PB-groupoids vs VB-groupoids

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joint work with Alfonso Garmendia

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Geometric structures and infinite-dimensional manifolds

Erwin Schrödinger Institute, Vienna, 13th January 2025

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vector bundle of rank k

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Vector bundles over M of rank k $\longleftrightarrow$ *principal bundles over M with structural Lie group $\mathrm{GL}(k)$*

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What about replacing M with a Lie groupoid $\mathcal{G} \rightrightarrows M$?

Outline

1. VB-groupoids
2. Frames of a VB-groupoid
3. PB-groupoids
4. PB-VB correspondence

VB-groupoid (Pradines, 1988) = vector bundle object in the category of Lie groupoids = Lie groupoid object in the category of vector bundles

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- blow-up constructions in index theory (Debord and Skandalis, 2021)

VB-groupoid (Pradines, 1988) = vector bundle object in the category of Lie groupoids = Lie groupoid object in the category of vector bundles

$$\begin{array}{ccc} E_{\mathcal{G}} & \xrightarrow{\pi_{\mathcal{G}}} & \mathcal{G} \\ \tilde{t}\,\downarrow\downarrow\,\tilde{s} & & t\,\downarrow\downarrow\,s \\ E_M & \xrightarrow{\pi_M} & M \end{array}$$

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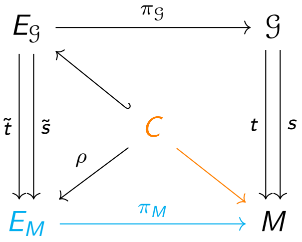

core $C := \ker(\tilde{s}) \cap \ker(\pi_{\mathcal{G}}) \subseteq E_{\mathcal{G}}$

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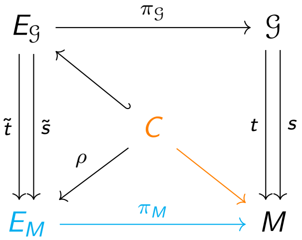

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core-anchor $\rho := \tilde{t}|_C : C \to E_M$

rank $(l, k) := \big(\mathrm{rank}(C \to M), \mathrm{rank}(E_M \to M)\big)$
(so that $\mathrm{rank}(E_{\mathcal{G}}) = l + k$)

Example (tangent VB-groupoid)

$$
\begin{array}{ccc}
T\mathcal{G} & \xrightarrow{\;\pi_{\mathcal{G}}\;} & \mathcal{G} \\
dt \big\downarrow\big\downarrow ds & & t \big\downarrow\big\downarrow s \\
TM & \xrightarrow{\;\pi_M\;} & M
\end{array}
$$

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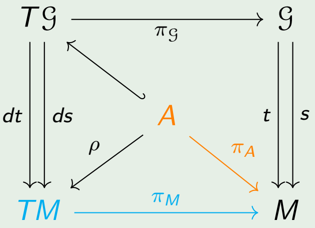

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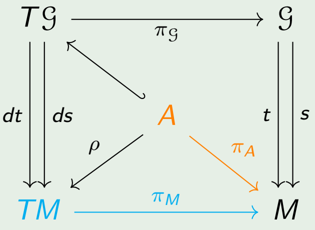

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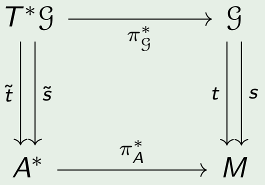

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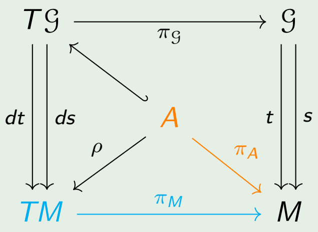

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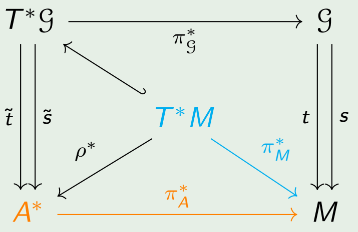

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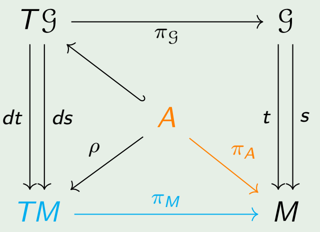

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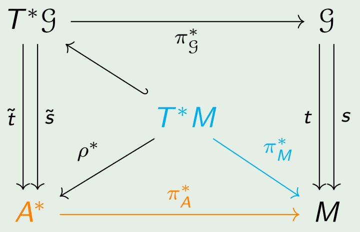

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$E_{\mathcal{G}} \rightrightarrows E_M$ VB-groupoid of rank (l, k) with core C

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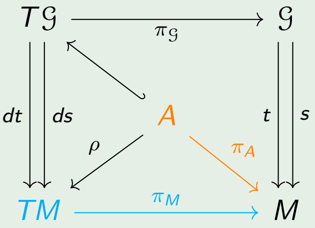

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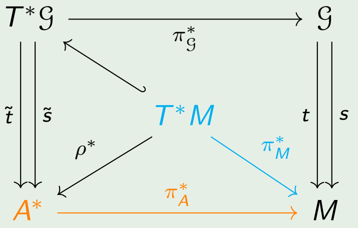

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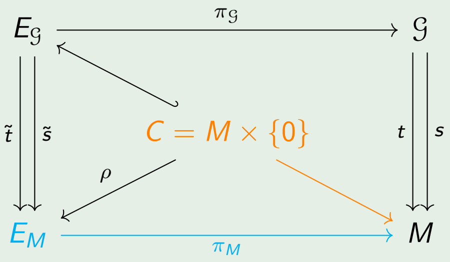
Remark (duality between VB-groupoids)

$E_{\mathcal{G}} \rightrightarrows E_M$ VB-groupoid of rank (l, k) with core C
$\Rightarrow E^*_{\mathcal{G}} \rightrightarrows C^*$ VB-groupoid of rank (k, l) with core E^*_M

Example (rank $(0, k)$ (trivial-core) and $(l, 0)$ (trivial-base))



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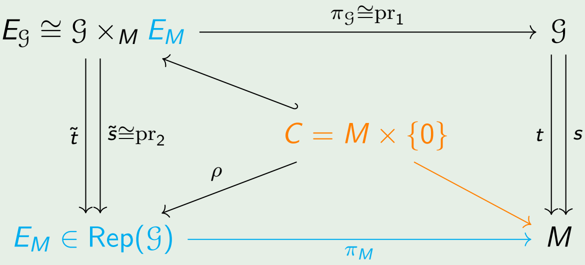

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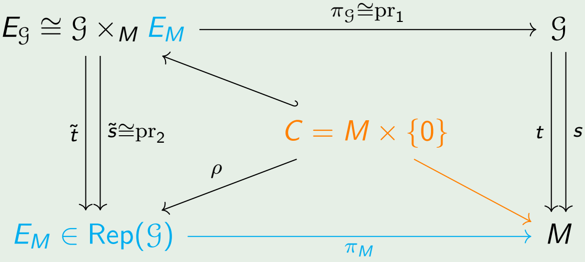

For $E_{\mathcal{G}} = T\mathcal{G}$ and $E_M = TM$ we get $\mathcal{G} \rightrightarrows M$ étale and its canonical representation on TM

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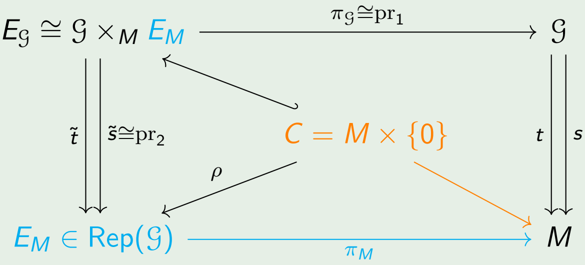

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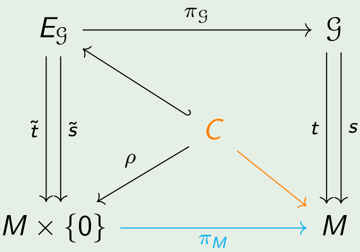

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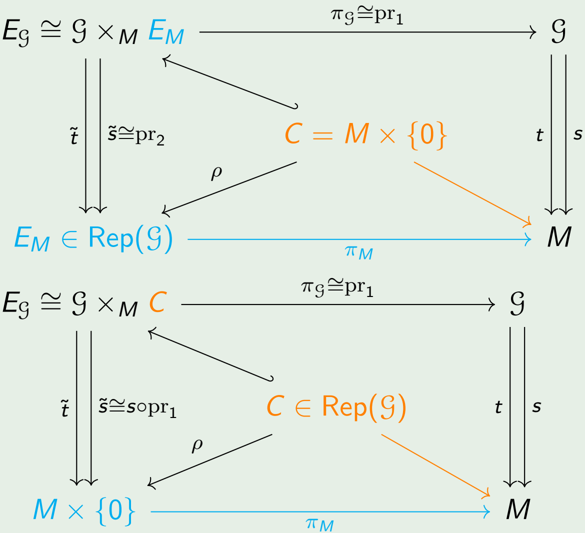

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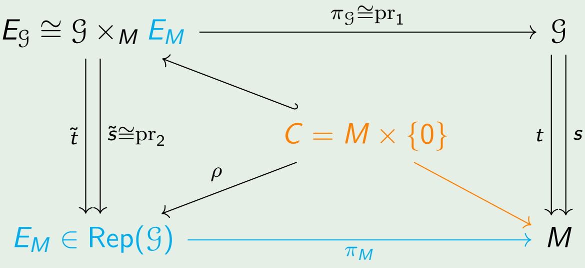

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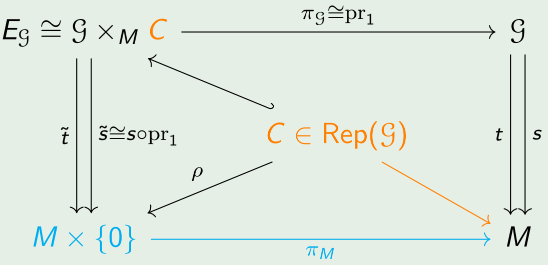

For $E_{\mathcal{G}} = T\mathcal{G}$ and $E_M = TM$ we get $M = 0$ and the adjoint representation on $C = \mathfrak{g}$

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with $d_{\phi_g} := (t(\phi_g)^b)^{-1} \circ \rho_{t(g)} \circ t(\phi_g)^c : \mathbb{R}^l \to \mathbb{R}^k$

Proposition (C. - Garmendia '24)

$\mathrm{Fr}^{ad}(E_{\mathfrak{g}}) \rightrightarrows \mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M)$ *is a Lie groupoid*

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- $\bar{s}(\phi_g) = \left(s(\phi_g)^c, s(\phi_g)^b\right)$
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- $\bar{m}(\phi_g, \phi_h) = \tilde{m}(\phi_g|_{\{0\}\times\mathbb{R}^k}, \phi_h|_{\{0\}\times\mathbb{R}^k}) + \tilde{m}(\cdot, 0_h) \circ \phi_g|_{\mathbb{R}^l\times\{0\}}$
- $\bar{u}(\varphi_x^c, \varphi_x^b) = \varphi_x^c \circ \mathrm{pr}_1 + \tilde{u} \circ \varphi_x^b \circ \mathrm{pr}_2$
- $\bar{i}(\phi_g) = \tilde{i}\left(\phi_g|_{\{0\}\times\mathbb{R}^k} + \phi_g(-\bullet, d_{\phi_g}(\bullet))\right)$

Proposition (C. - Garmendia '24)

There is a natural action of the strict Lie 2-groupoid $\mathrm{GL}_2(l,k) \rightrightarrows \mathrm{GL}_1(l,k) \rightrightarrows \mathrm{GL}_0(l,k)$ *on the Lie groupoid* $\mathrm{Fr}^{ad}(E_{\mathfrak{g}}) \rightrightarrows \mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M)$

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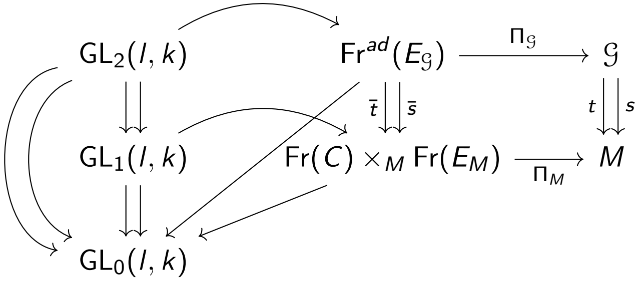
Any VB-groupoid induces therefore a diagram



(strict) Lie 2-groupoid = double Lie 2-groupoid over the unit groupoid

$$\begin{array}{ccc} \mathcal{H}_2 & \rightrightarrows & \mathcal{H}_1 \\ \downdownarrows & & \downdownarrows \\ \mathcal{H}_0 & \rightrightarrows & \mathcal{H}_0 \end{array}$$

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Example (General linear 2-groupoid of rank (l, k))

- $\mathrm{GL}_0(l, k) = \mathrm{Hom}(\mathbb{R}^l, \mathbb{R}^k)$
- $\mathrm{GL}_1(l, k) = \mathrm{Hom}(\mathbb{R}^l, \mathbb{R}^k) \times \mathrm{GL}(l) \times \mathrm{GL}(k)$
- $\mathrm{GL}_2(l, k) \subseteq \mathrm{Hom}(\mathbb{R}^l, \mathbb{R}^k) \times \mathrm{GL}(l + k)$

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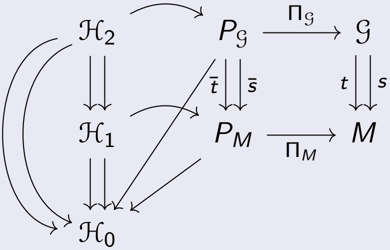
- $\mathrm{GL}_2(l,k) \subseteq \mathrm{Hom}(\mathbb{R}^l, \mathbb{R}^k) \times \mathrm{GL}(l+k)$

It is a particular case, for $E = (\mathbb{R}^l, \mathbb{R}^k) \to \{*\}$, of the general linear 2-groupoid $\mathrm{GL}(E)$ (del Hoyo and Stefani, 2019) and the 2-gauge groupoid 2-$\mathrm{Gau}(E)$ (Brahic and Ortiz, 2019) introduced to study 2-term RUTHs

Definition (C. - Garmendia '24)

PB-groupoid = diagram of Lie groupoids and principal bundles, with an action of a strict Lie 2-groupoid



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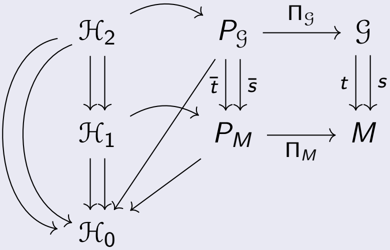

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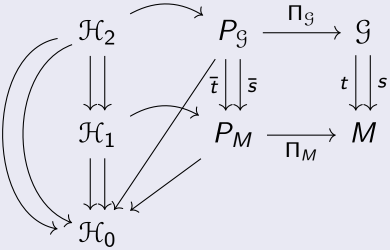

such that the action maps defines a Lie groupoid morphism $P_{\mathcal{G}} \times_{\mathcal{H}_0} \mathcal{H}_2 \to P_{\mathcal{G}}$, and the Lie groupoids $P_{\mathcal{G}}/\mathcal{H}_2 \rightrightarrows P_M/\mathcal{H}_1$ and $\mathcal{G} \rightrightarrows M$ are isomorphic

Example (particular cases)

When $\mathcal{H}_0 = 0$, the Lie 2-groupoid is a Lie 2-group and we recover,

- principal 2-bundles (Nikolaus and Waldorf, 2013)
- principal bundle groupoids (Garmendia and Paycha, 2023)
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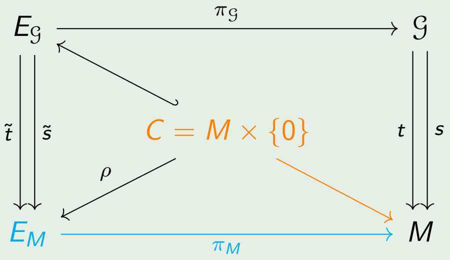
When $\mathcal{H}_2 = \mathcal{H}_1$ and $\mathcal{H}_0 = 0$, the Lie 2-groupoid is a Lie group and we recover

- PBG-groupoids (Mackenzie, 1987)
- principal bundles over a groupoid (Laurent-Gengoux, Tu and Xu, 2004)
- G-groupoids (Bruce, Grabowska and Grabowski, 2017)

Example (frames of VB-groupoids with trivial cores and trivial bases)



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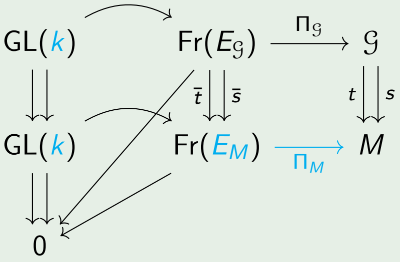

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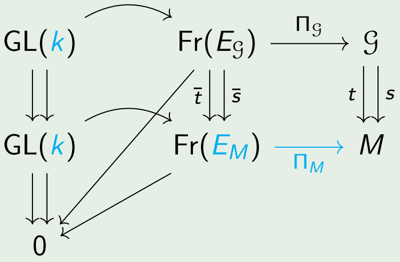

For $E_{\mathcal{G}} = T\mathcal{G}$ and $E_M = TM$ (hence $\mathcal{G} \rightrightarrows M$ étale), $\bar{m}(\phi_g, \phi_h) = d_{(g,h)} m(\phi_g(\cdot), \phi_h(\cdot))$

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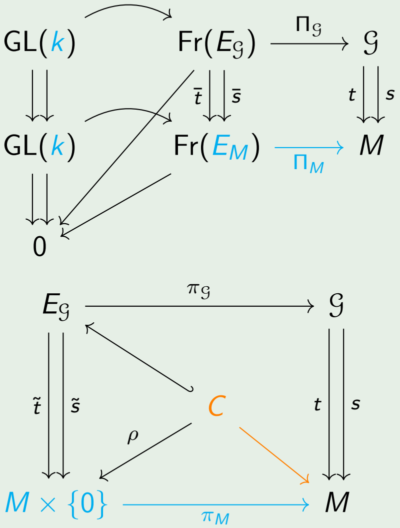

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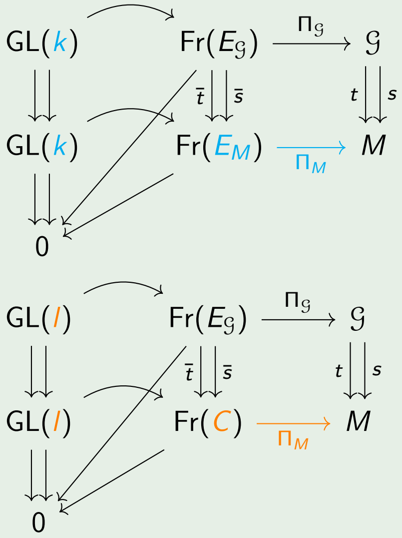

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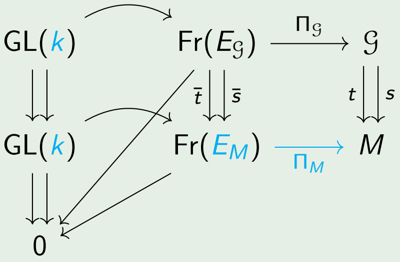

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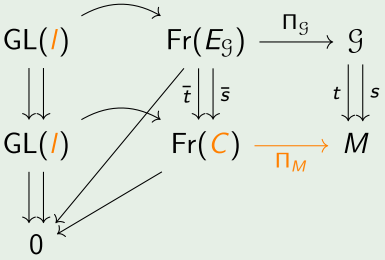

For $E_{\mathcal{G}} = T\mathcal{G}$ and $E_M = TM$ (hence $M = 0$ and $\mathcal{G} = G$ Lie group), $\bar{m}(\phi_g, \phi_h) = d_g R_h(\phi_g(\cdot))$

Example (adapted frames and duality of VB-groupoids)

Consider a VB-groupoid $E_{\mathcal{G}} \rightrightarrows E_M$ over $\mathcal{G} \rightrightarrows M$ of rank (l, k) with core $C \to M$, and its dual $E^*_{\mathcal{G}} \rightrightarrows C^*$ (of rank (k, l) with core $E^*_M \to M$).

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Then the following PB-groupoids

$$\begin{array}{ccc} \mathrm{Fr}^{ad}(E_{\mathcal{G}}) & \xrightarrow{\Pi_{\mathcal{G}}} & \mathcal{G} \\ \downarrow\downarrow & & \downarrow\downarrow \\ \mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M) & \xrightarrow[\Pi_M]{} & M \end{array} \qquad \begin{array}{ccc} \mathrm{Fr}^{ad}(E^*_{\mathcal{G}}) & \xrightarrow{\Pi_{\mathcal{G}}} & \mathcal{G} \\ \downarrow\downarrow & & \downarrow\downarrow \\ \mathrm{Fr}(E^*_M) \times_M \mathrm{Fr}(C^*) & \xrightarrow[\Pi_M]{} & M \end{array}$$

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Proposition (C. - Garmendia '24)

$E_{\mathcal{G}} \rightrightarrows E_M$
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Proposition (C. - Garmendia '24)

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Corollary

VB-groupoids over $\mathcal{G} \rightrightarrows M$ *of rank* (l,k) $\longleftrightarrow$ *PB-groupoids over* $\mathcal{G} \rightrightarrows M$ *with structural Lie 2-groupoid* $\mathrm{GL}(l,k)$

Open problems and future projects

- Frames of a VB-algebroid and infinitesimal counterpart of a PB-groupoid

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Thank you for your attention