations:

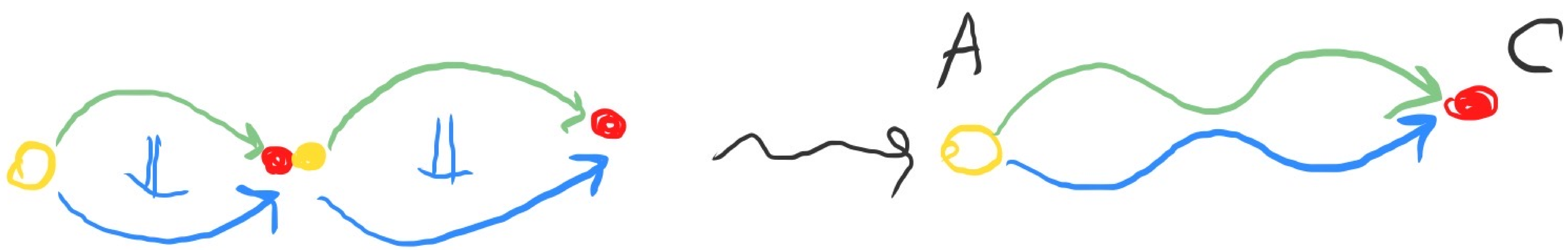


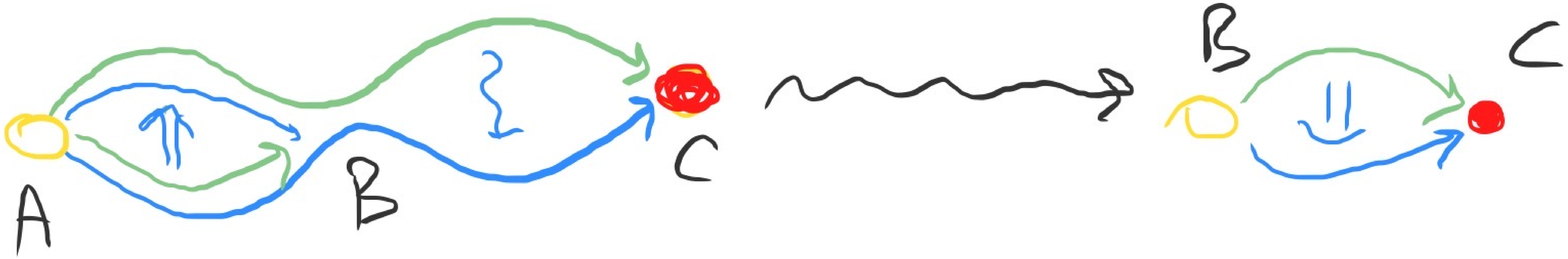


$\Rightarrow$ Two ways to compose in each case:

(homotopie)

1) 

2) 

3) 

Category in cocategories with a tetramodule

$A, B \to \mathbb{B}(A,B)$ dg cocategory

$\mathcal{M}(A,B)$ dg cobimodule

$$\mathbb{B}(A,B) \otimes \mathbb{B}(B,C) \xrightarrow{m_{ABC}} \mathbb{B}(A,C) \quad \text{assoc}$$

compatibilities
$$\left[\begin{array}{l} \mathcal{M}(A,B) \otimes \mathbb{B}(B,C) \to m^*_{ABC}\, \mathcal{M}(A,C) \\ \operatorname{cotr}_{\mathbb{B}(A,B)} m^*_{ABC}\, \mathcal{M}(A,C) \to \mathcal{M}(B,C) \end{array}\right.$$

Example: $\mathcal{M}(A,A)(\mathrm{id}_A, \mathrm{id}_A)$ is WHAT over the Hopf algebra $\mathbb{B}(A,A)(\mathrm{id}_A, \mathrm{id}_A)$?

Answer: twisted **tetramodule** (for untwisted: see B. Shoikhet).

U - Hopf algebra; S - antipode

U is an U-bimod in two ways:

1) $a \cdot u \cdot b = \begin{cases} aub & \text{— } U \\ S(b)uS(a) & \text{— } U^{+} \end{cases}$

a twisted tetramodule:

U-bimodule M and

$$\Delta : M \to U^{+} \otimes M \otimes U$$

morphism of bimodules

+ compatibilities

Motivation:

$$C^{\bullet}(A^{\otimes n}, {}_{\alpha}A^{\otimes n})$$

$$C_{\bullet}(A^{\otimes n}, {}_{\alpha}A^{\otimes n})$$

α - cyclic permut.

Kontsevich-Vlassopoulos bracket on those

Kaledin's Frobenius
(and others')

$$HH_{\bullet}(A, A) \xrightarrow{\simeq} HH_{\bullet}(A^{\otimes n}, {}_{\alpha}A^{\otimes n})$$

$$\begin{tikzcd}
A_1 \arrow[r, "f_{11}"] \arrow[d, "f_{12}"'] \arrow[dr, phantom, "\Phi"] & B_1 \\
B_2 & A_2 \arrow[l, "f_{22}"] \arrow[u, "f_{21}"']
\end{tikzcd}$$

$B_1 \otimes B_2$ is

: an $A_1 \otimes A_2$ - bimod

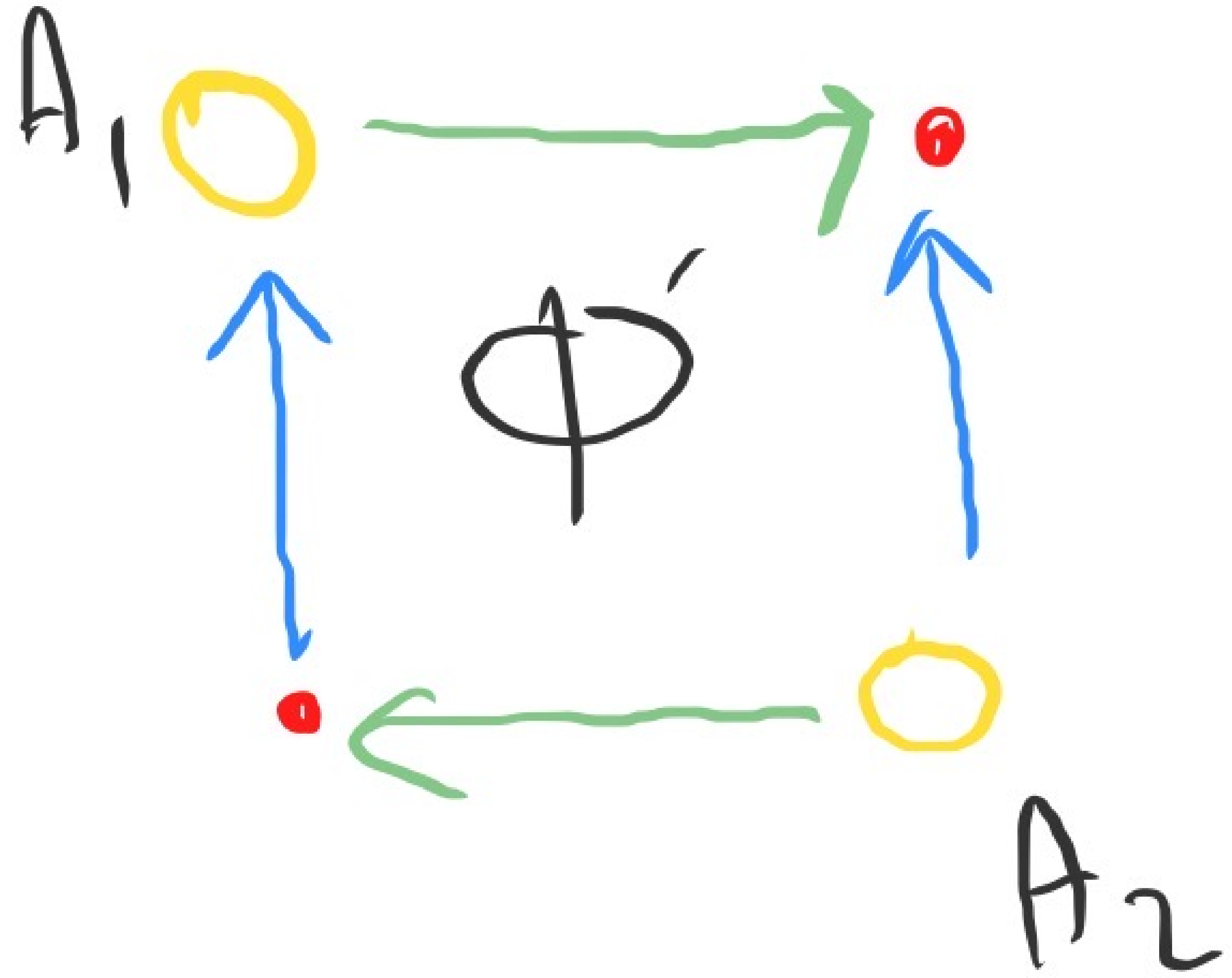
$\downarrow$ for right action

$\uparrow$ for left

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = f_{11}(a_1)b_1 \otimes f_{22}(a_2)b_2$$

$$(b_1 \otimes b_2)(a_1 \otimes a_2) = b_1 f_{21}(a_2) \otimes b_2 f_{12}(a_1)$$

$$C^{\bullet}(\Phi) = C^{\bullet}(A_1 \otimes A_2, B_1 \otimes B_2)$$

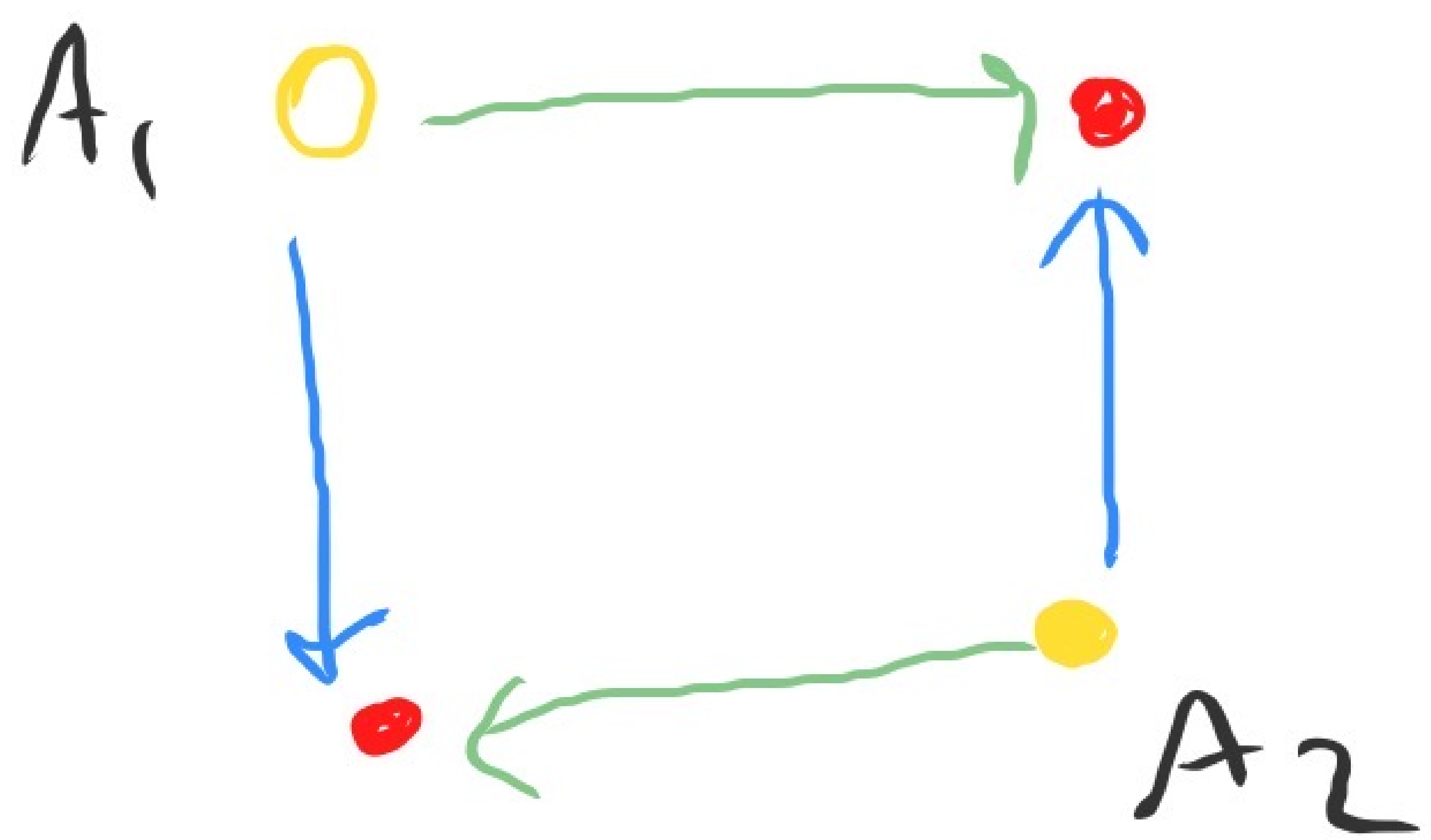


$$C_\circ(\phi') = C_\circ(A_1 \oplus A_2,\ B_1 \oplus B_2)$$

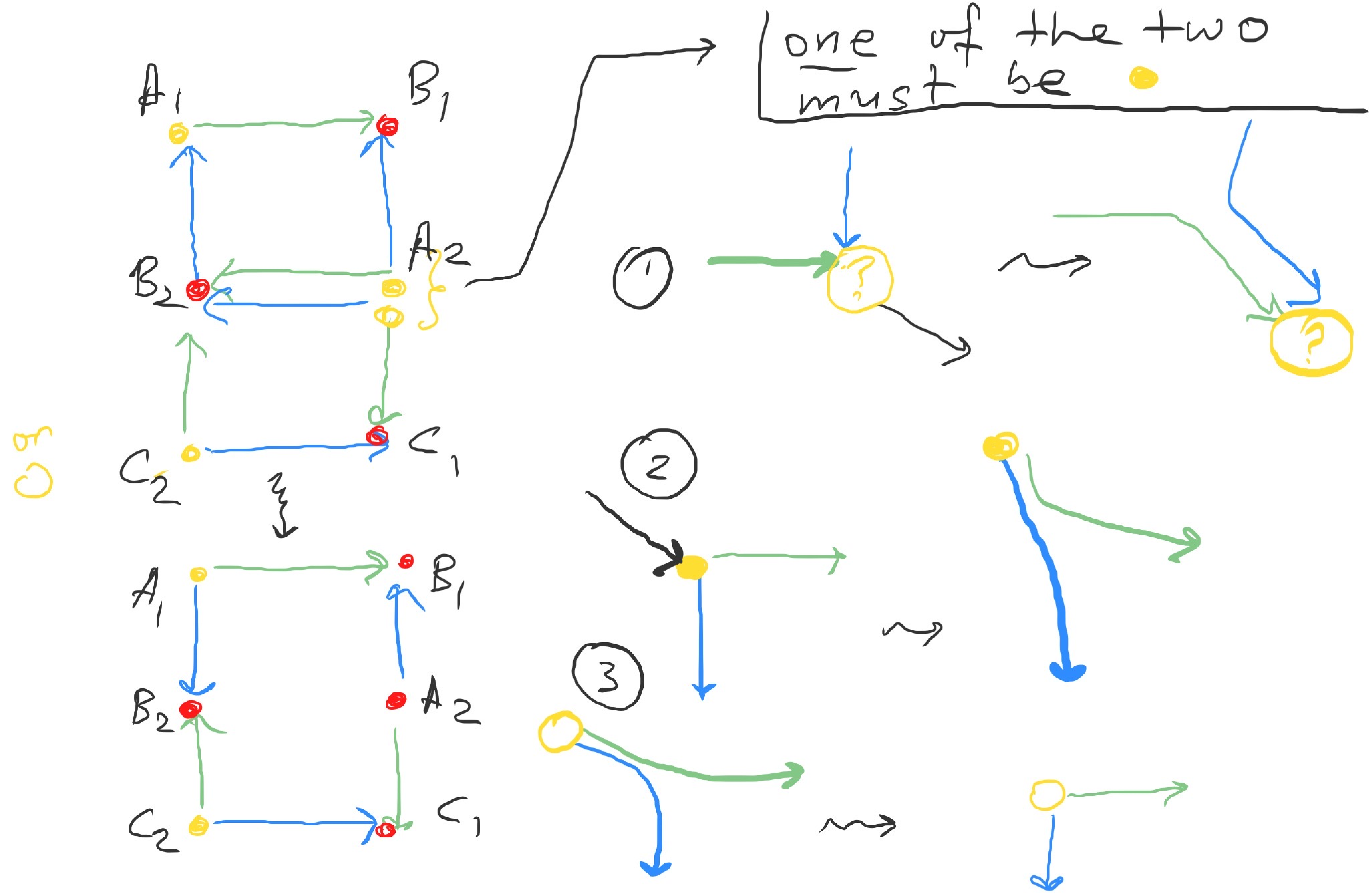
$\bullet$ for cochains

$\circ$ for chains

also hybrid pictures:



$\overset{\mathbb{L}}{\otimes} A_1$ combined with $C^\circ(A_2, -)$



one of the two must be
A1
B1
A2
B2
C1
C2
or
1
2
3
?

Claim: the cochain part extends very naturally;
"Category in (co)cats..."
chains?

The above ops,
+ Frobenius $\Longrightarrow$
? some new structure