A CATEGORIFICATION OF THE TUBE ALGEBRA

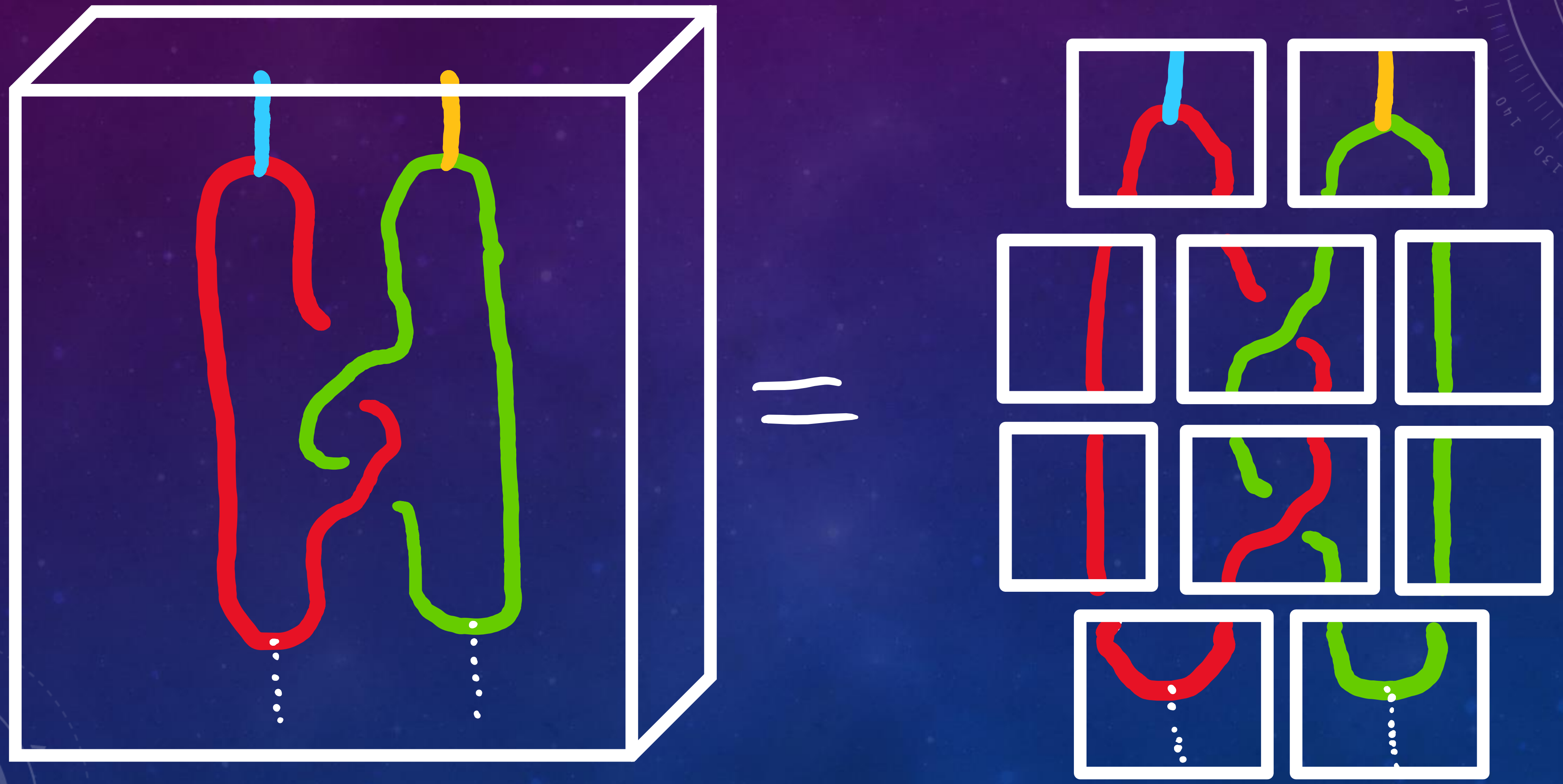
ALEX BULLIVANT
UNIVERSITY OF LEEDS

w. Delcamp
2006.06536
+ upcoming...

(Non-chiral) 2+1D Topological Phases of Matter

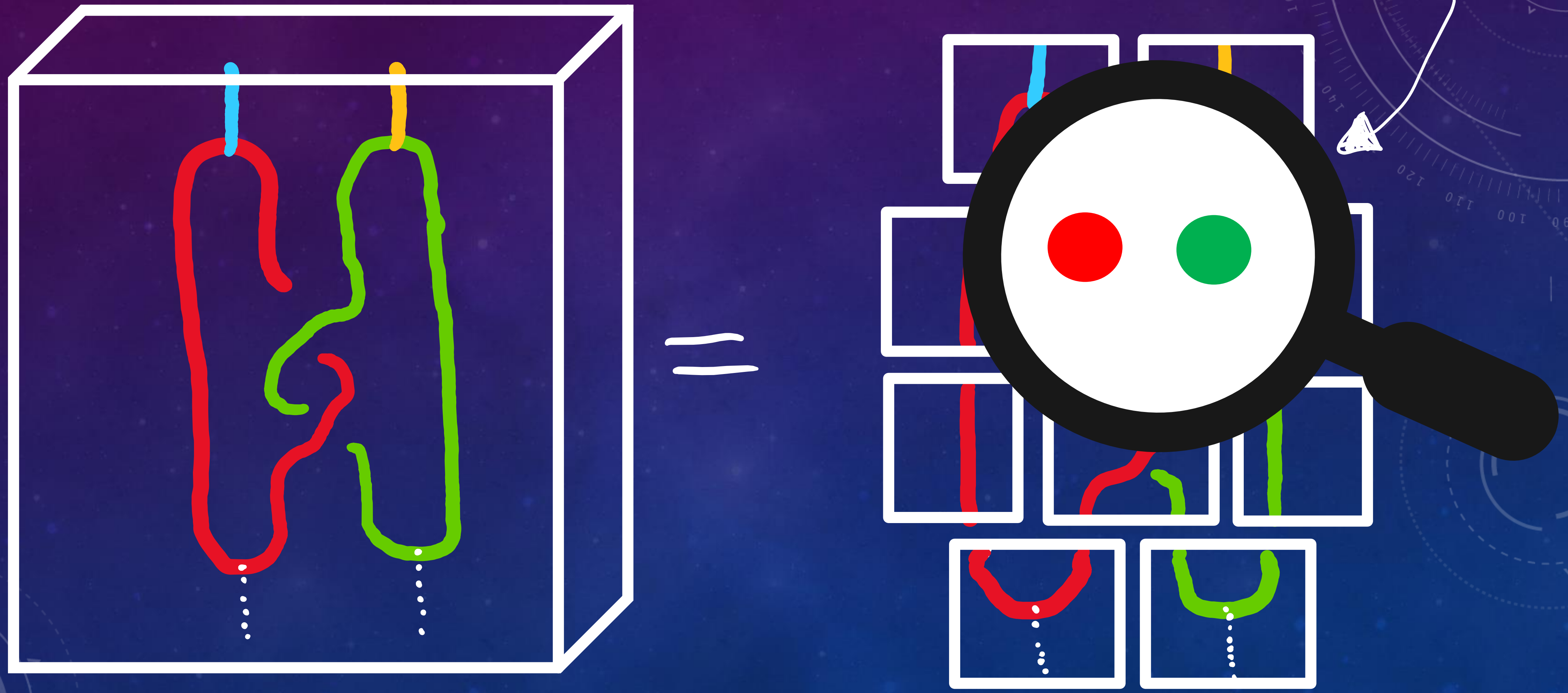
$$\begin{tikzcd}
\text{Unitary, spherical fusion category } \mathcal{C} \arrow[rr, "\text{String-net}"] \arrow[dr, "\text{Center}"'] & & \text{Topological lattice Hamiltonian} \arrow[dl, "\text{Topological excitations}"] \\
& \text{Anyon theory } Z(\mathcal{C}) \equiv \text{Modular tensor category} &
\end{tikzcd}$$

Modular tensor Category



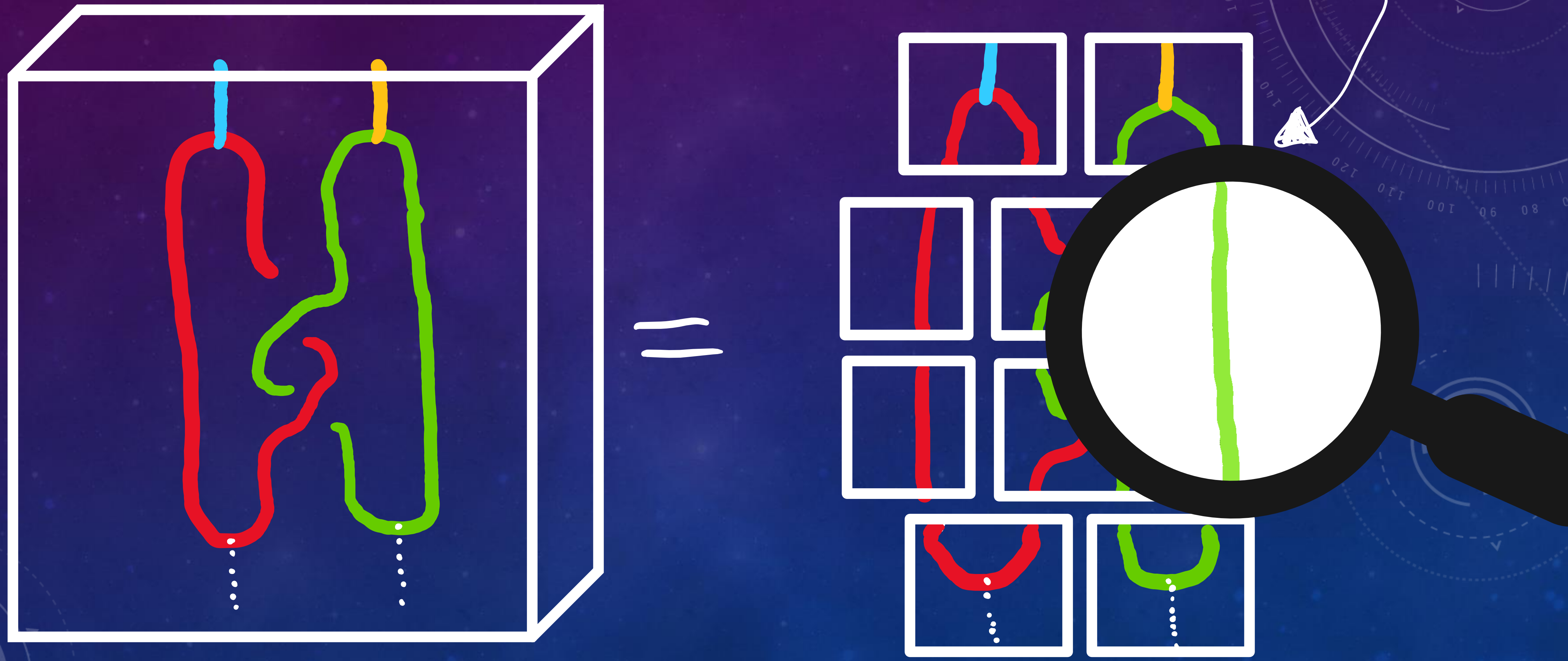
Modular tensor Category

Objects:
Anyons in 2-disk

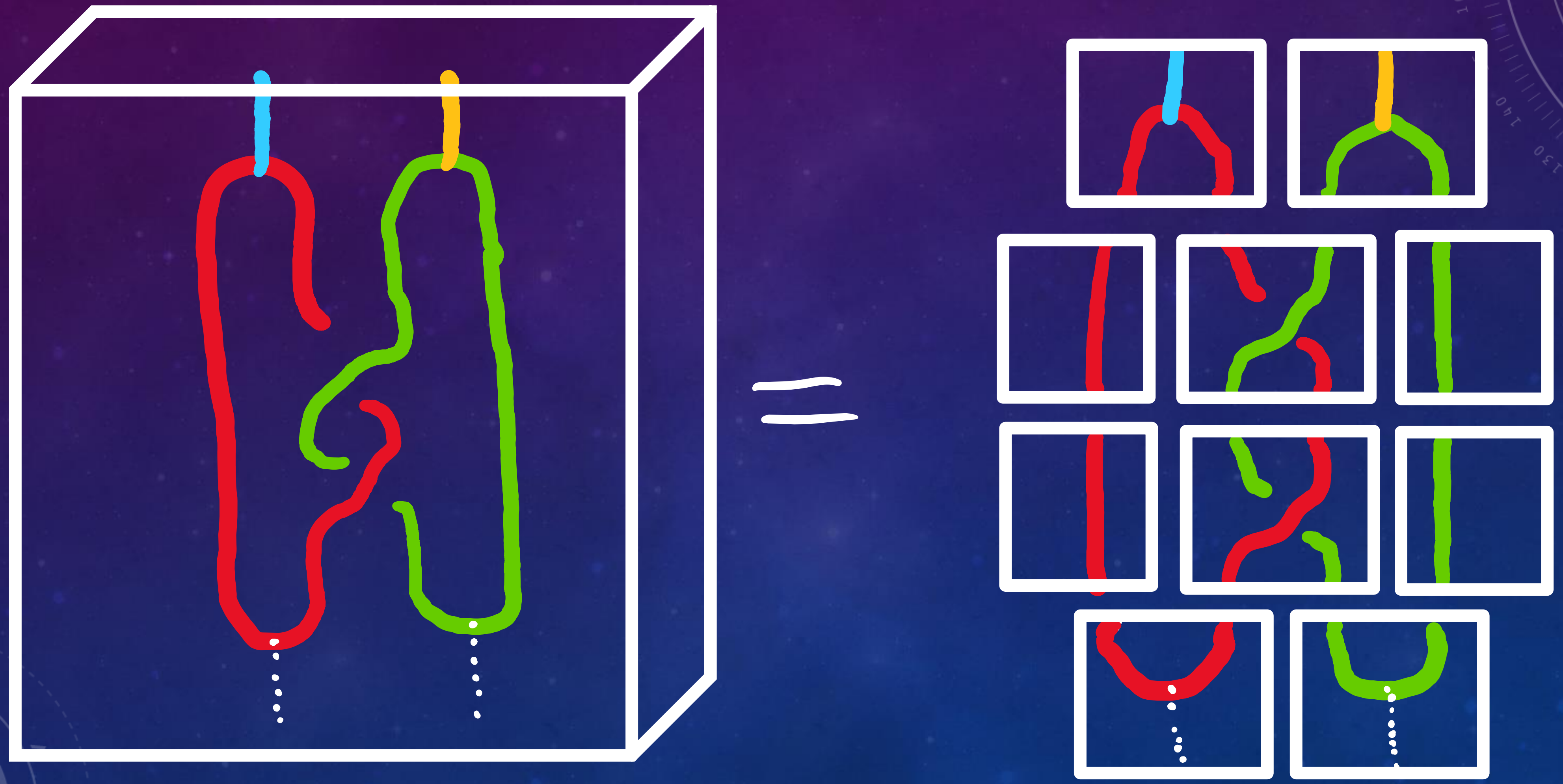


Modular tensor Category

Morphisms:
Isotopy classes of world-lines



Modular tensor Category



(Non-chiral) 3+1D Topological Phases of Matter

Unitary, spherical fusion 2-category $\mathcal{M}$

"Membrane-net"

Topological lattice Hamiltonian

???

Douglas + Reutter 1812.1193

Center

Topological excitations

"Extended" Anyon theory $Z(\mathcal{M})$

$\equiv$ "Modular tensor 2-category"

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"Extended" Anyon theory $Z(\mathcal{M})$

$\equiv$ "Modular tensor 2-category"

DW theory $2\mathrm{Vect}_G^{\pi}$

1905.04644

see Yin Tian talk

$Z(2\mathrm{Vect}_G^{\pi})$

(Non-chiral) 3+1D Topological Phases of Matter

Unitary, spherical fusion 2-category $\mathcal{M}$

"Membrane-net"

Topological lattice Hamiltonian

Center

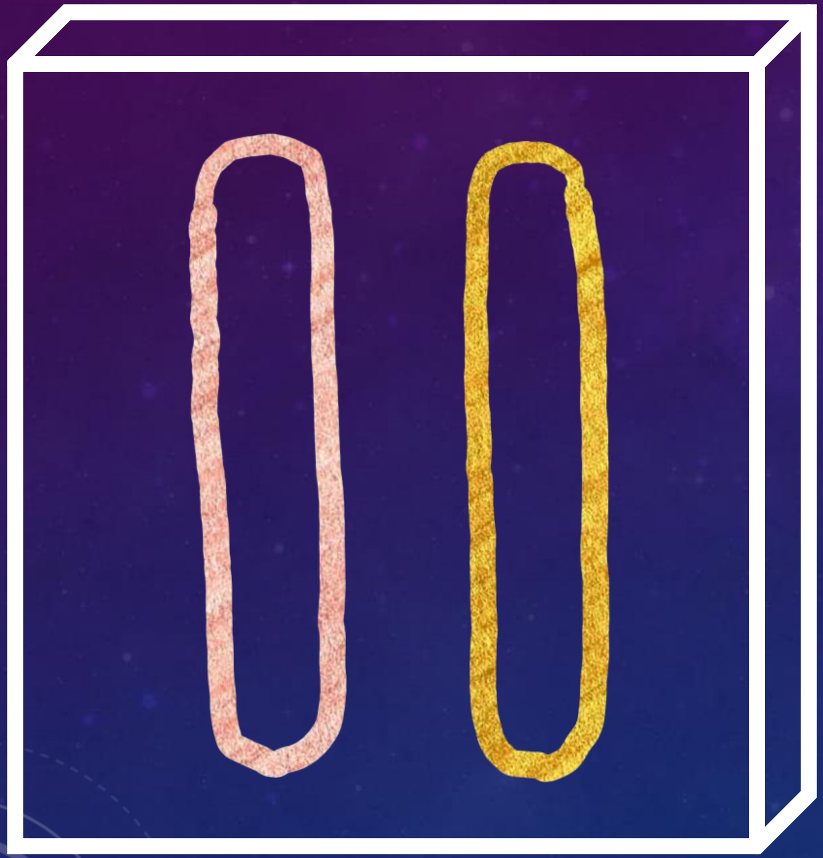
Topological excitations $\Rightarrow$ Categorified tube algebra.

"Extended" Anyon theory $Z(\mathcal{M})$

$\equiv$ "Modular tensor 2-category"

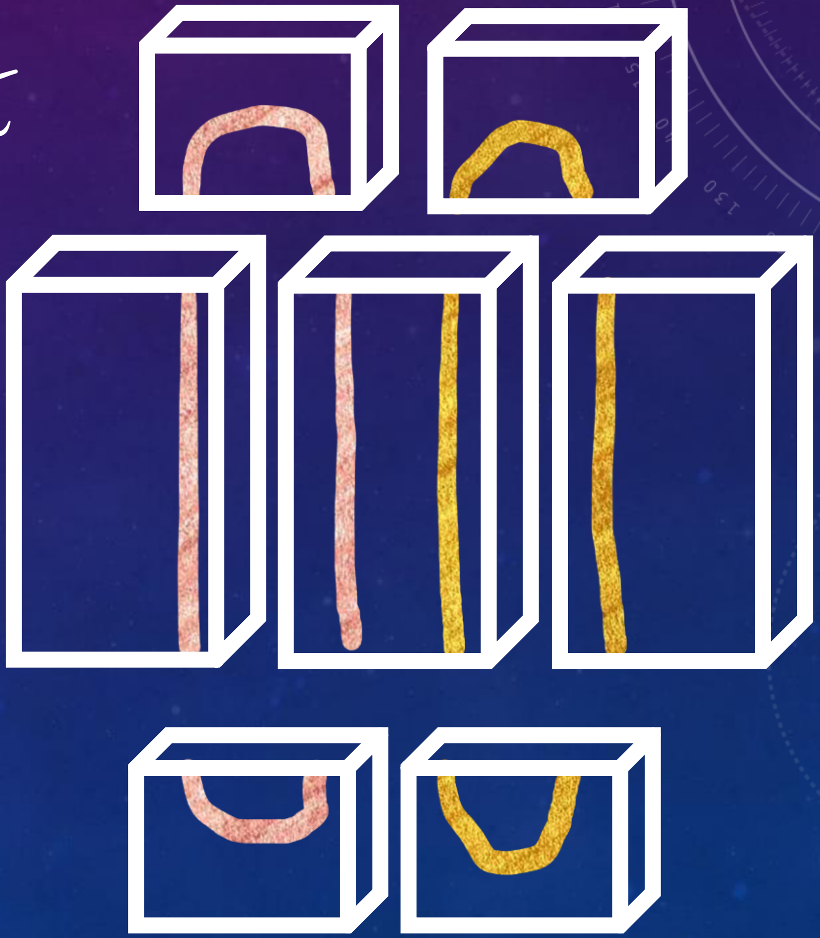
Todays talk

"Modular tensor" 2-category $\underset{\text{Roughly}}{\simeq}$ Braided monoidal 2-category + extra conditions

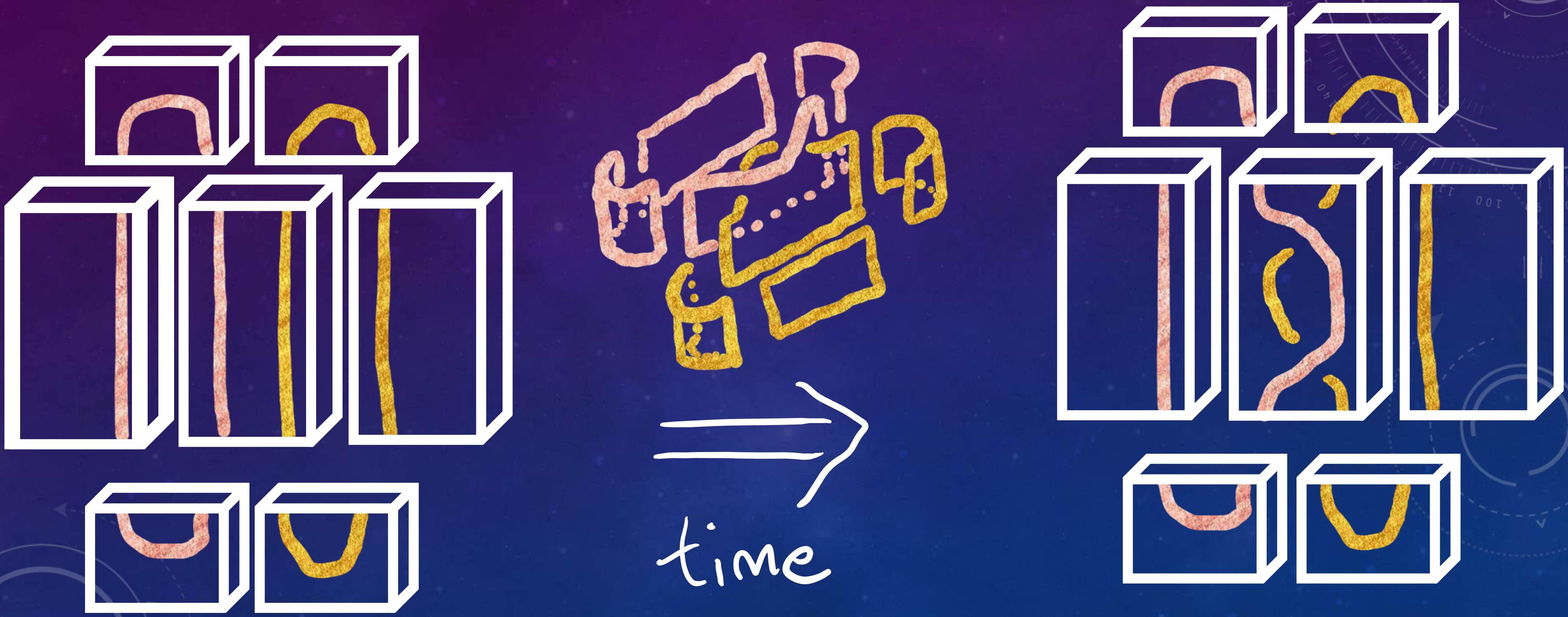


Cut Hilbert space into tensor factors

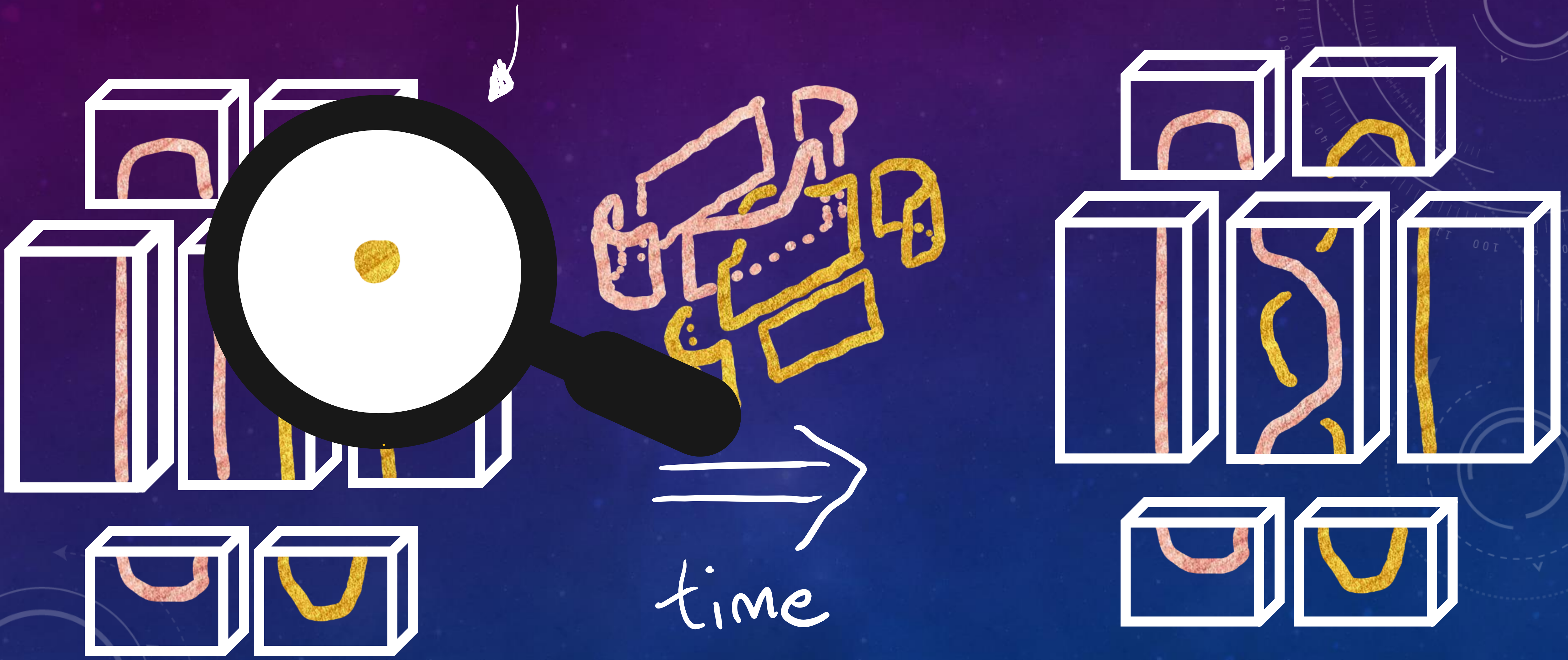
$\Longleftrightarrow$



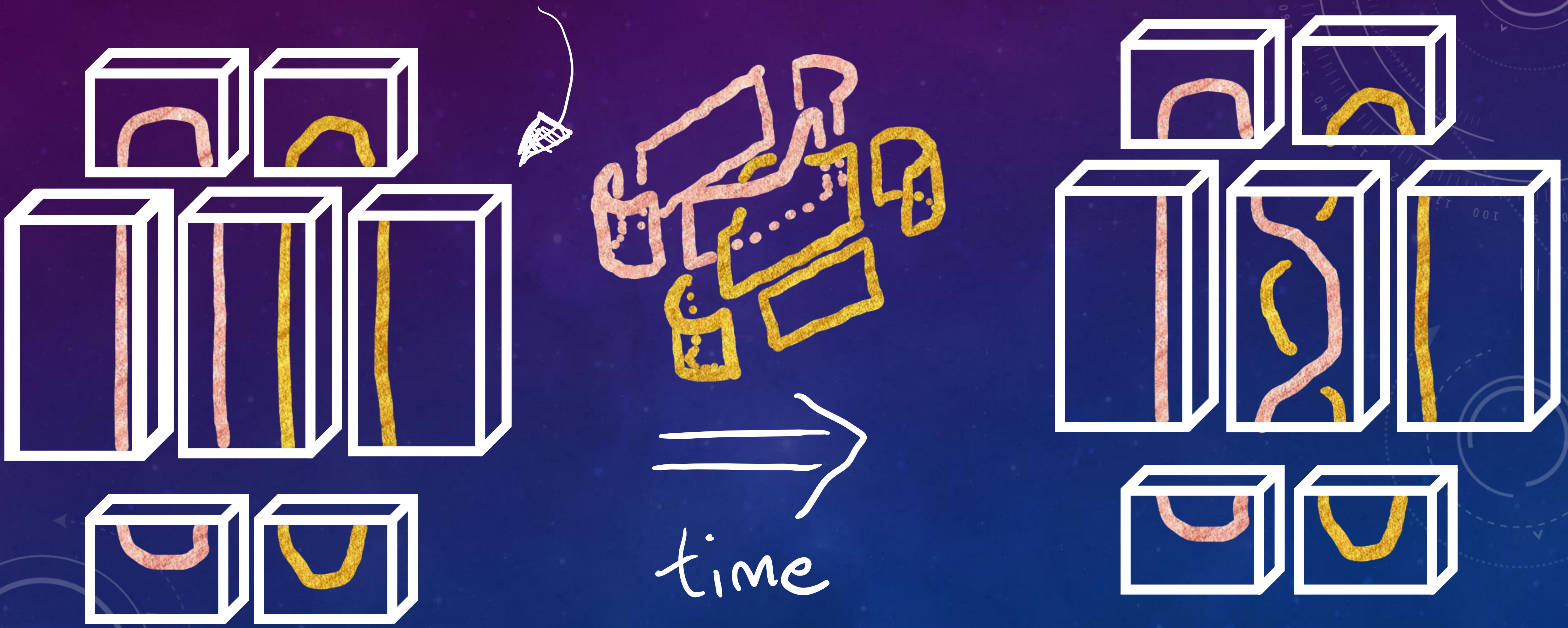
"Modular tensor 2-category"



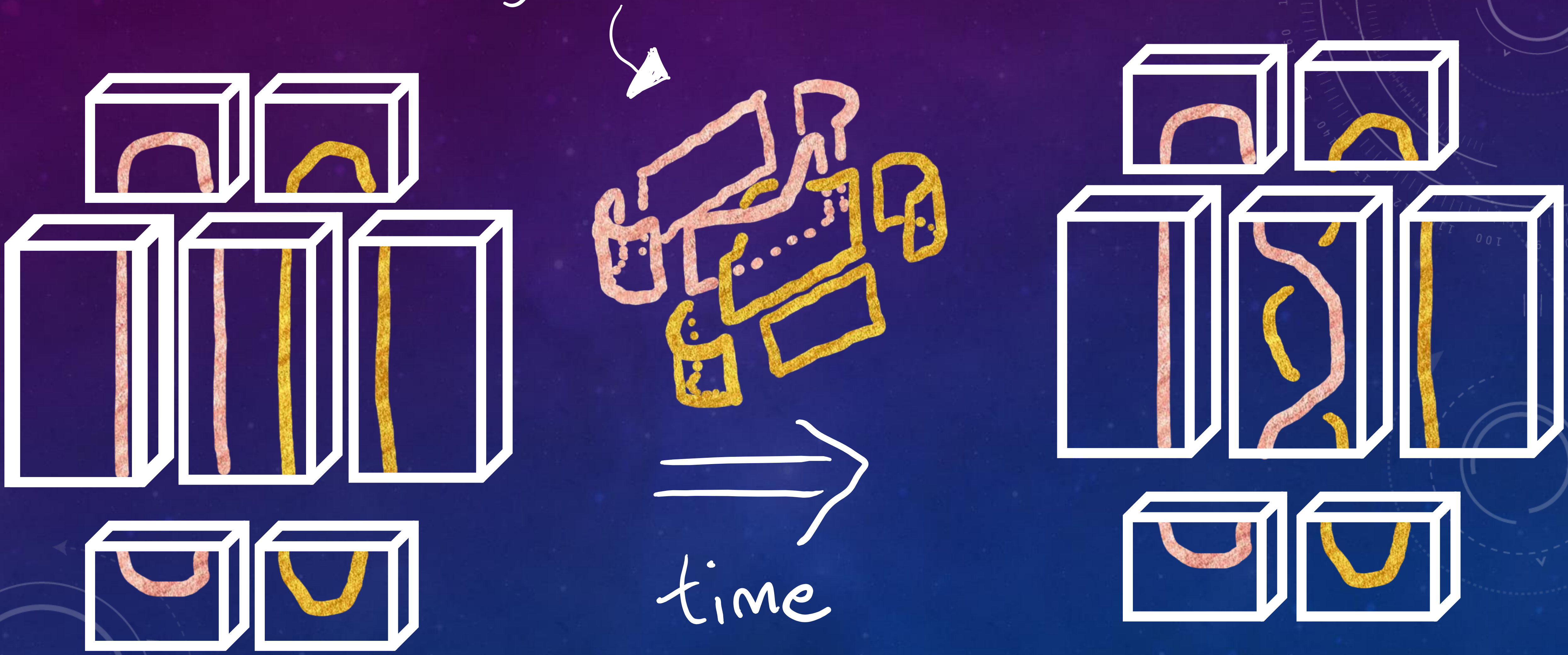
objects: Anyons in gapped boundary condition for 2-disk



1-morphisms: Hilbert space for 3-disk with open-string terminating on gapped boundary



2-Morphisms: Isotopy classes of string world-sheets



knotted surface in 4D

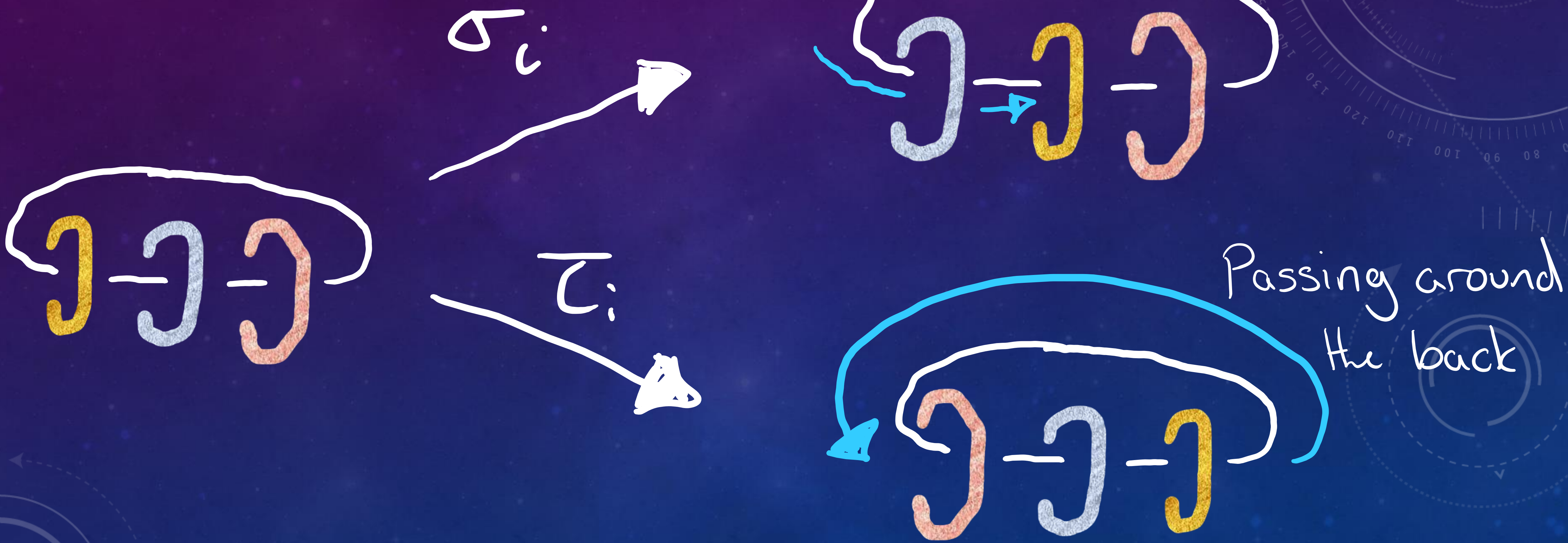
- Amplitude depends only on isotopy class

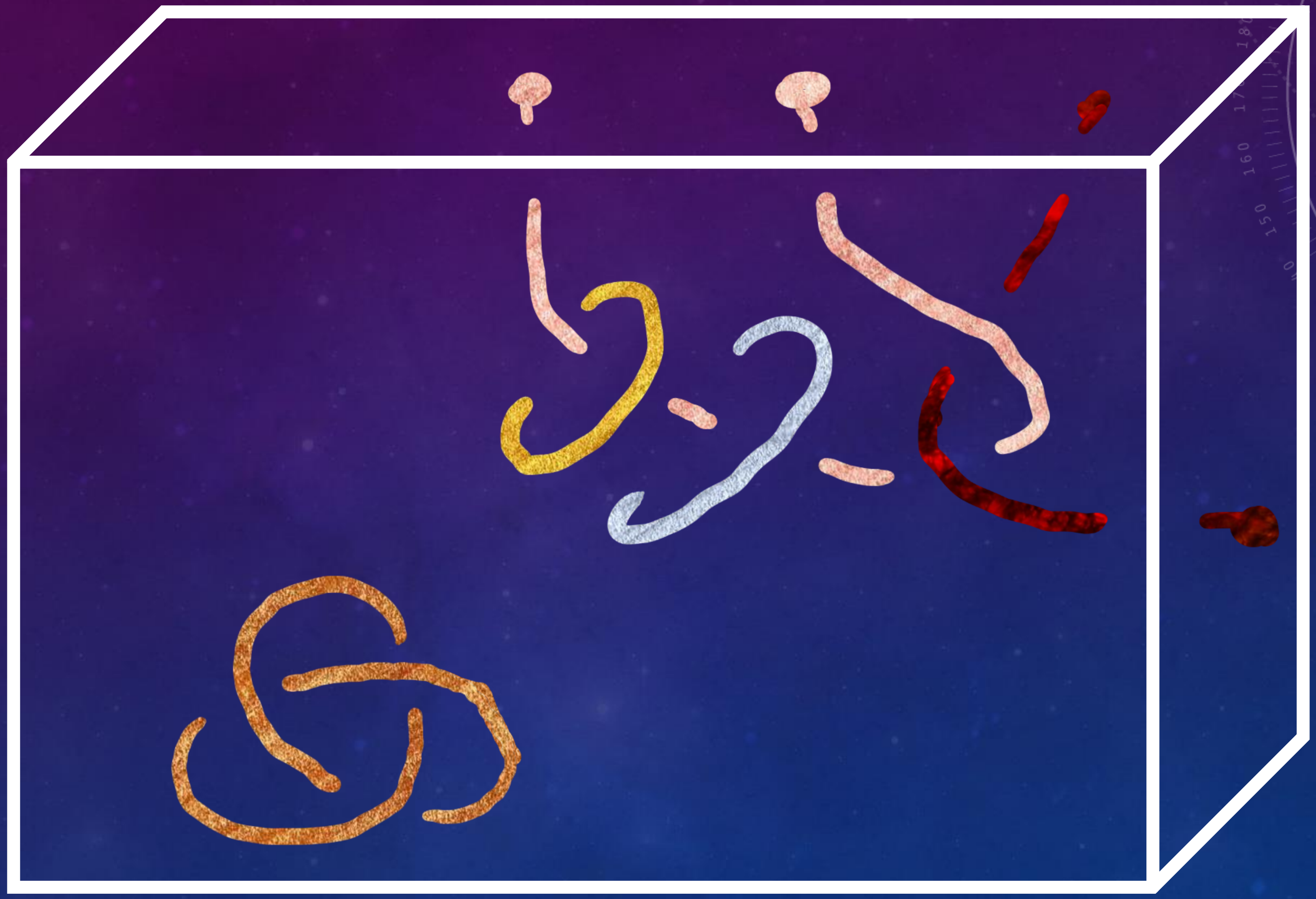
$\simeq$

Linking of excitations

AB, Kimball, Martin, Rowell, 2018, CMP

AB, Delcamp, 2019, JHEP





(Non-chiral) 3+1D Topological Phases of Matter

Unitary, spherical fusion 2-category $\mathcal{M}$

$\xrightarrow{\text{"Membrane-net"}}$

Topological lattice Hamiltonian

Center

"Extended" Anyon theory $Z(\mathcal{M})$

$\equiv$ "Modular tensor 2-category"

Topological excitations $\Rightarrow$ Categorified tube algebra.

Todays talk

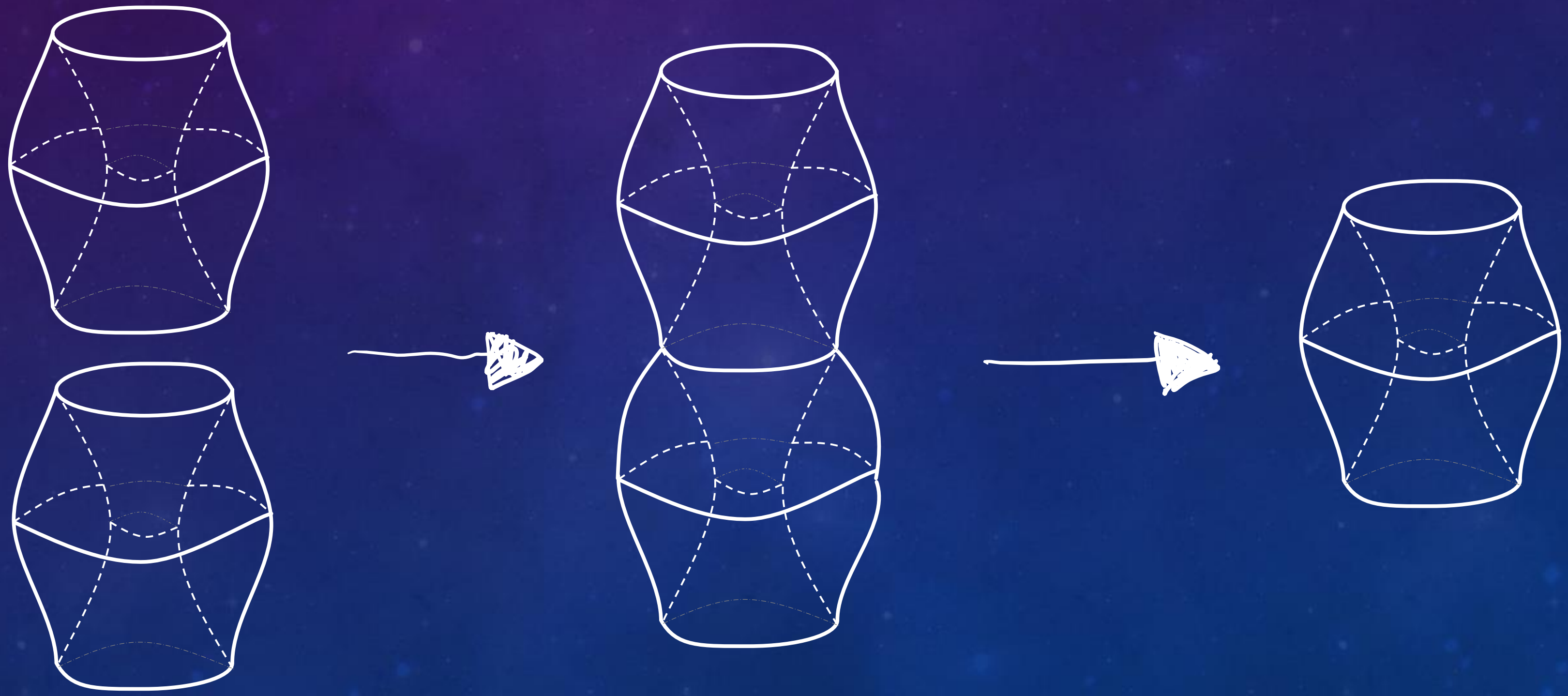
HAMILTONIAN MODELS OF TPM

Topological Lattice Models

* oriented d-manifold Σ equipped w. discretisation Σ_Δ
  - discretisation := Δ-complex + branching structure
* $\mathcal{H}[\Sigma_\Delta] := \mathrm{Span}_{\mathbb{C}}\{\phi : \Sigma_\Delta \to S\}$
* Local Hamiltonian $H = -\sum_{\Delta^\circ \subset \mathrm{Int}(\Sigma_\Delta)} H_{\Delta^\circ}$ s.t. $H_{\Delta^\circ} \cdot H_{\Delta^\circ} = H_{\Delta^\circ}$, $[H_{\Delta^\circ}, H_{\Delta^{\circ\prime}}] = 0$
* Family of unitary isomorphisms $U : GS[\Sigma_\Delta] \to GS[\Sigma_{\Delta'}]$ whenever $\partial \Sigma_\Delta = \partial \Sigma_{\Delta'}$
  - "symmetries" of ground state subspace

$$U \circ \Big[\prod_{\Delta^\circ \subset \mathrm{Int}(\Sigma_\Delta)} H_{\Delta^\circ}\Big] = \Big[\prod_{\Delta^\circ \subset \mathrm{Int}(\Sigma_{\Delta'})} H_{\Delta^\circ}\Big] \circ U$$

CATEGORIFIED TUBE ALGEBRAS



Algebra

Given a monoidal category $\mathcal{C}$ an Algebra (A, p) is an object $A \in \mathcal{C}^{o}$ and morphism $p : A \otimes A \to A$ s.t. following commutes

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[r, "p \otimes A"] \arrow[d, "\alpha_{A,A,A}"] & A \otimes A \arrow[dr, "p"] & \\
& & A \\
A \otimes (A \otimes A) \arrow[r, "A \otimes p"'] & A \otimes A \arrow[ur, "p"'] &
\end{tikzcd}$$

2-Algebra

Given a monoidal 2-category $\mathcal{B}$ a 2-algebra (A, P, Q) is an object $A \in \mathcal{B}^0$

a 1-morphism $P : A \boxtimes A \to A$ and 2-isomorphism

$$\begin{tikzcd}
(A \boxtimes A) \boxtimes A \arrow[r, "P \boxtimes A"] \arrow[d, "\alpha_{A,A,A}"'] & A \boxtimes A \arrow[dr, "P"] \arrow[d, phantom, "\Swarrow Q"] & \\
 & & A \\
A \boxtimes (A \boxtimes A) \arrow[r, "A \boxtimes P"'] & A \boxtimes A \arrow[ur, "P"'] &
\end{tikzcd}$$

satisfying some coherence data...

$$\begin{tikzcd}
A \boxtimes A \boxtimes A \boxtimes A \arrow[d, bend right] \arrow[d] \arrow[d, bend left] \\
A
\end{tikzcd}$$

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\end{tikzcd}
$$

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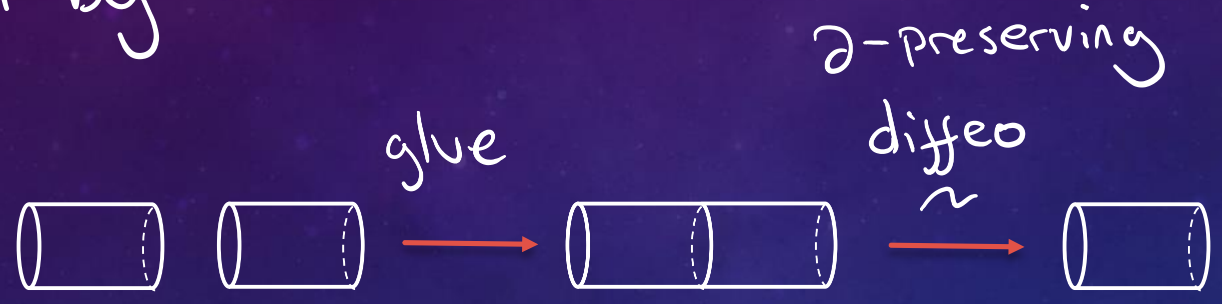
A
\end{tikzcd}
$$

Categorified tube algebra $\in$ 2Vect := symmetric monoidal 2-category of Vect-module categories

Construction outine:

1) Define 2Vector-space := semisimple Abelian category $\sim$ module category for semisimple algebra to cylinder $\mathrm{Mod}(\square)$

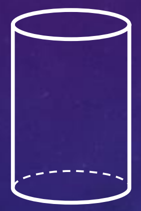
2) Form 2-algebra $\otimes : \mathrm{Mod}(\square) \boxtimes \mathrm{Mod}(\square) \longrightarrow \mathrm{Mod}(\square)$ inspired by



3) "Extended anyon theory" $\sim$ Representation theory for $\mathrm{Mod}^{\otimes}(\square)$

Step 1:

The 2-Vector space of the cylinder

Mod()

Given 3+1D TLM $\Rightarrow$

G.S. [] = G.S. [$S^1 \times$ ]

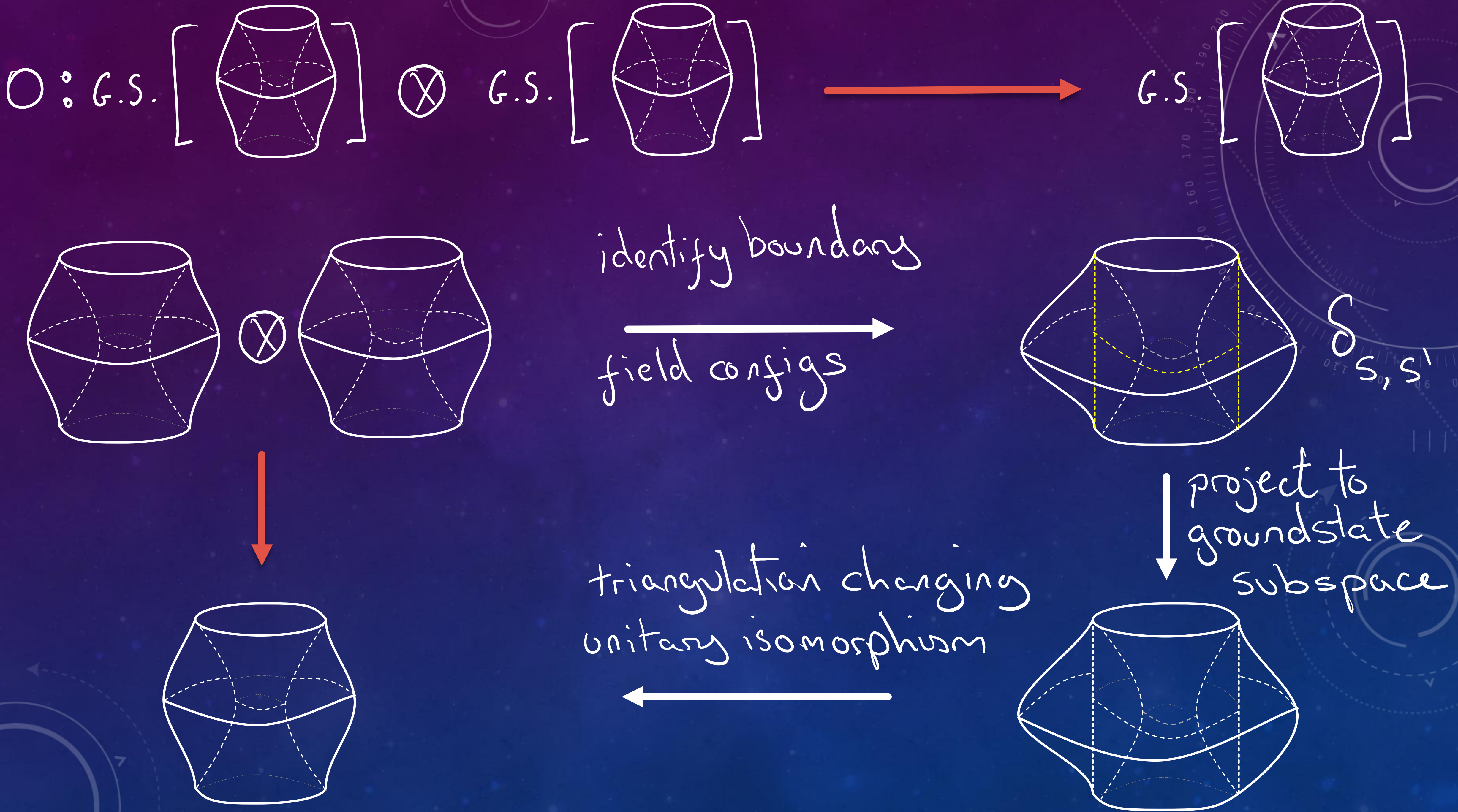
equipped w. triangulation

bigon $\simeq$ 2-disk

We can enrich this Hilbert space w.

finite-dimensional, associative, $*$-algebra in Vect

$\Rightarrow$ semisimple algebra


O : G.S. ⊗ G.S.
G.S.
identify boundary
field configs
δ s, s'
project to
groundstate
subspace
triangulation changing
unitary isomorphism

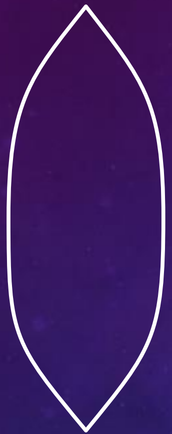
Mod() := Category of G.S.° [] – modules (right)

$\Rightarrow$ Semisimple Abelian category $\in$ 2Vect

Step 2:

Categorified tube algebra

$\otimes$: Mod() $\boxtimes$ Mod() $\longrightarrow$ Mod()

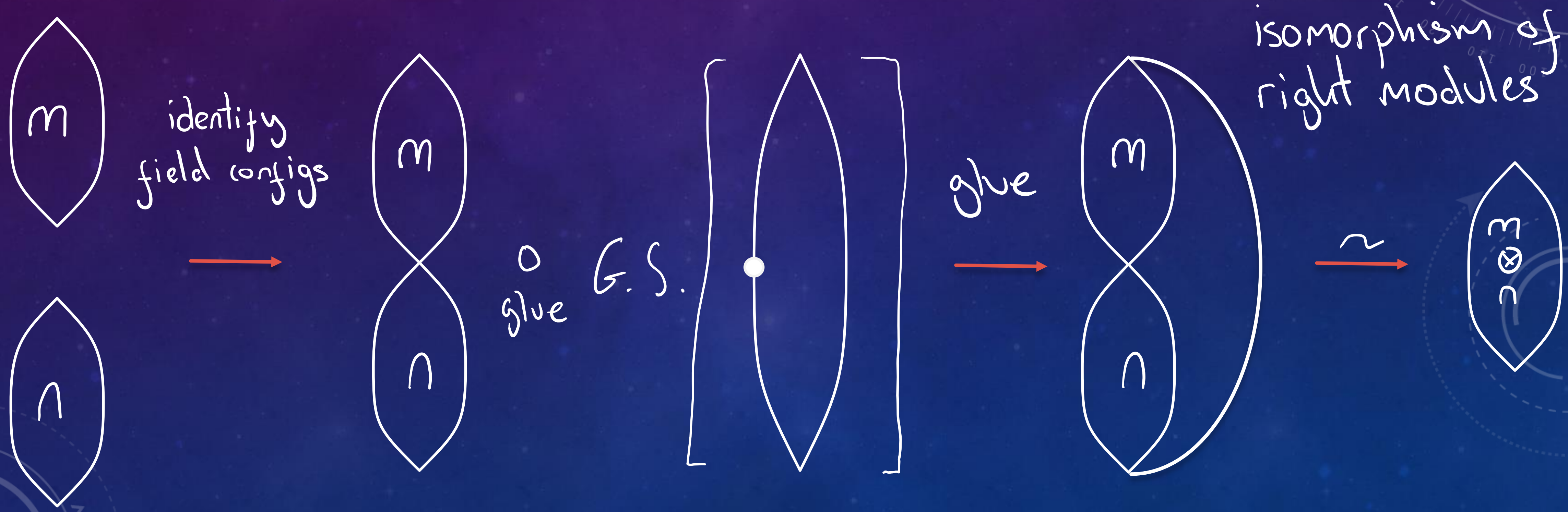
G.S.$\left[S^1 \times \right.$  $\left. \right]$

For ease of drawing $\Longrightarrow$

G.S.$\left[\right.$  $\left. \right]$

used for defining categorified tube algebra for pt in 2+1D

$$\mathrm{Mod}(\,|\,) \boxtimes \mathrm{Mod}(\,|\,) \xrightarrow{\text{linear bifunctor}} \mathrm{Mod}(\,|\,)$$



Associator

Q :

triangulation invariance of ground state subspaces

$\xrightarrow{\sim}$

$$(M_1 \otimes M_2) \otimes M_3 \quad \cong \quad M_1 \otimes (M_2 \otimes M_3)$$

Example: Topological gauge theory w. finite group G

– theory of flat G-connections + gauge transformations

$\mathrm{Mod}^{\otimes}($  $) \Rightarrow \mathrm{Vect}^{\otimes}_{G//G}$ (multifusion category)

objects: finite vector spaces graded by $\{g \xrightarrow{h} h^{-1}gh\}_{\forall g,h\in G} := V_{g \xrightarrow{h}}$

morphisms: grading preserving linear maps

2-product: $\otimes : V_{g\xrightarrow{h}} \boxtimes W_{g'\xrightarrow{h'}} \mapsto \begin{cases} (V\otimes W)_{g\xrightarrow{hh'}} & \text{if } g' = h^{-1}gh \\ \text{zero vector space} & \text{else} \end{cases}$

Conjecture:

$\mathrm{Mod}^{\otimes}($  $)$ is multifusion for all 3+1D topological lattice models.

$\Longrightarrow$ should follow from a 2-inner product structure, really talking about 2-Hilbert spaces not just 2-Vector space.

Conjecture: Assume true from now on

$\mathrm{Mod}^{\otimes}($  $)$ is multifusion for all 3+1D topological lattice models.

$\Longrightarrow$ should follow from a 2-inner product structure, really talking about 2-Hilbert spaces not just 2-Vector space.

Step 3:

$\mathrm{Mod}^{\otimes}($  $)$ - Representation theory

+ extended anyon theory

2-Category of module categories

- similarly to modules of an algebra, we can define <u>module categories</u> for a 2-algebra

$\triangleright$: $\mathrm{Mod}^{\otimes}($  $) \boxtimes \mathcal{M} \longrightarrow \mathcal{M}$ (linear functor + coherence)

$\mathrm{MOD}($  $) :=$ objects: $\mathrm{Mod}^{\otimes}($  $)$ – module categories

morphisms: $\mathrm{Mod}^{\otimes}($  $)$ – module functors

2-morphisms: $\mathrm{Mod}^{\otimes}($  $)$ – module natural trans

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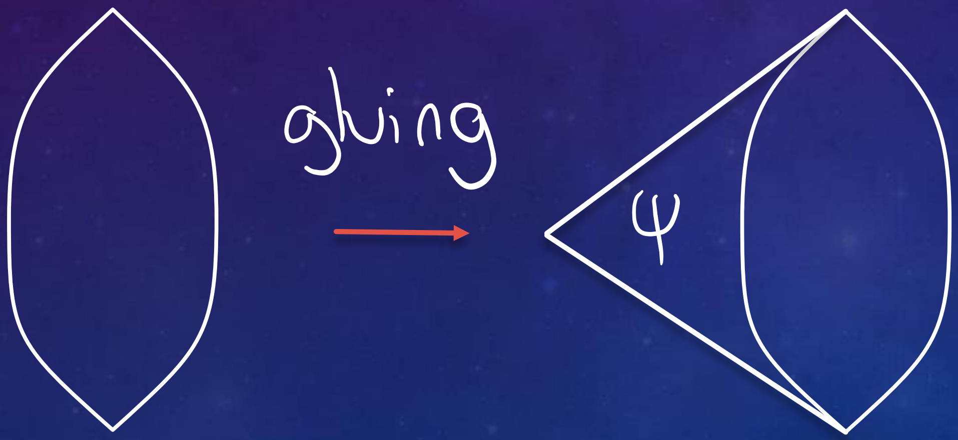
$\mathrm{Mod}^{\otimes}(\;)$ – module categories ~ Category of modules for a separable algebra in $\mathrm{Mod}^{\otimes}(\;)$

$\underbrace{\mathrm{Hom}(\mathrm{Mod}(A), \mathrm{Mod}(B))}_{\text{hom-category}}$ ~ Category of A-B-bimodules in $\mathrm{Mod}^{\otimes}(\;)$

see Douglas + Reutter 1812.1193
and Gaiotto + Johnson-Freyd 1905.09566
+ Refs for nice intro

Morphisms in Mod(|)

U:  , Unitary isomorphism for triangulation change of interior

$|\psi\rangle$: 

, $|\psi\rangle \in$ G.S.

We can define A-B-bimodules in terms of modules for an algebra

$$\mathrm{Alg}_{A|B} \equiv \bigoplus_{X,Y \in \mathrm{Mod}^{\otimes}(\ )} \mathrm{Hom}_{\mathrm{Mod}^{\otimes}(\ )}(A \otimes X \otimes B, Y)$$

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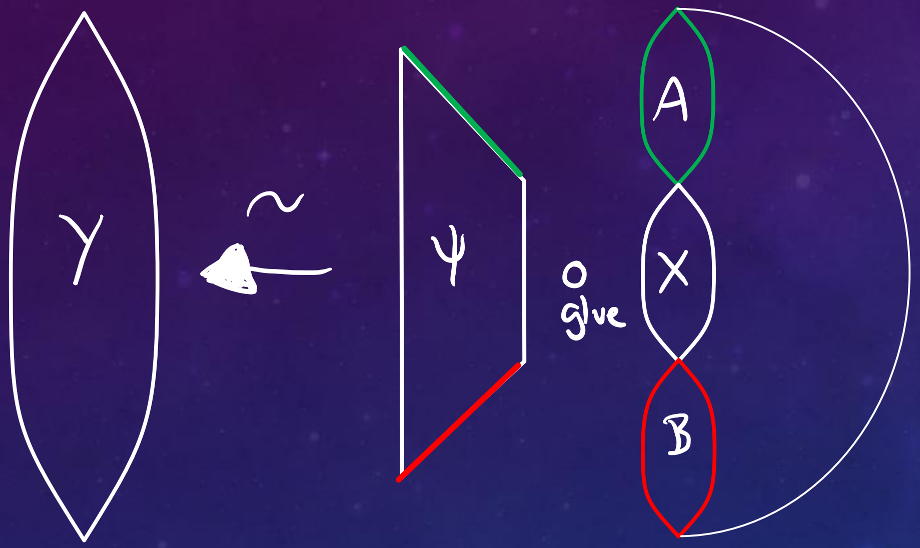

,

$$|\psi\rangle \in \mathrm{Hom}_{\mathrm{Mod}^{\otimes}(\square)}(A \otimes X \otimes B,\, Y) \subseteq \text{G.S.}$$

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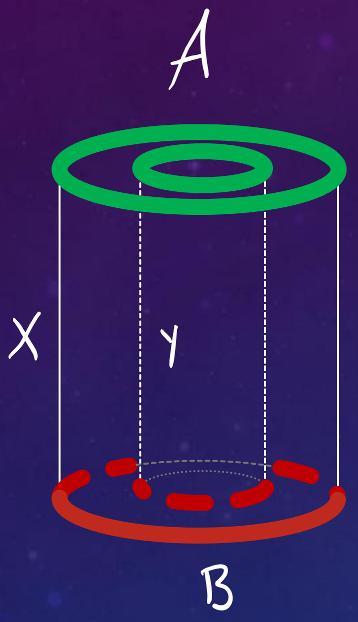
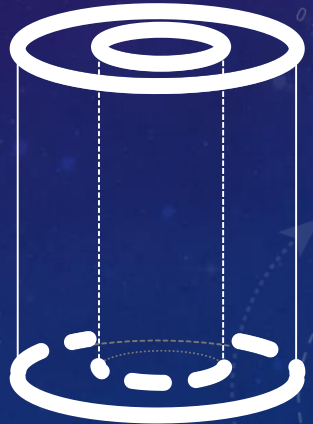
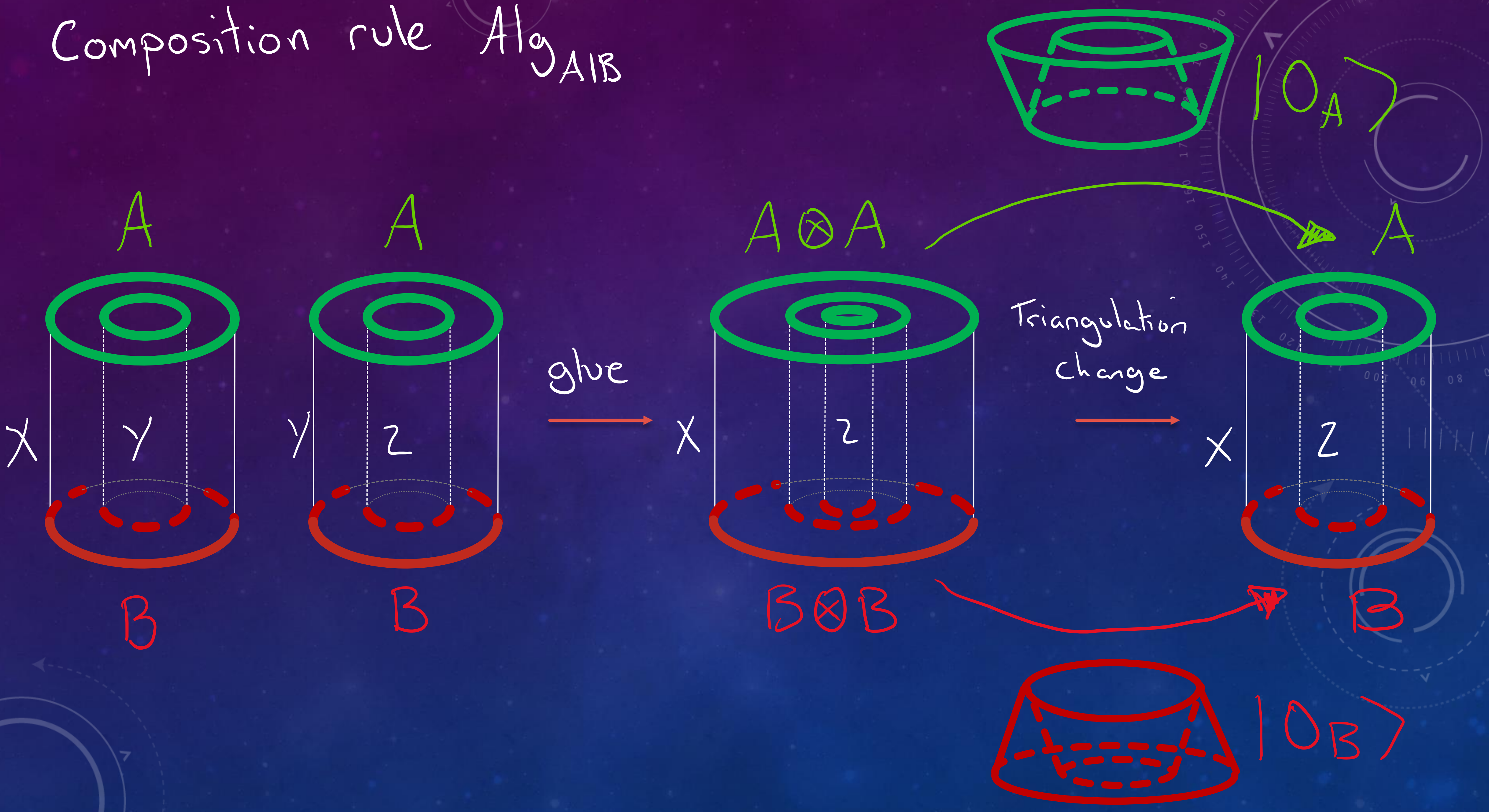


$\in \mathrm{Hom}_{\mathrm{Mod}^{\otimes}(\square)}\left(A\otimes X\otimes B,\, Y\right) \subseteq G.S.\Big[$  $\Big]$

Composition rule $Alg_{A|B}$

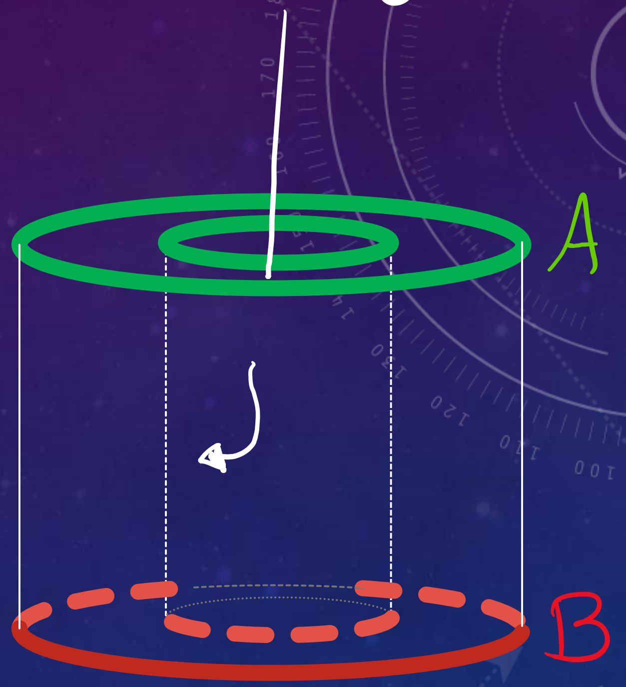


For gapped system

$$S_A \propto |\partial A| - \gamma + S_{excite}$$

cut away a local neighbourhood of excitation

topological properties of excitation encoded as boundary conditions



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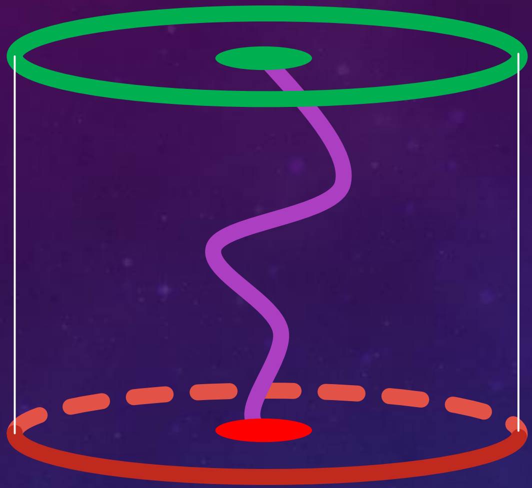

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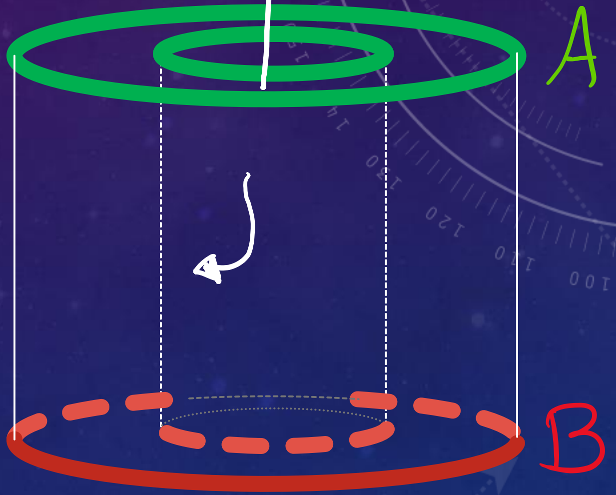


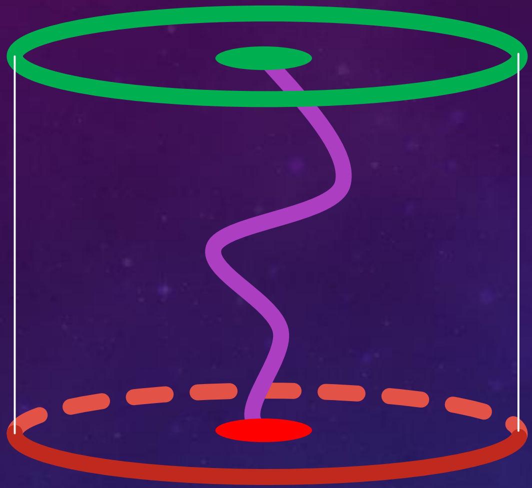
$\mathrm{sAlg}_{A|B}$-Module : Length scale invariant Hilbert space for open string

~ topological string upto local excitation

For gapped system

$$S_A \propto |\partial A| - \gamma + S_{excite}$$

topological properties of excitation encoded as boundary conditions



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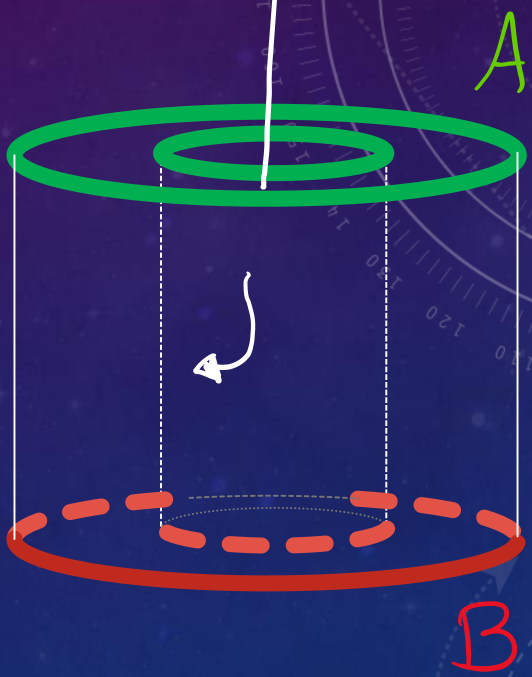
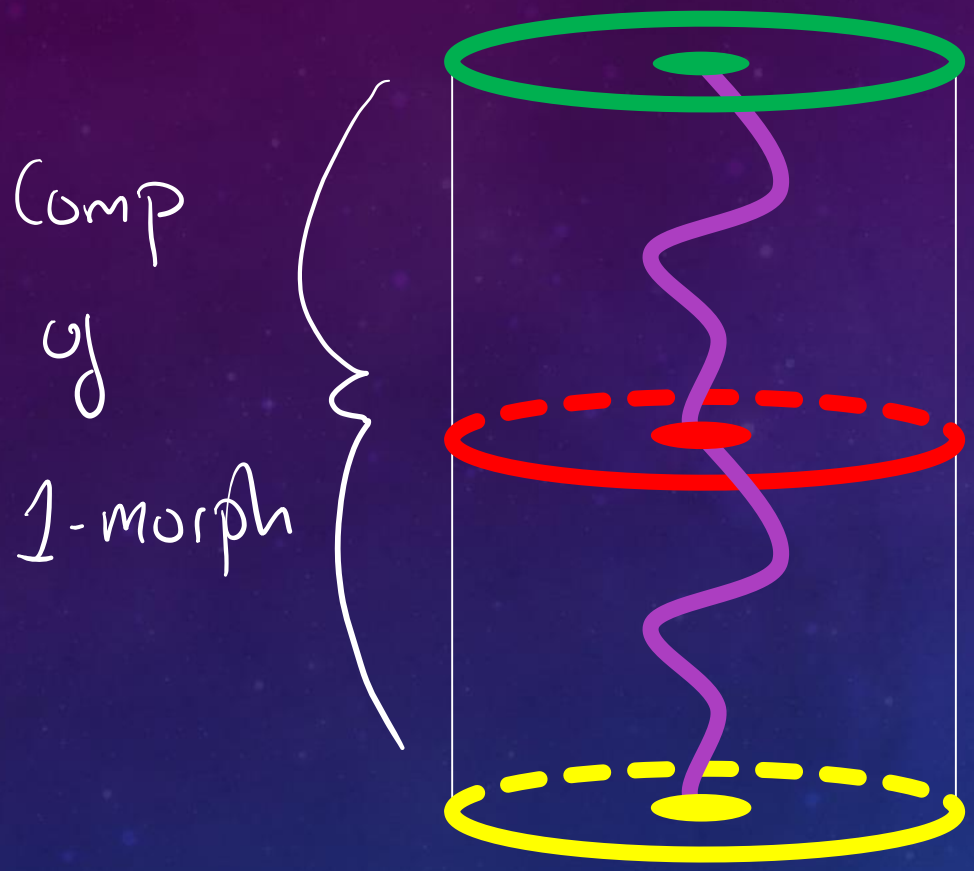


Mod(A) $\Rightarrow$ Boundary conditions for endpoint of an open string

$\sim$ boundary anyons upto local excitation.

MOD($\bigcirc$)



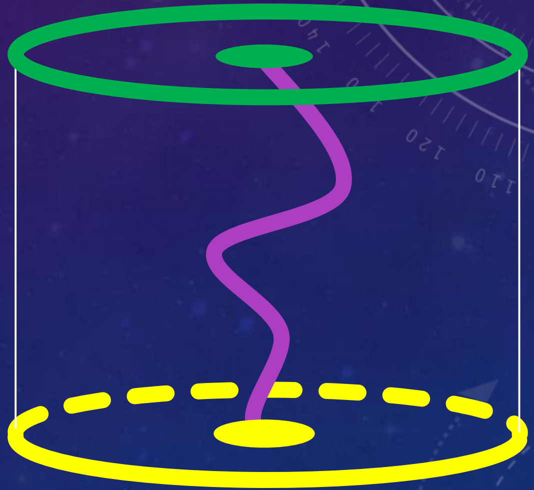
MOD(○)

Comp of 1-morph

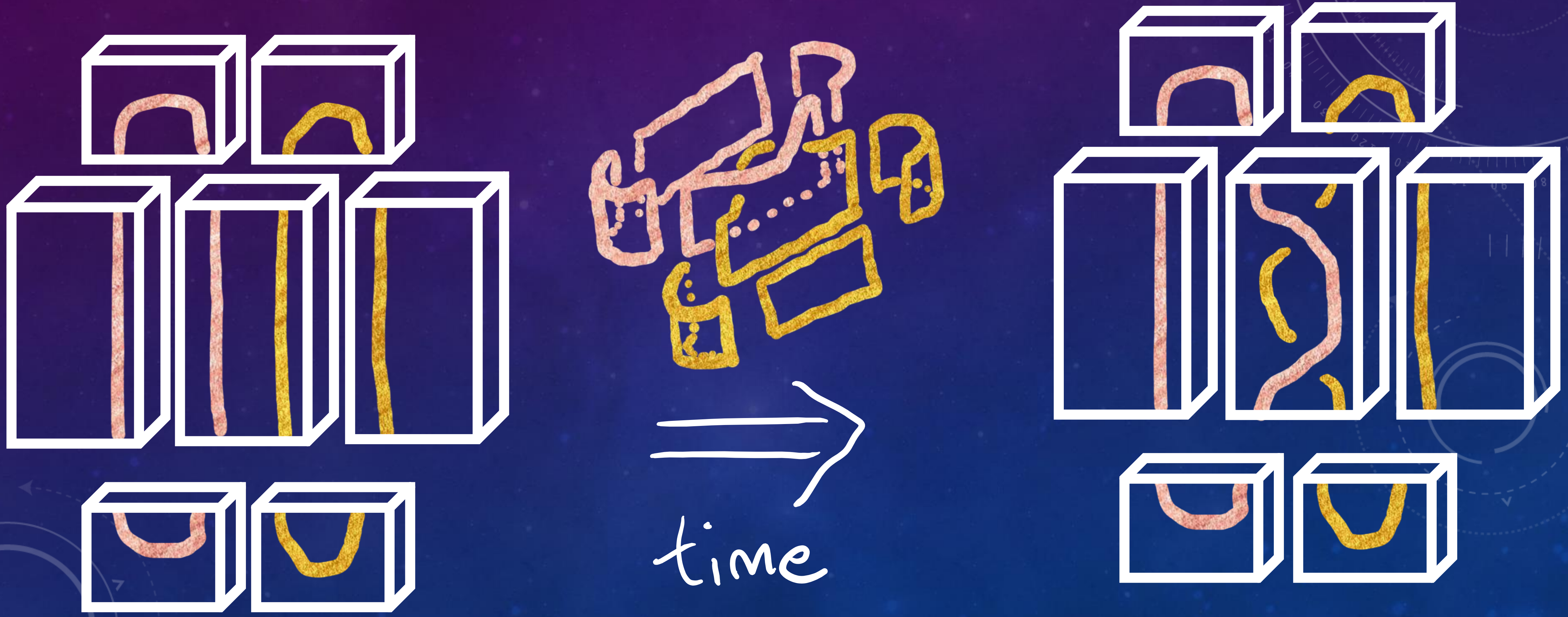
2-morphism

$\Longrightarrow$

"renormalisation"



What about monoidal + braiding for MOD($\bigcirc$)



Braiding + Monoidal Structure on $\mathrm{MOD}(\bigcirc)$

* For topological gauge theory $\mathrm{Mod}^{\otimes}($  $) = \mathrm{Vect}^{\otimes}_{G//G}$
* $\mathrm{Vect}^{\otimes}_{G//G}$ defines "quasitriangular Hopf category"
  - see thesis of Neuchl
* $\Rightarrow \mathrm{MOD}\left(\mathrm{Vect}^{\otimes, R}_{G//G}\right)$ is braided + monoidal 2-category
* Kong, Tian, Zhou $\boxed{Z\left(2\mathrm{Vect}^{\pi}_{G}\right) \simeq \mathrm{MOD}\left(\mathrm{Vect}^{\tau\pi, \otimes, R}_{G//G}\right)}$

Open questions:

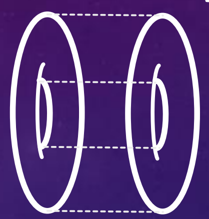
1) Can we canonically put quasi-triangular hopf category (or weakened variant) on $\mathrm{Mod}^{\otimes}($  $)$?

2) Assuming true:

Given spherical fusion 2-category $\mathcal{B}$
$\Rightarrow$ We can define 3+1D TLM canonically

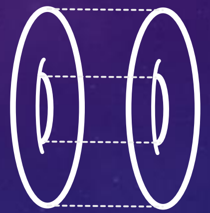
$Z(\mathcal{B}) \overset{?}{\simeq} \mathrm{MOD}^{\otimes, Br}($  $)$

Following Delcamp's talk: For 3+1D TLM we can classify loop-like excitations using

Mod() $\equiv$ Category of (right) G.S.$^{\circ}$[]-modules

$T^2 \times I$

Dimension and crossing w. circle

$\dim\Big[\mathrm{Mod}\Big($  $\Big)\Big] \simeq Z\Big[\mathrm{G.S.}\Big[$  $\Big]\Big] \simeq \mathrm{G.S.}\Big[$  $\times S^1\Big]$

$:= \mathrm{Nat}(id, id)$

$:=$ center of algebra

3-Torus

$\Rightarrow$ Ground state-degeneracy of 3-Torus = # of simple loop-like excitations

Dimension and crossing w. circle

$\mathrm{Dim}[\mathrm{MOD}(\bigcirc)] \simeq Z[\mathrm{Mod}^{\otimes}($  $)]$

$\equiv$ pseudoNat(id,id) $\equiv$ Drinfeld center

$\simeq \mathrm{Mod}^{Br}($  $)$

"3-loop braiding"

Thanks For Listening!