Approximately differentiable homeomorphisms: derivatives and Jacobians

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Thematic programme: Infinite-dimensional Geometry: Theory and Applications

Zofia Grochulska (University of Jyväskylä, Finland)

Research from my PhD at University of Warsaw, Poland: joint work with Paweł Goldstein (University of Warsaw) and Piotr Hajłasz (University of Pittsburgh)

John Ball's nonlinear elasticity

- Aim: model deformations of materials
- Ω a domain in $\mathbb{R}^n$,
- Family of admissible mappings: $f : \Omega \to \mathbb{R}^n$ (for example: homeomorphisms)
- Energy of the form: $\int_\Omega W(Df(x))\,dx$, so we need a derivative of f
- Minimize among admissible mappings

The minimizer: does it still describe a physical deformation?

- Does it preserve orientation?
- Does it map sets of measure zero onto sets of measure zero (*the Lusin condition (N)*)?

John Ball's nonlinear elasticity gives rise to questions

- What are the properties of limits (in different metrics) of homeomorphisms?
- How can we easily check if a homeomorphism preserves or reverses orientation?

Easy recipe for diffeomorphisms:

positive Jacobian → orientation—preserving

negative Jacobian → orientation—reversing

- What can we say about Jacobians of homeomorphisms?
- And about derivatives?

Approximate derivative

$$\lim_{r\to 0} \frac{|B(x,r)\cap E_x|}{|B(x,r)|} = 1$$

A function $f:\Omega\to\mathbb{R}$ is approximately differentiable at a point $x\in\Omega\subset\mathbb{R}^n$ if there exist a measurable set E_x of which x is a density point and a linear mapping L s. t.

$$\lim_{y\to x,\, y\in E_x} \frac{|f(x)-f(y)-L(x-y)|}{|x-y|} = 0.$$

Notation: $L = D_{\mathrm{a}} f(x)$

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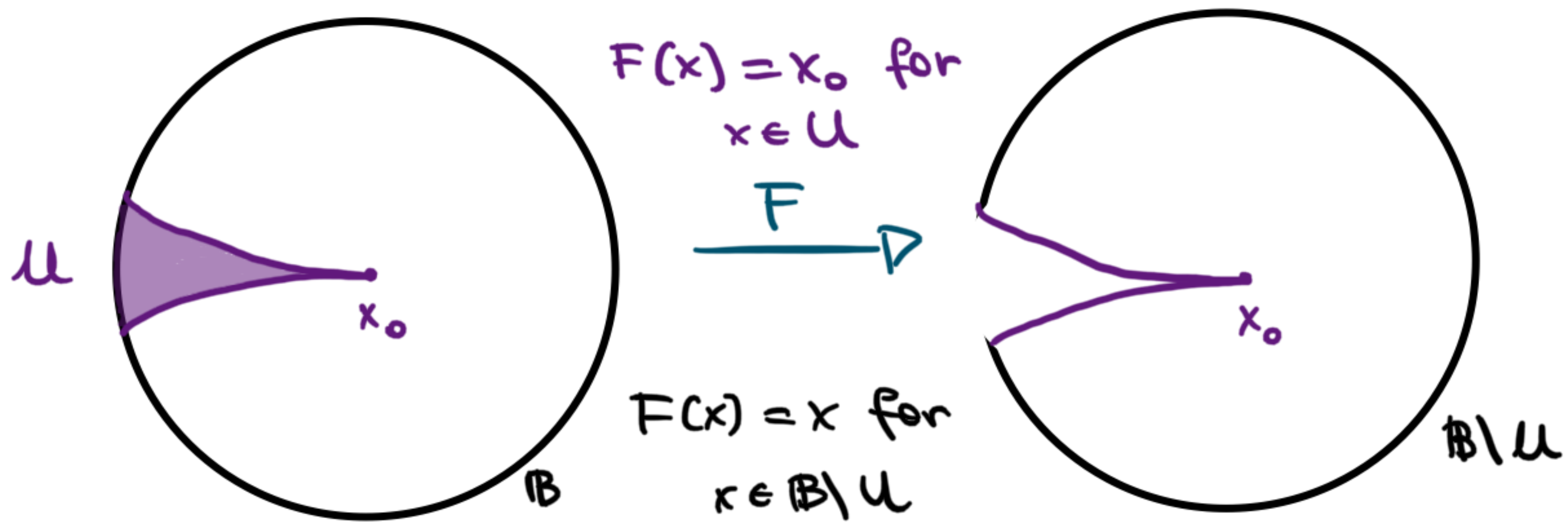
(Whitney) A function f is approximately differentiable a.e. on $\Omega\subset\mathbb{R}^n$ if for any $\varepsilon>0$ there exists a C^1 function $g:\mathbb{R}^n\to\mathbb{R}$ s. t.

$$|\{x\in\Omega : f(x)\neq g(x)\}| < \varepsilon.$$

- The mapping $F : \mathbb{B} \to \mathbb{R}^2$ is approximately differentiable at x_o but it is not classically differentiable at x_0.



- The Weierstrass nowhere differentiable function is also nowhere approximately differentiable.
- Old concept: appeared in 1916 in the works of Khintchine and Denjoy.
- For a while: contender for the best non-classical derivative.

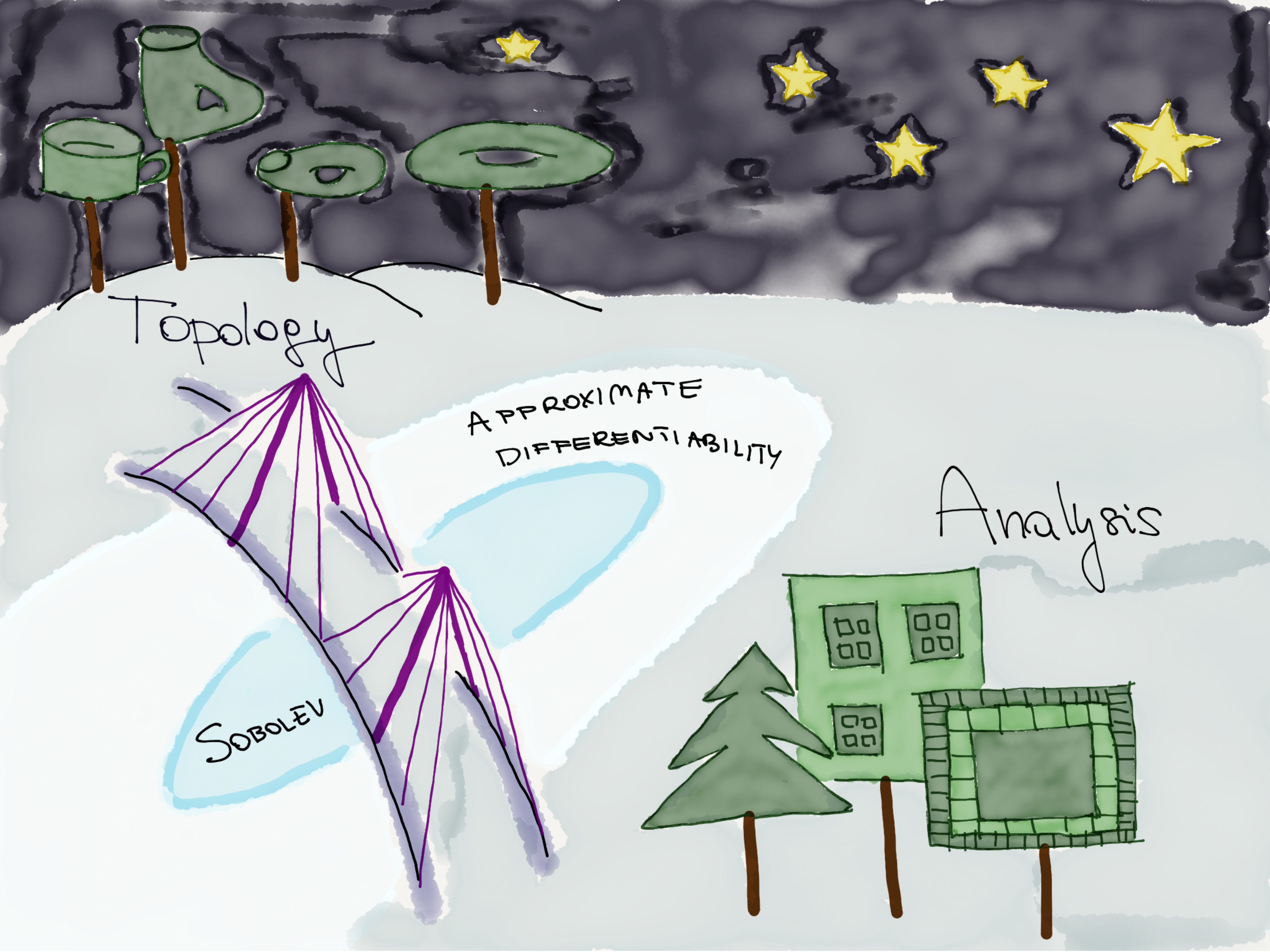
Approximate derivative is important

- Sobolev functions and functions of bounded variation are a.e. approximately differentiable.
- Federer's change of variables theorem: $J_{\mathrm{a}} f := \det D_{\mathrm{a}} f$

(Federer) Let Ω be an open set in $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^n$ be an a.e. approximately differentiable homeomorphism, which satisfies the Lusin condition (N). Then for any measurable function $\varphi : \mathbb{R}^n \to \mathbb{R}$, we have

$$\int_\Omega (\varphi \circ f)(x)\,|\,J_{\mathrm{a}} f(x)\,|\,dx = \int_{f(\Omega)} \varphi(y)\,dy.$$

- A.e. approximately differentiable mappings: limits of C^1-mappings in *the Lusin metric* $d_L(f, g) = |\,\{x : f(x) \neq g(x)\}\,|$
- Used for regularity results in Heisenberg groups (Capolli, Pinamonti, Speight).


Topology
APPROXIMATE DIFFERENTIABILITY
Analysis
SOBOLEV

Theorem 1. (Goldstein, Hajłasz; ARMA 2017) There is an a.e. approximately differentiable homeomorphism $\Phi : [0,1]^n \to [0,1]^n$, such that

- $\Phi = id$ on $\partial[0,1]^n$, → orientation-preserving
- $$D_a\Phi = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & -1 \end{bmatrix} \text{ a. e.,}$$ → negative Jacobian
- Φ satisfies the Lusin condition (N).
- Φ is a uniform limit of measure-preserving C^∞-diffeomorphisms.

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Hajłasz's question: does there exist a $W^{1,p}(\Omega, \mathbb{R}^n)$-homeomorphism which preserves orientation and whose Jacobian changes sign?

Depends on p and n! Answers *centered around* S. Hencl (Charles University, Prague). Not yet fully answered.

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Conjecture: it is possible to prescribe more general derivatives than the reflection matrix

Theorem 2. (P. Goldstein, Z.G., P. Hajłasz; AiM 2025) Let $Q = [0,1]^n$. For any measurable map $T : Q \to GL(n)$ that satisfies

$$\int_Q |\det T(x)|\, dx = 1,$$

Necessary because of Federer's Change of variables

there exists an a.e. approximately differentiable homeomorphism $\Phi : Q \to Q$ s. t. $\Phi|_{\partial Q} = \mathrm{id}$ and $D_\mathrm{a}\Phi = T$ a.e.

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- Basically any *sensible* mapping can be the approximate derivative of an a.e. approximately differentiable homeomorphism.
- But: assumption $T(x) \in GL(n)$ implies that $\det T(x) \neq 0$. This is not necessary, there are even Sobolev homeomorphisms with zero Jacobian a.e. (S. Hencl 2011).
- Is it more difficult to prescribe general T instead of a reflection matrix?

Theorem 2. **(P. Goldstein, Z.G., P. Hajłasz; AiM 2025)** Let $Q = [0,1]^n$. For any measurable map $T : Q \to GL(n)$ that satisfies

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Moreover,

- Φ^{-1} is approximately differentiable a.e. and $D_{\mathrm{a}}\Phi^{-1} = T^{-1}(\Phi^{-1}(y))$ for almost all $y \in Q$;

- Φ preserves sets of measure zero, i.e., for any $A \subset Q$,

$$|A| = 0 \text{ if and only if } |\Phi(A)| = 0;$$

- Φ is a limit of C^∞-diffeomorphisms $\Phi_k : Q \twoheadrightarrow Q$ in the uniform metric, i.e., $||\Phi - \Phi_k||_\infty + ||\Phi^{-1} - \Phi_k^{-1}||_\infty \to 0$ as $k \to 0$.

The moral: uniform limits of diffeomorphisms can be horrible.

Diffeomorphisms with prescribed derivative

Theorem 3. (P.G., Z.G., P.H.) Suppose that

- $\Omega \subset \mathbb{R}^n$ is a bounded domain;
- $F : \Omega \to \mathbb{R}^n$ is an orientation preserving diffeomorphism onto the bounded image $F(\Omega)$;
- $T : \Omega \to GL(n)^+$ is a measurable mapping s.t.

$$\int_{\Omega} \det T(x)\, dx \leq |F(\Omega)|.$$

Necessary because of Change of variables

Then for any $\varepsilon > 0$, there exists a diffeomorphism $\Phi : \Omega \to F(\Omega)$

- with $\Phi = F$ near $\partial\Omega$
- and a compact set $K \subset \Omega$ s.t.

$D\Phi = T$ on K and $|\Omega \backslash K| < \varepsilon$.

Idea of the proof of Theorem 3

A constructive proof: a lot of different tools

- Alberti's theorem (1991) about Lusin type theorem for gradients
- Many explicit constructions of diffeomorphisms
- Dacorogna-Moser theory of mappings with prescribed Jacobian
- Topological arguments to guarantee global injectivity

Sketch of the proof of Theorem 2

Theorem 2. (P. Goldstein, Z.G., P. Hajłasz; AiM 2025) Let $Q = [0,1]^n$. For any measurable map $T : Q \to GL(n)$ that satisfies

$$\int_Q |\det T(x)|\, dx = 1,$$

there exists an a.e. approximately differentiable homeomorphism $\Phi : Q \to Q$ s. t. $\Phi|_{\partial Q} = \mathrm{id}$ and $D_a\Phi = T$ a.e.

Step 1

- Reduction to the case $\det T > 0$.
- Smart use of the a.e. approximately differentiable homeomorphism with $J_a\Phi = -1$ (Theorem 1).

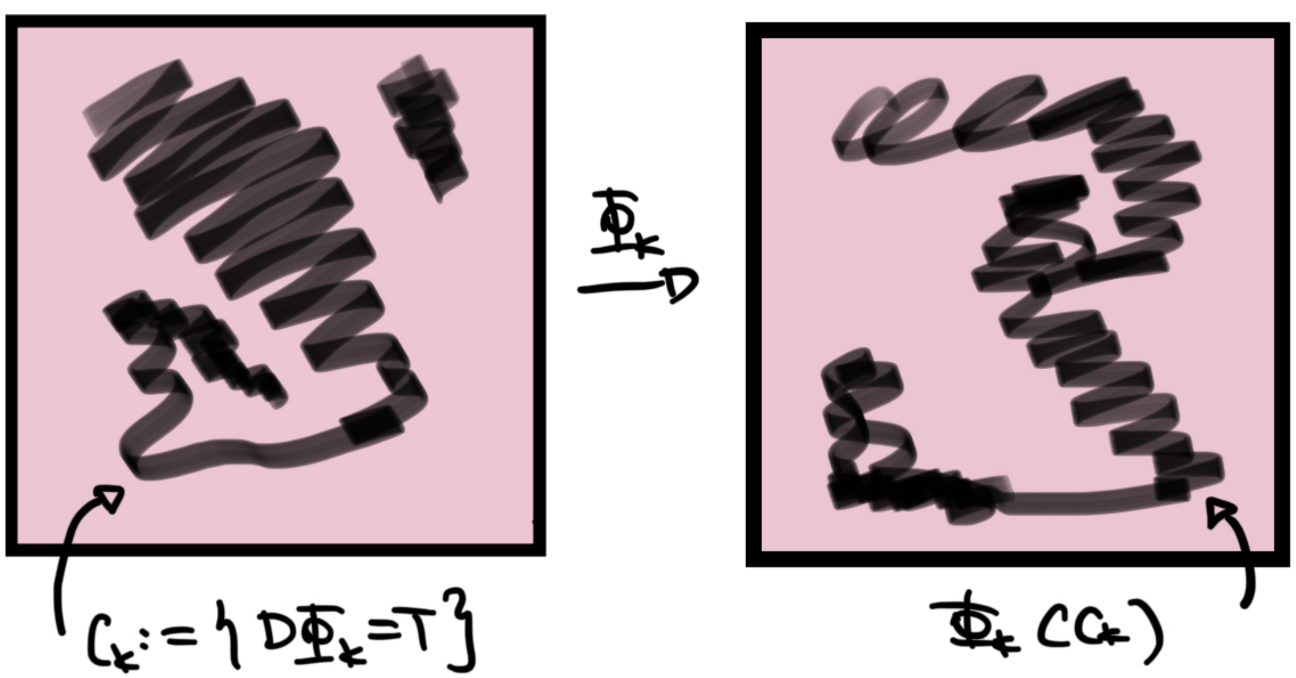
Step 2

- Prescribing the derivative of a diffeomorphism by Theorem 3.
- Why not just use Theorem 3?
- A naive iteration does not guarantee injectivity in the limit…

Sketch of the proof of Theorem 2

Step 2

- Prescribing the derivative of a diffeomorphism by Theorem 3.
- Why not just use Theorem 3? We get a sequence of diffeomorphisms Φ_k...
- A naive iteration does not guarantee injectivity in the limit...



$$\int_{Q \setminus C_k} \det T(x)\, dx = |\, \Phi_k(Q \setminus C_k) \,|$$

Applying Theorem 3 to $Q \setminus C_k$ means moving around points within the entire cube!

- So: difficult iteration scheme to ensure that the approximating sequence of diffeomorphisms converges uniformly.

Iteration scheme

- Aim: prescribe the derivative *at small scales*
- We need the volume constraint to hold *at small scales*
- We construct a sequence of diffeomorphisms $\Phi_k : Q \to Q$ and of partitions $\mathscr{P}_k = \{P_{ki}\}_i$ of Q such that
    - $D\Phi_k = T$ on a *large* part of Q,
    - $\operatorname{diam}(P_{ki})$ is *small*,
    - $\displaystyle\int_{P_{ki}} \det T(x)\, dx = |\Phi_k(P_{ki})|$.

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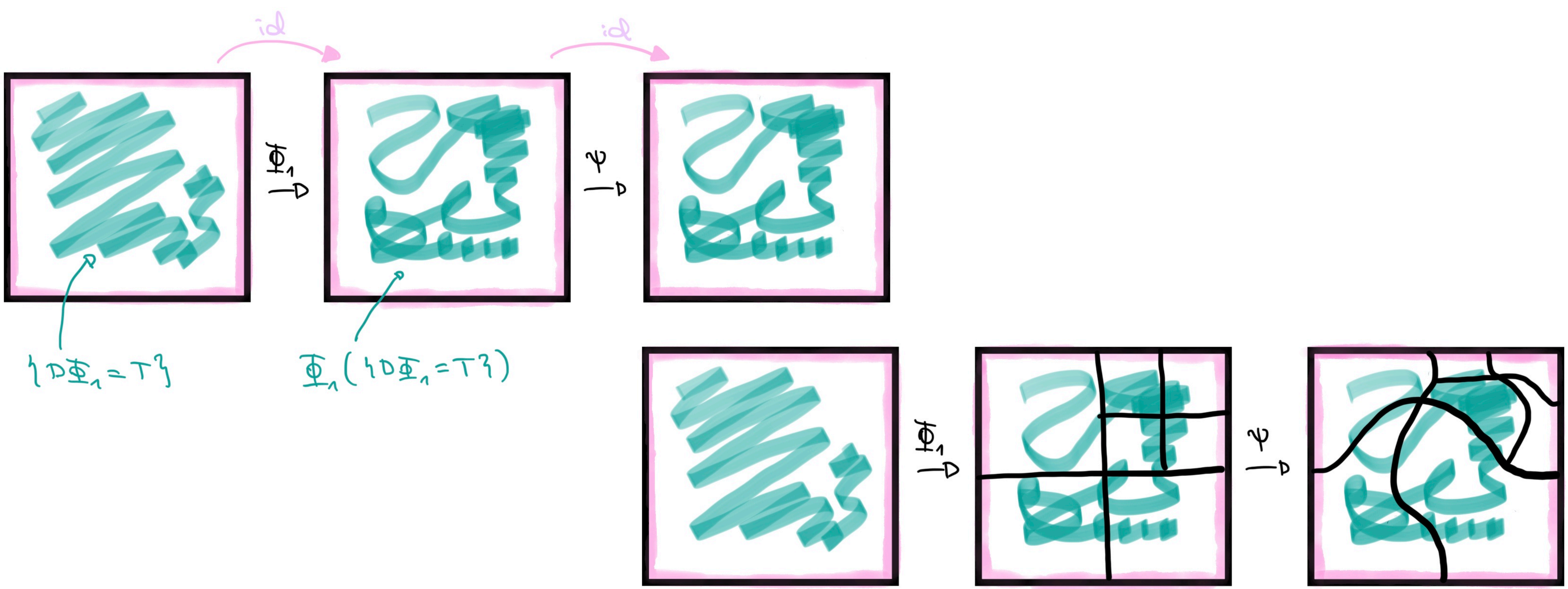
- Φ_1 — directly from Theorem 3:
  - $\Phi_1 = \mathrm{id}$ near ∂Q,
  - $D\Phi_1 = T$ on a compact set E_1,
  - the partition: the entire Q.

Construction of Φ_2: part 1

Correct the way in which Φ_1 distributes the measure (≈ correct its Jacobian)

- We find $\widetilde{\Phi}_2 := \Psi \circ \Phi_1$ and the partition $\mathscr{P}_2 = \{P_{2i}\}_i$,
- $\widetilde{\Phi}_2 = \Phi_1$ near ∂Q and near a *large* part of the set $\{D\Phi_1 = T\}$,
- $\displaystyle\int_{P_{2i}} \det T(x)\, dx = |\widetilde{\Phi}_2(P_{2i})|$.



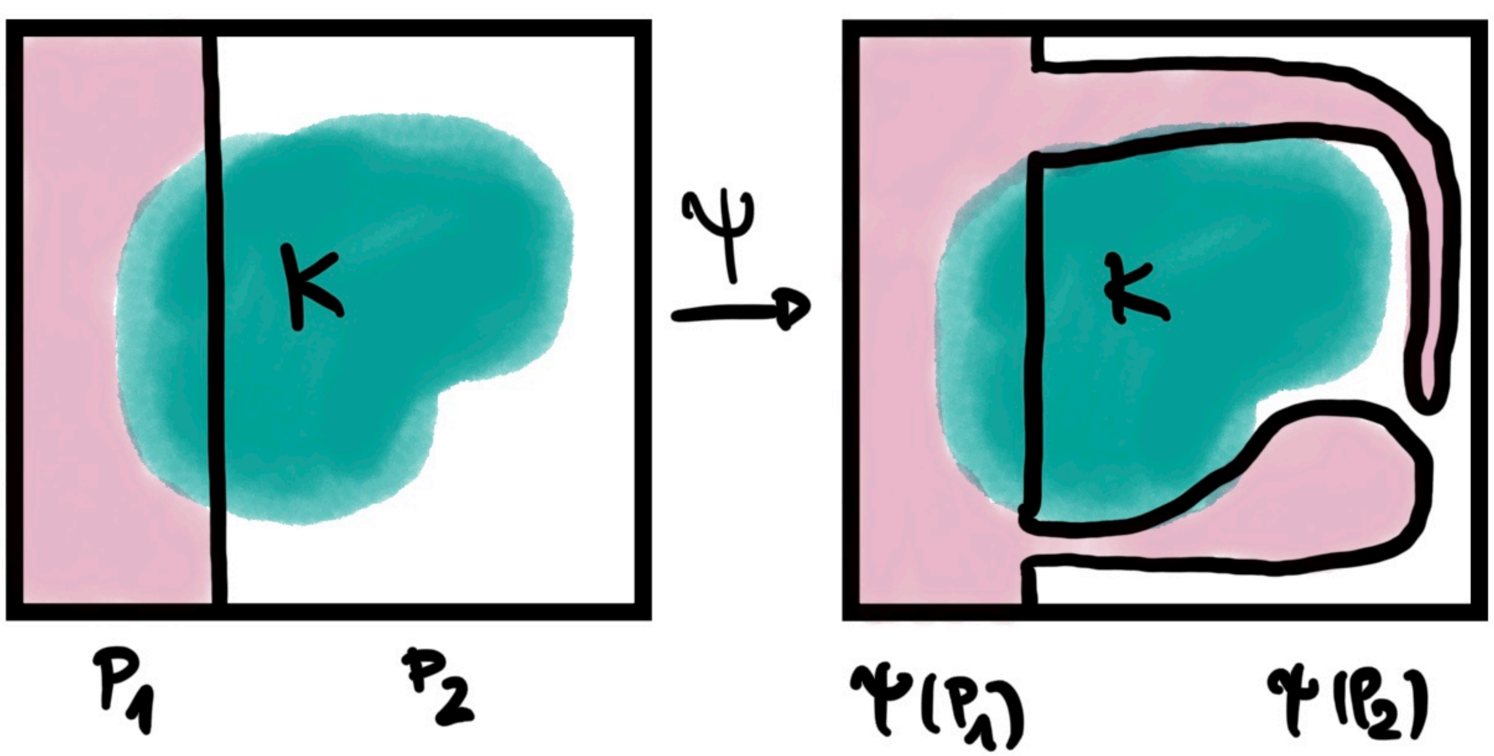
Idea behind Ψ

Let $f : Q \to [0,\infty)$ s.t.

- $\int_Q f(x)\,dx = |Q|$,
- K compact subset of Q,
- $f = 1$ a.e. on K,
- $Q = P_1 \cup P_2$.

Then there is a diffeomorphism $\Psi : Q \to Q$ s.t.

- $\Psi = \mathrm{id}$ near ∂Q and near a *large* part of K
- $\int_{P_i} f = |\Psi(P_i)|$ for $i = 1,2$.



Inspired by the Homeomorphic measures theorem of Oxtoby and Ulam (1944)

$$\mu(E) = |h(E)|$$

Construction of Φ_2: part 2

Reminder of part 1:

Correct the way in which Φ_1 distributes the measure ($\approx$ correct its Jacobian)

- We find $\widetilde{\Phi}_2 := \Psi \circ \Phi_1$ and the partition $\mathscr{P}_2 = \{P_{2i}\}_i$,
- $\widetilde{\Phi}_2 = \Phi_1$ near ∂Q and near a *large* part of the set $\{D\Phi_1 = T\}$,
- $\displaystyle\int_{P_{2i}} \det T(x)\, dx = |\,\widetilde{\Phi}(P_{2i})\,|$.

Part 2: Correct the derivative of $\widetilde{\Phi}_2$ inside each P_{2i} using Theorem 3.

→ This yields Φ_2 s.t. $D\Phi_2 = T$ on a larger set.

The end of the sketch of the proof

Topology helps

Topology helps

A very important corollary from Brouwer's Invariance of Domain theorem

Given two bounded domains Ω, Ω' in $\mathbb{R}^n$ and a homeomorphism $f : \overline{\Omega} \to \overline{\Omega'}$, we have

$$f(\partial\Omega) = \partial\Omega' \text{ and } f(\Omega) = \Omega'.$$

Homeomorphisms map boundaries to boundaries and interiors to interiors!

What is the use of Theorem 2?

- Gives little hope for the use of a.e. approximately differentiable mappings in nonlinear elasticity.
- Makes you appreciate *positive* results more.
- Could be useful in other constructions.
- Gives rise to further questions, for example:

What is the use of Theorem 2?

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- Gives rise to further questions, for example:

Question. Let $Q = [0,1]^n$ and $T : Q \to GL(n)^+$ be measurable with $\int_Q \det T(x)\, dx = 1$.

Does there exist a homeomorphism $\Phi : Q \to Q$, $\Phi|_{\partial Q} = \mathrm{id}$ which is **differentiable** a.e. on Q with $D\Phi = T$ a.e.?

Thank you!