Lie algebroids on (pre-mutli)symplectic manifolds and sigma models

Noriaki Ikeda
Ritsumeikan University, Kyoto, Japan

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§1. Introduction

Purpose

Higher generalizations of the twisted Poisson structure

Generalization of AKSZ sigma models

To do

Compatibility with Lie algebroids and (pre)-multisymplectic manifolds

Lie algebroid sigma models with WZ term

Plan of Talk

Geometry of Lie algebroids with multisymplectic manifolds

Q-manifold descriptions

Higher dimensional sigma sigma models

§2. Preliminary

Lie algebroids

Definition 1. *A Lie algebroid* $(E, \rho, [-,-])$ *is a vector bundle* E *over* M *with a bundle map* $\rho : E \to TM$ *called the anchor map, and a Lie bracket* $[-,-] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ *satisfying the Leibniz rule,*

$$[e_1, fe_2] \;=\; f[e_1, e_2] + \rho(e_1)f \cdot e_2,$$

where $e_i \in \Gamma(E)$ *and* $f \in C^\infty(M)$*.*

Example 1. [Lie algebra] *Let a manifold M be one point $M = \{pt\}$. Then it is a Lie algebra $\mathfrak{g}$.*

Example 2. [Tangent Lie algebroid] $E = TM$ *and* $\rho = \mathrm{id}$, $[-,-]$ *is a normal Lie bracket on the space of vector fields* $\mathfrak{X}(M)$.

Example 3. [Action Lie algebroid] *Assume a smooth action of a Lie group G, $M \times G \to M$. The differential of the map induces an infinitesimal action of the Lie algebra $\mathfrak{g}$ of G on the manifold M. It induces a bundle map $\rho : M \times \mathfrak{g} \to TM$. Consistency of a Lie bracket requires that ρ is a Lie algebra morphism such that*

$$[\rho(e_1), \rho(e_2)] \;=\; \rho([e_1, e_2]).$$

Lie algebroid differentials

$\Gamma(\wedge^\bullet E^*)$ is the space of E-differential forms.

Definition 2. *A Lie algebroid differential* ${}^E\mathrm{d} : \Gamma(\wedge^m E^*) \to \Gamma(\wedge^{m+1}E^*)$ *such that* $({}^E\mathrm{d})^2 = 0$ *is defined by*

$$
\begin{aligned}
{}^E\mathrm{d}\alpha(e_1, \ldots, e_{m+1}) &= \sum_{i=1}^{m+1} (-1)^{i-1} \rho(e_i)\alpha(e_1, \ldots, \check{e}_i, \ldots, e_{m+1}) \\
&+ \sum_{1\le i<j\le m+1} (-1)^{i+j} \alpha([e_i, e_j], e_1, \ldots, \check{e}_i, \ldots, \check{e}_j, \ldots, e_{m+1}),
\end{aligned}
$$

where $\alpha \in \Gamma(\wedge^m E^*)$ *and* $e_i \in \Gamma(E)$.

(Pre-)n-plectic structure

Definition 3. *A closed* $(n+1)$*-form* $\omega \in \Omega^{n+1}(M)$ *is called a pre-*n*-plectic form.* (M, ω) *is called a pre-*n*-plectic manifold.*

An 1-plectic structure is a symplectic structure.

§3. Compatible E-n-form

Notation

$\rho : E \to TM$ is regarded as $\rho \in \Gamma(TM \otimes E^*)$.
$\iota_\rho^k \omega \in \Omega^{n+1-k}(M, \wedge^k E^*)$ is defined by

$$\begin{aligned}
&\iota_\rho^k \omega(v_{k+1}, \ldots, v_{n+1})(e_1, \ldots, e_k) \\
&= \iota_{\rho(e_1)} \ldots \iota_{\rho(e_k)} \omega\big(v_{k+1}, \ldots, v_{n+1}\big) \\
&= \langle \otimes^k \rho, \, \omega \rangle \big(e_k, \ldots, e_1, v_{k+1}, \ldots, v_{n+1}\big) \\
&:= \omega(\rho(e_k), \ldots, \rho(e_1), v_{k+1}, \ldots, v_{n+1}).
\end{aligned}$$

for $e_1, \ldots, e_k \in \Gamma(E)$ and $v_{k+1}, \ldots, v_{n+1} \in \mathfrak{X}(M)$.

Definition 4. *Let E be a Lie algebroid on a pre-n-plectic manifold (M, ω). If an E-n-form $J \in \Gamma(\wedge^n E^*)$ satisfies*

$$ {}^{E}\mathrm{d}J = -\iota_{\rho}^{n+1}\omega (= -\langle \otimes^{n+1}\rho, \omega \rangle), \tag{1} $$

J is called $compatible$ with the Lie algebroid structure and the pre-n-plectic structure.

Since ${}^{E}\mathrm{d}^2 = 0$, ${}^{E}\mathrm{d}(\iota_{\rho}^{n+1}\omega) = 0$ must be satisfied for consistency.

Proposition 1. ${}^{E}\mathrm{d}(\iota_{\rho}^{n+1}\omega) = 0$ *is satisfied if* $\mathrm{d}\omega = 0$ *and* $[\rho(e_1), \rho(e_2)] = \rho([e_1, e_2])$.

Lemma 1. (1) *is consistent with a Lie algebroid.*

§4. Examples

Example 4. [Twisted Poisson structure] *Klimcik-Strobl '01, Park '00, Ševera-Weinstein '01*
Let $\pi \in \Gamma(\wedge^2 TM)$ *and* $H \in \Omega^3(M)$ *be a closed* 3*-form. If* (π, H) *satisfies*

$$\frac{1}{2}[\pi, \pi]_S = \langle \otimes^3 \pi,\ H \rangle, \tag{2}$$

it is called a twisted Poisson structure.

For given (π, H)*,* T^*M *is a Lie algebroid. The anchor map is*

$\pi^\sharp : T^*M \to TM$ *and the Lie bracket is*

$$[\alpha, \beta]_{\pi, H} = \mathcal{L}_{\pi^\sharp(\alpha)}\beta - \mathcal{L}_{\pi^\sharp(\beta)}\alpha - \mathrm{d}(\pi(\alpha, \beta)) + \iota_{\pi^\sharp(\alpha)}\iota_{\pi^\sharp(\beta)}H.$$

(2) *is equivalent to*

$${}^{E}\mathrm{d}\pi = -\langle \otimes^3 \pi,\ H \rangle.$$

$J = \pi$ *with the pre-*2*-plectic form* $\omega = H$*.*

Example 5. [twisted R-Poisson structure] *Chatzistavrakidis '21*

Let $\pi \in \Gamma(\wedge^2 TM)$ *be a Poisson bivector field,* $H \in \Omega^{n+1}(M)$ *be a closed* $(n+1)$*-form, and* $R \in \Gamma(\wedge^n TM)$*.*

Under the Lie algebroid structure on T^*M *induced from the Poisson bivector field* π*,* (π, H, R) *is called a twisted* R*-Poisson structure if*

$$[\pi, R]_S = (-1)^n \langle \otimes^{n+1} \pi,\, H \rangle.$$

The equation is equivalent to ${}^E\mathrm{d}R = (-1)^n \langle \otimes^{n+1} \pi,\, H \rangle$*. Thus,* $R = J$ *is a compatible* E*-*n*-form with the pre-*n*-plectic form* $\omega = (-1)^{n+1} H$*.*

Example 6. [Momentum map] *Let* (M, ω) *be a symplectic manifold* $(n = 1)$ *with an action of a Lie group* G. *The action induces an action Lie algebroid structure on* $E = M \times \mathfrak{g}$ *with a Lie algebra* $\mathfrak{g}$ *of* G *with the action* $\rho : M \times \mathfrak{g} \to TM$. *We take* $\nabla = \mathrm{d}$. $\mu_0 \in \Gamma(M, M \times \mathfrak{g}^*)$ *is a momentum map if*

$$\mathrm{d}\mu_0(e) = -\iota_{\rho(e)}\omega, \qquad \mu_0([e_1, e_2]) = \rho(e_1)\mu_0(e_2).$$

for $e, e_1, e_2 \in \mathfrak{g}$. *They are equivalent to*

$$\mathrm{d}\mu_0(e) = -\iota_{\rho(e)}\omega, \qquad {}^E\mathrm{d}\mu_0(e_1, e_2) = -\iota_\rho^2\omega(e_1, e_2).$$

Example 7. [Homotopy moment(um) map] *Callies-Fregier-Rogers-Zambon '13 Let (M,ω) be an n-plectic manifold. Assume an action of a Lie group G on M.*

Let Let $\mu = \sum_{k=0}^{n-1} \mu_k$ with $\mu_k \in \Omega^k(M, \wedge^{n-k}\mathfrak{g}^*)$, where $k = 0, \ldots, n-1$.

μ is a homotopy momentum map if it satisfies

$$(\mathrm{d} + \mathrm{d}_{CE})\mu = -\sum_{k=0}^{n-1} (\iota_\rho)^k \omega.$$

Here d_{CE} is the Chevalley-Eilenberg differential on $\wedge^\bullet \mathfrak{g}^*$.

The 0*-form part of the equation is* $\mathrm{d}_{CE}\mu_0 = -(\iota_\rho)^{n+1}\omega$. *It is equivalent to*

$$^{E}\mathrm{d}\mu_0 = -(\iota_\rho)^{n+1}\omega,$$

if we use equations for higher order μ_k. $J = -\mu_0$ *is a compatible* E*-*n*-form.*

Example 8. [Momentum section] *Blohmann-Weinstein '18, Kotov-Strobl '16*

(M, ω) *is a pre-symplectic manifold and* $(E, \rho, [-,-])$ *is a Lie algebroid over* M. *Assume a connection* ∇ *on* E.

Definition 5. *A section* $\mu \in \Gamma(E^*)$ *is called a* $momentum\ section$ *if* $\mu \in \Gamma(E^*)$ *satisfies the following two conditions,*

$$\nabla\mu \;=\; -\iota_\rho\omega, \qquad {}^{E}\mathrm{d}\mu = -(\iota_\rho)^2\omega.$$

The second condition is (1) *for* $n = 1$ *with* $\mu = J$.

Example 9. [Homotopy momentum section] *Hirota-NI '21*

(M, ω) *is a pre-n-plectic manifold and* $(E, \rho, [-,-])$ *is a Lie algebroid over* M. *Let* $\mu_k \in \Omega^k(M, \wedge^{n-k}E^*)$, *where* $k = 0, \ldots, n-1$.

Definition 6. *A sum* $\mu = \sum_{k=0}^{n-1} \mu_k$ *is called a homotopy momentum section if* μ *satisfies*

$$(\nabla + {}^E\mathrm{d}^{\nabla})\mu = -\sum_{k=0}^{n} \iota_\rho^{n+1-k}\omega.$$

μ_0 *satisfies* ${}^E\mathrm{d}\mu_0 = -\iota_\rho^{n+1}\omega$. *Thus,* $\mu_0 = J$.

§5. Higher Dirac structure

Graded manifold

A nonnegatively graded manifolds is called an N-manifold.

Definition 7. *If an N-manifold $\mathcal{M}$ has a vector field Q of degree $+1$ satisfying $Q^2 = 0$, it is called a Q-$manifold$.*

Q-manifold description of Compatible E-n-form

We consider $\mathcal{M} = T^*[n-1]E[1]$.

We take local coordinates on $T^*[n-1]E[1]$, (x^i, a^a, z_i, y_a) of degree $(0, 1, n-1, n-2)$.

$$\rho(e_a) := \rho_a^i(x)\partial_i \qquad J(e_{a_1}, \cdots, e_{a_n}) := \frac{1}{n!} J_{a_1 \dots a_n}(x),$$
$$[e_a, e_b] := C_{ab}^c(x) e_c, \qquad (\omega :=) H = \frac{1}{n!} H_{i j_1 \dots j_n}(x) \mathrm{d}x^{j_1} \dots \mathrm{d}x^{j_n},$$

for the basis e_a of $\Gamma(E)$,

We define

$$
\begin{aligned}
Q = {} & \rho^i_a(x) a^a \frac{\partial}{\partial x^i} + \frac{(-1)^n}{2} C^a_{bc}(x) a^b a^c \frac{\partial}{\partial a^a} \\
& + \left((-1)^n \rho^i_a z_i + C^c_{ab}(x) a^b y_c + J_{ab_2 \ldots b_n} a^{b_2} \ldots a^{b_n}\right) \frac{\partial}{\partial y_a} \\
& + (-1)^n \Bigg(\partial_i \rho^j_a z_j a^a - \frac{1}{2} \partial_i C^a_{bc}(x) a^b a^c y_a \\
& + \frac{1}{n!} (\partial_i J_{a_1 \ldots a_n} - \rho^{j_1}_{a_1} \ldots \rho^{j_n}_{a_n} H_{i j_1 \ldots j_n}) a^{a_1} \ldots a^{a_n} \Bigg) \frac{\partial}{\partial z_i},
\end{aligned}
$$

Proposition 2. *$Q^2 = 0$ is equivalent to the condition of the compatible E-n-form under a Lie algebroid E.*

QP-manifold

Definition 8. *If an N-manifold $\mathcal{M}$ has a graded symplectic form ω_{grad} of degree n and a vector field Q of degree $+1$ satisfying $Q^2 = 0$ such that $\mathcal{L}_Q \omega_{grad} = 0$, it is called a QP-$manifold$.*

For any QP-manifold of degree $n \neq 0$, there exists a function $\Theta \in C^\infty(\mathcal{M})$ such that $Q = \{\Theta, -\}$ satisfying

$$\{\Theta, \Theta\} = 0.$$

Note: QP-manifolds $\to$ AKSZ sigma models

Note: If $H \neq 0$, the previous Q is not QP since $\mathcal{L}_Q \omega_{grad} \neq 0$.

Higher Dirac structure

Hagiwara '02, Wade '02, NI-Uchino '10, Zambon '12, Bi-Sheng '15, Bursztyn-Martinez-Rubio '16, Cueca '19,,,

QP-manifold

Choose the canonical graded symplectic form of degree n on $\mathcal{M} = T^*[n-1]E[1]$ as

$$\omega_{grad} = \delta x^i \wedge \delta z_i + \delta a^a \wedge \delta y_a,$$

where δ is the graded de Rham differential.

Define

$$\Theta = \rho_a^i(x) z_i a^a + \frac{1}{2} C_{bc}^a(x) a^b a^c y_a + \frac{1}{(n+1)!} \rho_{a_1}^{i_1} \ldots \rho_{a_{n+1}}^{i_{n+1}} H_{i_1 \ldots i_{m+1}}(x) a^{a_1} \ldots a^{a_{n+1}}.$$

Θ gives a Lie n-algebroid structure on $E \oplus \wedge^{n-1} E^*$.

E is a Lie algebroid and $\mathrm{d}H = 0 \Rightarrow \{\Theta, \Theta\} = 0$.

Lie n-algebroid induced from QP-manifold

A Lie n-algebroid on $E \oplus \wedge^{n-1} E^*$ is an algebroid with three operations, $((-,-), \rho, [-,-]_D)$.

$(-,-) : \Gamma(E \oplus \wedge^{n-1}E^*) \otimes \Gamma(E \oplus \wedge^{n-1}E^*) \to \Gamma(\wedge^{n-2}E^*)$ is a symmetric paring. The bundle map $\rho : E \oplus \wedge^{n-1}E^* \to TM$ is the anchor map, and the bilinear bracket $[-,-]_D : \Gamma(E \oplus \wedge^{n-1}E^*) \times \Gamma(E \oplus \wedge^{n-1}E^*) \to \Gamma(E \oplus \wedge^{n-1}E^*)$ is called the (higher) Dorfman bracket.

A map $j_* : \Gamma(E \oplus \wedge^{n-1}E^*) \to (C_0^\infty \oplus C_1^\infty)(T^*[n-1]E[1])$ is induced from the map

$$j : E \oplus \wedge^{n-1}E^* \oplus TM \to T^*[n-1]E[1],$$

$j : (x^i, e^a, e_a, \partial_i) \mapsto (x^i, a^a, y_a, z_i)$, where ∂_i, e^a and e_a is basis of TM, E^* and E.

Operations are given by derived brackets,

$$
\begin{aligned}
(e_1, e_2) &= j^*\{\underline{e}_1, \underline{e}_2\}, \\
\rho(e)f &= j^*\{\{\underline{e}, \Theta\}, f\}, \\
[e_1, e_2]_D &= j^*\{\{\underline{e}_1, \Theta\}, \underline{e}_2\},
\end{aligned}
$$

for $e, e_1, e_2 \in \Gamma(E \oplus \wedge^{n-1}E^*)$, $\underline{e}, \underline{e}_1, \underline{e}_2 \in C^\infty(T^*[n-1]E[1])$ and $f \in C^\infty(M)$.

Three operations of this Lie n-algebroid are as follows. Let $u+\alpha, v+\beta \in \Gamma(E \oplus \wedge^{n-1}E^*)$, where $u, v \in \Gamma(E)$ and $\alpha, \beta \in \Gamma(\wedge^{n-1}E^*)$.

$$
\begin{aligned}
(u+\alpha, v+\beta) &= (u, \beta) + (\alpha, v), \\
\rho(e)f &= \rho(u)f, \\
[u+\alpha, v+\beta]_D &= [u, v] + \mathcal{L}_u\beta - \iota_v{}^E\mathrm{d}\alpha + \iota_u\iota_v(\iota_\rho^{n+1}H),
\end{aligned}
$$

where the interior product ι_v is the contraction with respect to E and E^*, and the Lie derivative is $\mathcal{L}_u = \iota_u{}^E\mathrm{d} + {}^E\mathrm{d}\iota_u$.

Higher Dirac structure

Definition 9. *A Lagrangian Q-submanifold $\mathcal{N}$ is a sub graded manifold satisfying the conditions, $(e_1, e_2) = 0$ for all $e_1, e_2 \in C^\infty(\mathcal{N})$, and $[C^\infty(\mathcal{N}), C^\infty(\mathcal{N})]_D \subset C^\infty(\mathcal{N})$.*

A higher Dirac structure is a subbundle L of $E \oplus \wedge^{n-1} E^*$ induced from a Lagrangian Q-submanifold.

Proposition 3. [NI] *Let $J \in \Gamma(\wedge^n E^*)$. Define*

$$\Gamma(L_J) = \{u + (J, u) \in \Gamma(E \oplus \wedge^{n-1} E^*) | u \in \Gamma(E)\}.$$

Then, ${}^E\mathrm{d}J = -(\iota_\rho)^{n+1} H$ iff L_J is a higher Dirac structure.

§6. Lie algebroid sigma model with WZ term

Action functional

Chatzistavrakidis ' 21, NI '21

Let Ξ be an $n+1$ dimensional manifold with n dimensional boundary, $\Sigma = \partial\Xi$. Choose a Lie algebroid E over a d-dimensional target space M.

$\langle -, - \rangle$: pairing of TM and T^*M.
$(-,-)$: pairing of E and E^*.

$X : \Xi \to M$ is a smooth map.
$A \in \Omega^1(\Sigma, X^*E)$, $Y \in \Omega^{n-2}(\Sigma, X^*E^*)$, $Z \in \Omega^n(\Sigma, X^*T^*M)$.

The action functional is as follows,

$$S = \int_\Sigma \Big[\langle Z, \mathrm{d}X \rangle + (Y, \mathrm{d}A) - \langle Z,\, X^* \rho(A) \rangle + \frac{1}{2}(Y, X^*[A, A])$$

$$+ X^* J(A, \ldots, A) \Big] + \int_\Xi X^* H.$$

$$= \int_\Sigma \Big[Z_i \wedge \mathrm{d}X^i + Y_a \wedge \mathrm{d}A^a - \rho_a^i(X) Z_i \wedge A^a + \frac{1}{2} C_{ab}^c(X) Y_c \wedge A^a \wedge A^b$$

$$+ \frac{1}{n!} J_{a_1 \ldots a_n}(X) A^{a_1} \wedge \ldots \wedge A^{a_n} \Big]$$

$$+ \int_\Xi \frac{1}{(n+1)!} H_{i_1 \ldots i_{n+1}}(X) \mathrm{d}X^{i_1} \wedge \ldots \wedge \mathrm{d}X^{i_{n+1}}.$$

Note: If $H = 0$, it is an AKSZ sigma model.

Hamiltonian formalism

Take $\Sigma = \boldsymbol{R} \times T^n$.

Let $\mathcal{H}$ be a Hamiltonian such that $\mathcal{H} = \int_{T^n}(p_A \dot{q}^A - \mathcal{L})$, where $q = (X, A, Y, Z)$.

The symplectic form is given by

$$\omega_{cl} = \int_{T^n} \delta q^A \wedge \delta p_A = \omega_{AKSZ-BFV}|_0.$$

The space of constraints is $\mathcal{E} = \{G_I | I = 1, 2, \ldots, m\}$, which gives a Lie algebroid structure as a Poisson algebra. The dynamics is consistent with constraints if

$$\{\mathcal{H}, \mathcal{E}\}_{PB} \subset \mathcal{E},$$
$$\{\mathcal{E}, \mathcal{E}\}_{PB} \subset \mathcal{E}.$$

Then G_I are called first class constraints.

The Hamiltonian is proportional to constraints,

$$\mathcal{H} = \int_{T^n} d^{n+1}\sigma (Z_{0i} G_X^i + Y_{0a} G_A^a + A_0^a G_{Ya}).$$

Here G's are constraints without time derivatives,

$$
\begin{aligned}
G_X^i &:= (\mathrm{d}X^i - \rho_a^i(X)A^a)^{(s)}\,, \\
G_A^a &:= (\mathrm{d}A^a + \tfrac{1}{2}C_{bc}^a(X)A^b \wedge A^c)^{(s)}\,, \\
G_{Ya} &:= \Big(\mathrm{d}Y_a + (-1)^n \rho_a^i(X)Z_i + (-1)^{n-1}C_{ab}^c(X)Y_c \wedge A^b \\
&\qquad + \frac{1}{n!}J_{ab_2\ldots b_{n+1}}(X)A^{b_2} \wedge \ldots \wedge A^{b_{n+1}}\Big)^{(s)}\,,
\end{aligned}
$$

which are spatial parts of equations of motion.

Theorem 1. *Suppose that E is a Lie algebroid and $\omega = H$ is a pre-n-plectic form. Then, G^i_X, G^a_A and G_{Ya} are the first class constraints, $\{\mathcal{H}, \mathcal{E}\}_{PB} \subset \mathcal{E}$ and $\{\mathcal{E}, \mathcal{E}\}_{PB} \subset \mathcal{E}$ if and only if J is a compatible E-n-form* (1)*.*

BFV works! In fact, we obtain the following Poisson brackets,

$$\{G^i_X(\sigma), G^j_X(\sigma')\}_{PB} = \{G^i_X(\sigma), G^a_A(\sigma')\}_{PB} = \{G^a_A(\sigma), G^b_A(\sigma')\}_{PB} = 0,$$

$$\{G^i_X(\sigma), G_{Ya}(\sigma')\}_{PB} = (-1)^{n-1}\partial_j \rho^i_a G^j_X(\sigma)\delta^n(\sigma - \sigma'),$$

$$\{G^a_A(\sigma), G_{Yb}(\sigma')\}_{PB} = (-1)^n[\partial_i C^a_{bc} A^c \wedge G^i_X(\sigma) + C^a_{bc} G^c_A(\sigma)]^{(s)} \times \delta^n(\sigma - \sigma'),$$

$$
\begin{aligned}
&\{G_{Ya}(\sigma), G_{Yb}(\sigma')\}_{PB} = \Big[(\partial_i C_{ab}^c Y_c \\
&+\frac{(-1)^{n-1}}{n!}\partial_i J_{abc_3\ldots c_{n+1}} A^{c_3}\wedge\ldots\wedge A^{c_{n+1}}\Big) \wedge G_X^i \\
&+(-1)^{n-1} C_{ab}^c G_{Yc} + \frac{(-1)^{n-2}}{(n-1)!} J_{abce_4\ldots e_{n+1}} A^{e_4}\wedge\ldots\wedge A^{e_{n+1}}\wedge G_A^c \\
&+\frac{(-1)^{n-1}}{(n+1)!}\sum_{m=1}^{n}\rho_a^i\rho_b^j H_{ijk_1\ldots k_m k_{m+1}\ldots k_n}\mathrm{d}X^{k_1}\wedge\ldots\wedge \mathrm{d}X^{k_{m-1}}\wedge G_X^{k_m} \\
&\wedge\rho_{c_{m+1}}^{k_{m+1}} A^{c_{m+1}}\wedge\ldots\wedge\rho_{c_n}^{k_n} A^{c_n}\Big]^{(s)}(\sigma)\delta^n(\sigma-\sigma'),
\end{aligned}
$$

Here all the fields are spatial components.

§7. Lagrangian formalism

Consistency of gauge transformations

A gauge transformation of a field Φ is computed by $\delta\Phi = \left\{\int \mathrm{d}\sigma' \epsilon^I(\sigma') G_I(\sigma'), \Phi(\sigma)\right\}_{PB} + \tau^I(\Phi(\sigma)) G_I(\Phi(\sigma))$. Gauge transformations are consistent if

$$\delta S = 0, \qquad [\delta_1, \delta_2] \sim \delta_3$$

We need three gauge parameters, $c^a \in \Gamma(\Sigma, X^*E)$, $t_a \in \Gamma(\wedge^{n-2}T^*\Sigma, X^*E^*)$, $w_i \in \Gamma(\wedge^{n-1}T^*\Sigma, X^*T^*M)$.

Gauge transformations of fundamental fields are given by

$$\delta X^i = \rho_a^i(X) c^a, \qquad \delta A^a = \mathrm{d}c^a + C_{bc}^a(X) A^b c^c,$$

$$\delta Y_a = \mathrm{d}t_a + (-1)^n \rho_a^i(X) w_i + C_{ab}^c(X)(-Y_c c^b + (-1)^n t_c \wedge A^b)$$

$$+ \tfrac{(-1)^n}{(n-1)!} J_{ab_2 \ldots b_{n+1}}(X) A^{b_2} \wedge \ldots \wedge A^{b_n} c^{b_{n+1}},$$

$$\delta Z_i = \mathrm{d}w_i + \partial_i \rho_a^j(-Z_j \wedge c^a + (-1)^n w_j \wedge A^a)$$

$$+ \tfrac{1}{2} \partial_i C_{bc}^a (2Y_a \wedge A^b c^c + (-1)^n t_a \wedge A^b \wedge A^c)$$

$$+ \tfrac{1}{n!} \partial_i J_{a_1 \ldots a_{n+1}}(X) A^{a_1} \wedge \ldots \wedge A^{a_n} c^{a_{n+1}}$$

$$-\tfrac{1}{(n+1)!} H_{ij_1 \ldots j_n k} \sum_{m=0}^{n} \mathrm{d}X^{j_1} \wedge \ldots \wedge \mathrm{d}X^{j_m}$$

$$\wedge \rho_{a_{m+1}}^{j_{m+1}} A^{a_{m+1}} \wedge \ldots \wedge \rho_{a_n}^{j_n} A^{a_n} \rho_b^k c^b.$$

Theorem 2. *Suppose that E is a Lie algebroid and $\omega = H$ is a pre-n-plectic form. Then, the action functional is gauge invariant and the gauge algebra is closed, $\delta S = 0, [\delta_1, \delta_2] \sim \delta_3$ if and only if J is a compatible E-n-form* (1).

Covariant gauge transformations

Let ∇ be a connection on E.

Definition 10. *An E-$connection$ on TM with respect to the Lie algebroid E is a map ${}^E\nabla : \Gamma(TM) \to \Gamma(TM \otimes E^*)$ satisfying ${}^E\nabla_e(fv) = f\,{}^E\nabla_e v + (\rho(e)f)v$, for $e \in \Gamma(E)$, $v \in \Gamma(TM)$ and $f \in C^\infty(M)$.*

If a normal connection ∇ on E is given, a (canonical) E-connection on a tangent bundle, an E-connection is given by

$$ {}^{E}\nabla_e v \;:=\; \mathcal{L}_{\rho(e)} v + \rho(\nabla_v e) = [\rho(e), v] + \rho(\nabla_v e), $$

where $e \in \Gamma(E)$ and $v \in \mathfrak{X}(M)$. Additional to the ordinary curvature, $R(e,e') := [\nabla_e, \nabla_{e'}] - \nabla_{[e,e']}$, the E-torsion T and the basic curvature S are defined as

$$ \begin{aligned} T(e,e') &:= {}^{E}\nabla_e e' - {}^{E}\nabla_{e'} e - [e,e'], \\ S(e,e') &:= \mathcal{L}_e(\nabla e') - \mathcal{L}_{e'}(\nabla e) - \nabla_{\rho(\nabla e)} e' + \nabla_{\rho(\nabla e')} e - \nabla[e,e'] \\ &= (\nabla T + 2\mathrm{Alt}\, \iota_\rho R)(e,e'). \quad ({}^{E}R = \iota_\rho S.) \end{aligned} $$

Gauge transformations of fundamental fields are given by

$$\begin{aligned}
\delta^{\nabla} X^i &= \delta X^i = \rho^i(c), \\
\delta^{\nabla} A &= \nabla c - X^*T(A,c), \\
\delta^{\nabla} Y &= \nabla t + (-1)^n \iota_{X^*\rho} w^{\nabla} + X^*T(Y,c) - X^*T(A,t) \\
&\quad + X^*J(A,\ldots,A,c), \\
\delta^{\nabla} Z &= \nabla w^{\nabla} - \iota_{X^*\nabla\rho(c)} Z + \iota_{X^*\nabla\rho(A)} w^{\nabla} - X^*S(Y,A,c) \\
&\quad + (-1)^n X^*S(t,A,A) + X^*\nabla J(A,\ldots,A,c) - \iota_{X^*\rho(c)} \iota^n_{X^*\rho(A)} H \\
&\quad + \sum_{m=1}^{n} (n-m+1)(-1)^n \iota_{X^*\rho(c)} \iota_{F_X} \iota^{(n-m)}_{X^*\rho(A)} H \,.
\end{aligned}$$

§Conclusions

• We considered geometry compatible with a Lie algebroid and the pre-multisymplectic structure.

• It has many examples.

• A topological sigma model with WZ term is constructed. Consistency of the mechanics suggests that the BFV and BV work.

• The BV formalism has been concretely constructed for $E = T^*M, n = 2$ case. Chatzistavrakidis-NI-Šimunić, '22

Outlook

- AKSZ type construction of the BV formalism
- Multisymplectic reduction Blacker '21, Blacker-Miti-Ryvkin '22
- Quantization (a generalization of the deformation quantization)
- A generalization to general Lie n-algebroids

Thank you for your attention!