Uniformization of metric surfaces

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University of Firbourg

Friday 17th January, 2025

Geometric structures and infinite-dimensional manifolds - ESI Vienna

Uniformization problem

Uniformization problem: Find conditions on a metric space X homeomorphic to a model space M such that there exists a mapping

$$u\colon M \to X$$

with good geometric and analytic properties.

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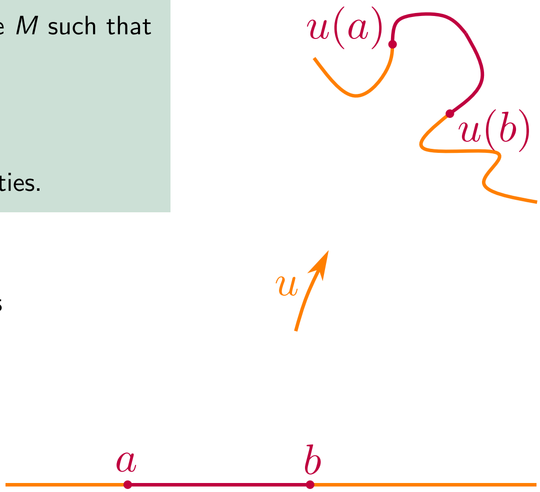

Dimension 1:

- Every locally rectifiable curve admits a parametrization by **arclength**.
  - u is 1-Lipschitz, i.e.

$$d(u(a), u(b)) \leq L \cdot |a - b|$$

for $L = 1$.

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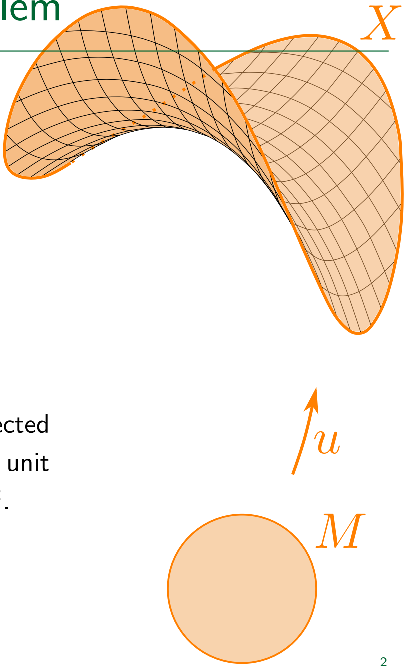
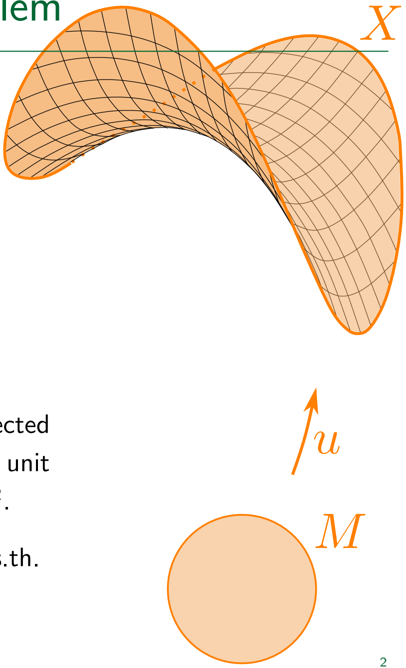


Dimension 2:

- **Classical uniformization theorem:** Every simply connected Riemann surface X is conformally equivalent to the open unit disc D, the complex plane $\mathbb{C}$, or the Riemann sphere $\mathbb{S}^2$.

Uniformization problem



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Dimension 2:

- **Classical uniformization theorem:** Every simply connected Riemann surface X is conformally equivalent to the open unit disc D, the complex plane $\mathbb{C}$, or the Riemann sphere $\mathbb{S}^2$.
  - Conformal map is locally bi-Lipschitz, i.e. $\exists L \geq 1$ s.th.

$$L^{-1} \cdot |a-b| \leq d(u(a), u(b)) \leq L \cdot |a-b|.$$

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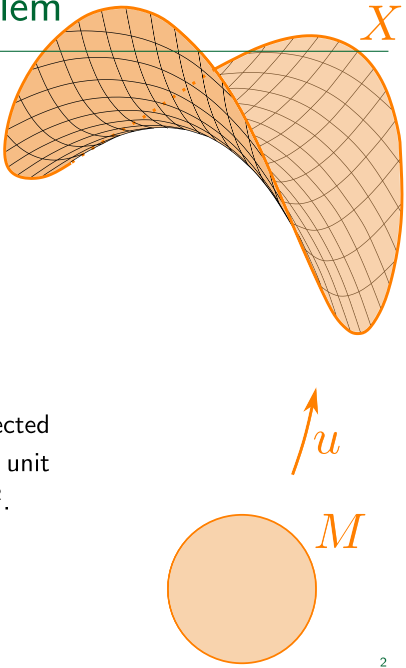


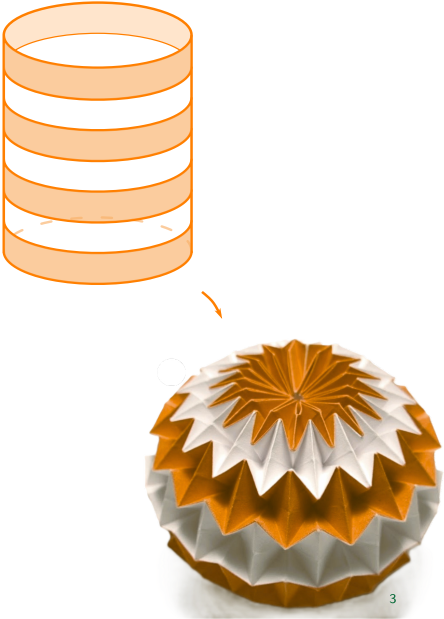
Dimension 2:

- **Classical uniformization theorem:** Every simply connected Riemann surface X is conformally equivalent to the open unit disc D, the complex plane $\mathbb{C}$, or the Riemann sphere $\mathbb{S}^2$.
  - Conformal map is locally bi-Lipschitz.
  - Maps infinitesimal balls to balls.

Metric surfaces

Definition: A metric space X is a *metric surface* if X is homeomorphic to a 2-dimensional manifold M.

- Non-smooth metric surfaces appear naturally as
  - deformations of smooth surfaces,
  - limits of sequences of Riemannian surfaces,
  - boundaries of Gromov hyperbolic groups.



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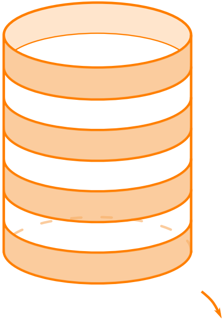
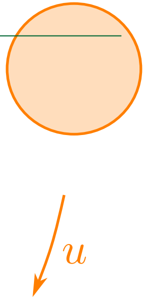


Goal: Find conditions on X such that there exists a parametrization $u\colon M \to X$ satisfying certain properties.

Uniformization of metric surfaces



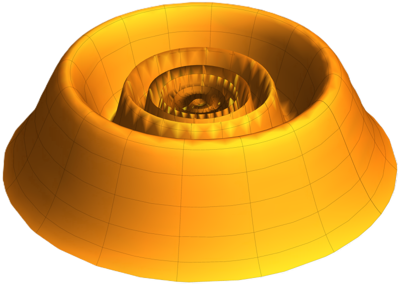
Let X be a metric surface homeomorphic to a Riemannian surface M.

Question: What type of parametrization $u\colon M \to X$ can we expect?

- If $u\colon M \to X$ is *Lipschitz*, then

$$\ell(u \circ \gamma) \leq L \cdot \ell(\gamma) \quad \text{for every curve } \gamma \text{ in } M.$$

$\Rightarrow$ Every pair of points in X can be joined by a curve of finite length.



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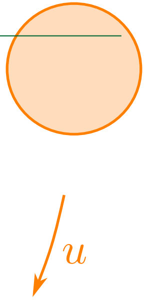

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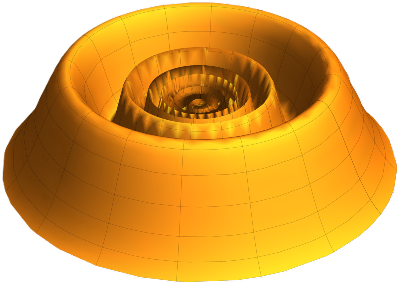


Example: Surface of revolution X

- Possesses finite Hausdorff 2-measure,
- Smooth except for 0,
- Every curve passing through 0 has infinite length.

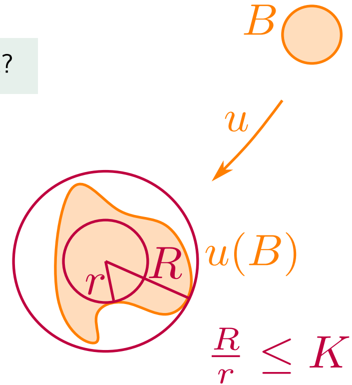
$\Rightarrow$ X does not possess a Lipschitz parametrization $u\colon D \to X$.

Uniformization of non-smooth metric surfaces

Let X be a metric surface homeomorphic to a Riemannian surface M.

Question: What type of parametrization $u\colon M \to X$ can we expect?

1. **Quasisymmetric uniformization:** A homeomorphism $u\colon M \to X$ is *quasisymmetric* if it distorts shapes of sets in a controlled manner on all scales.



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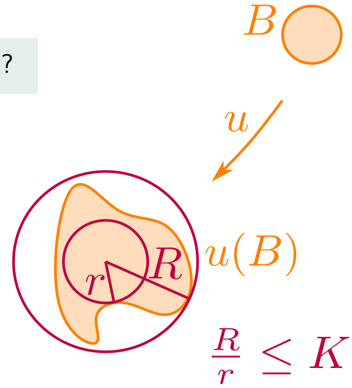
2. **Quasiconformal uniformization:** A homeomorphism $u\colon M \to X$ is *quasiconformal* if it distorts shapes of sets in a controlled manner on infinitesimal scales.

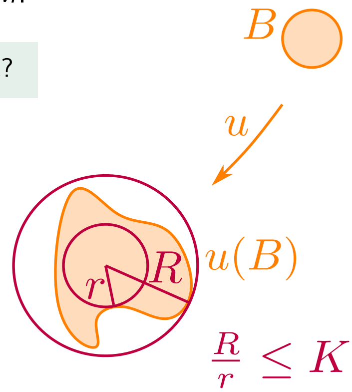


Uniformization of non-smooth metric surfaces

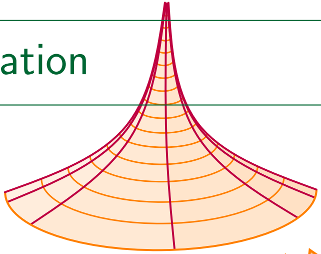
Let X be a metric surface homeomorphic to a Riemannian surface M.

Question: What type of parametrization $u: M \to X$ can we expect?

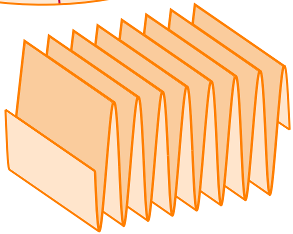
1. **Quasisymmetric uniformization:** A homeomorphism $u: M \to X$ is *quasisymmetric* if it distorts shapes of sets in a controlled manner on all scales.
2. **Quasiconformal uniformization:** A homeomorphism $u: M \to X$ is *quasiconformal* if it distorts shapes of sets in a controlled manner on infinitesimal scales.
3. **Weakly quasiconformal uniformization:** An almost homeomorphism $u: M \to X$ is *weakly quasiconformal* if it distorts shapes of sets in a controlled manner on infinitesimal scales.



Quasisymmetric uniformization



Theorem [Bonk–Kleiner 2002]: Let $X \approx S^2$ be an *Ahlfors 2-regular* metric surface. There exists a quasisymmetric map $u\colon S^2 \to X$ if and only if X is *linearly locally contractible*.



Ahlfors 2-regularity: $\mathcal{H}^2(B(x,r))$ is comparable to r^2.

Linear local contractibility (LLC): $B(x,r)$ is contractible in $B(x,\lambda r)$.

$\Rightarrow$ Prevent surface from having cusps, thin bottlenecks, dense wrinkles.

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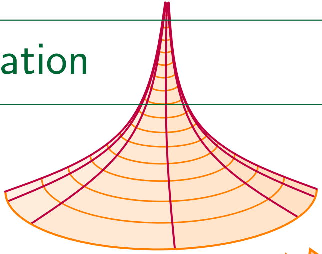

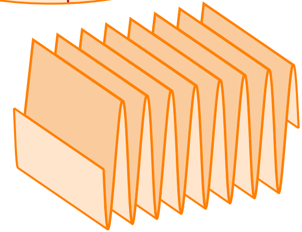

- Theorem does not generalize to higher dimensions (Semmes, Heinonen-Wu, Pankka-Wu).

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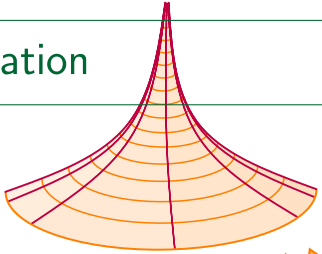

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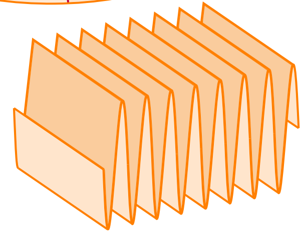
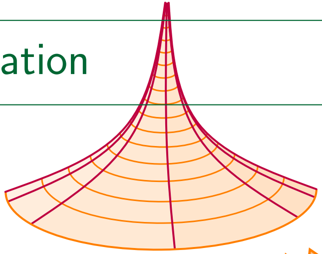


- Theorem does not generalize to higher dimensions (Semmes, Heinonen-Wu, Pankka-Wu).
- Ahlfors 2-regularity is not a quasisymmetric invariant.
  - $\mathrm{id}\colon S^2 \to (S^2, d_{S^2}^{\alpha})$ for $\alpha \in (0,1)$ is quasisymmetry.

Quasisymmetric uniformization



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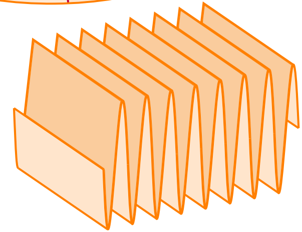

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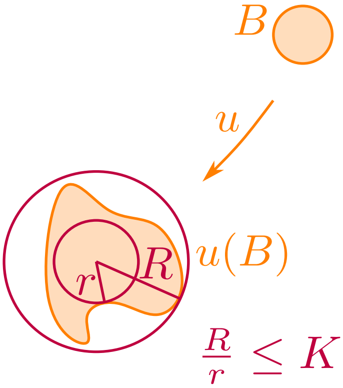
- Same statement without Ahlfors 2-regularity would solve:

Cannon's conjecture: Let G be a Gromov hyperbolic group whose boundary at infinity $\partial_\infty G$ is homeomorphic to S^2. Then G is a Kleinian group, i.e. G admits an isometric, properly discontinuous, and cocompact action on $\mathbb{H}^3$.

Geometric Quasiconformality

Let X, Y be metric surfaces.

- A homeomorphism $u\colon X \to Y$ is *quasiconformal* if it distorts **shapes of sets** in a controlled manner on infinitesimal scales.



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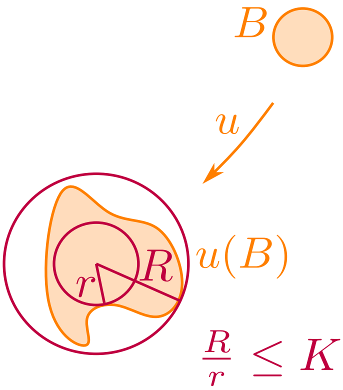
- A homeomorphism $u\colon X \to Y$ is *geometrically quasiconformal* if it distorts **families of curves** in a controlled manner.



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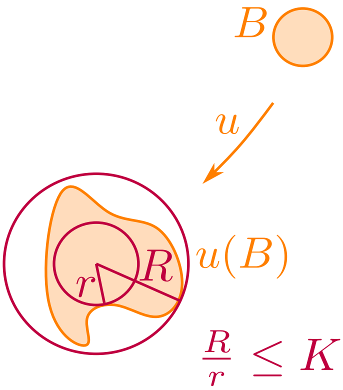
Question: How can we measure "largeness" of families of curves?



Geometric Quasiconformality

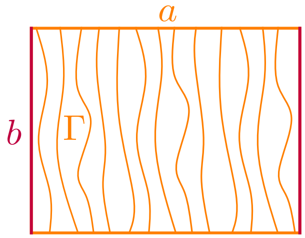
Let X, Y be metric surfaces and Γ a family of curves in X.

- The *(conformal) modulus* of Γ is

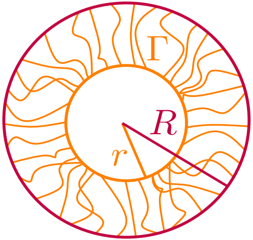
$$\operatorname{mod}(\Gamma) := \inf \int_X \rho^2 \, d\mathcal{H}^2,$$

where the infimum is taken over all Borel functions $\rho\colon X \to [0, \infty]$ with

$$\int_\gamma \rho \geq 1 \quad \text{for every locally rectifiable } \gamma \in \Gamma.$$



$\operatorname{mod}(\Gamma) = \frac{a}{b}$



$\operatorname{mod}(\Gamma) = 2\pi \left(\log\left(\frac{R}{r}\right)\right)^{-1}$

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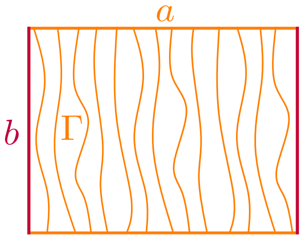
- $u \colon X \to Y$ is *geometrically quasiconformal* if $\exists K \geq 1$ s.th.

$$K^{-1} \operatorname{mod}(\Gamma) \leq \operatorname{mod}(u \circ \Gamma) \leq K \operatorname{mod}(\Gamma)$$

  for every family Γ of curves in X.



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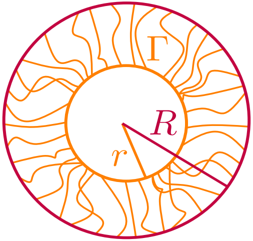

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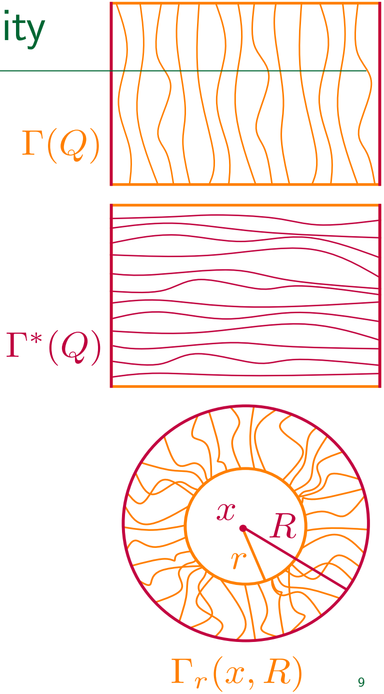
Geometric Quasiconformality

Modulus in the plane:

(1) If $Q \subset \mathbb{R}^2$ is a quadrilateral, then

$$\mathrm{mod}(\Gamma(Q)) \cdot \mathrm{mod}(\Gamma^*(Q)) = \frac{a}{b} \cdot \frac{b}{a} = 1.$$



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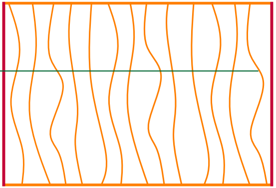
(2) If $x \in \mathbb{R}^2$ and $0 < r < R < \infty$, then

$$\lim_{r \to 0} \mathrm{mod}(\Gamma_r(x, R)) = \lim_{r \to 0} 2\pi \left(\log \left(\frac{R}{r} \right) \right)^{-1} = 0.$$

$\Gamma(Q)$



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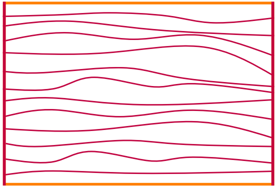

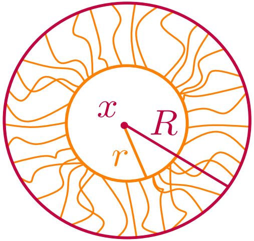

$\Gamma_r(x, R)$

Geometric Quasiconformality

If $u\colon U \subset \mathbb{R}^2 \to X$ is geometrically quasiconformal, then

(1) For every quadrilateral $Q \subset X$

$$\kappa^{-1} \leq \operatorname{mod}(\Gamma(Q)) \cdot \operatorname{mod}(\Gamma^*(Q)) \leq \kappa.$$

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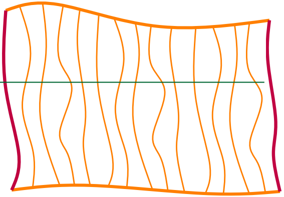

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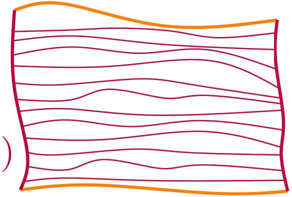

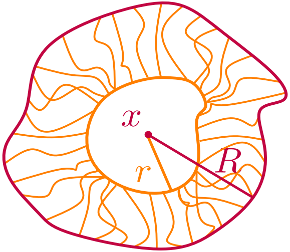

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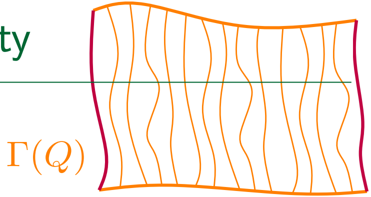
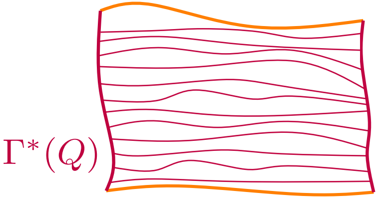
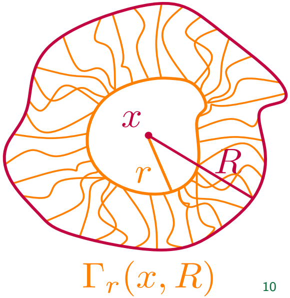
For example: $\mathbb{R}^2$ and Ahlfors 2-regular metric spaces.







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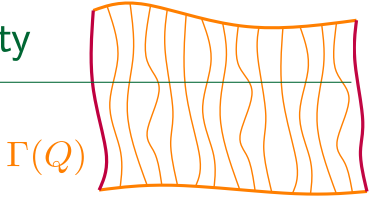
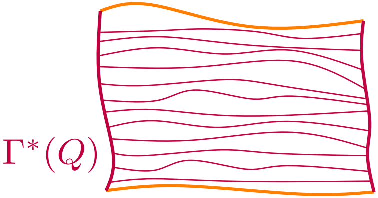
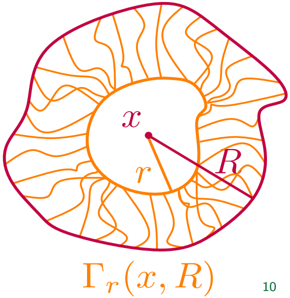
$$\lim_{r \to 0} \operatorname{mod}(\Gamma_r(x, R)) = 0.$$

For example: $\mathbb{R}^2$ and Ahlfors 2-regular metric spaces.

Theorem [Rajala 2017]: Let $X \approx \mathbb{R}^2$ be a metric surface of locally finite $\mathcal{H}^2$. There exists a (geometrically) quasiconformal map from a domain $U \subset \mathbb{R}^2$ onto X if and only if X satisfies (1) and (2).







Geometric Quasiconformality

Definition: A metric surface X is *reciprocal* if X satisfies

(1) For every quadrilateral $Q \subset X$

$$\kappa^{-1} \leq \text{mod}(\Gamma(Q)) \cdot \text{mod}(\Gamma^*(Q)) \leq \kappa.$$

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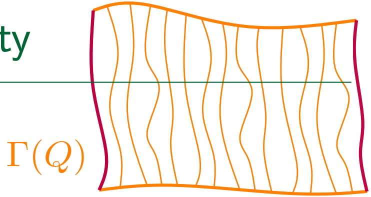
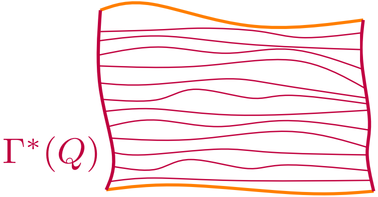
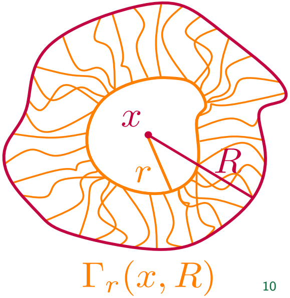
$$\lim_{r \to 0} \text{mod}(\Gamma_r(x, R)) = 0.$$

For example: $\mathbb{R}^2$ and Ahlfors 2-regular metric spaces.

Theorem [Rajala 2017]: Let $X \approx \mathbb{R}^2$ be a metric surface of locally finite $\mathcal{H}^2$. There exists a (geometrically) quasiconformal map from a domain $U \subset \mathbb{R}^2$ onto X if and only if X is reciprocal.







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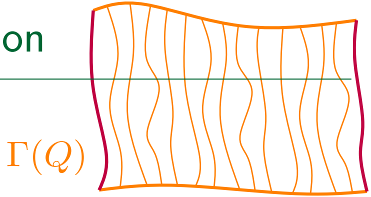

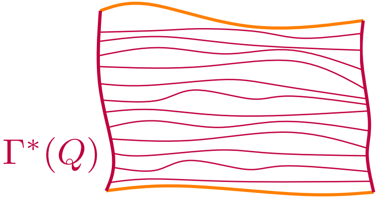

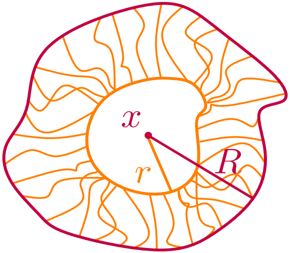

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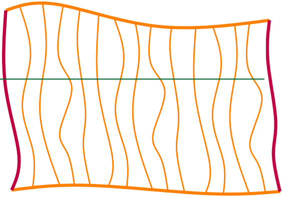
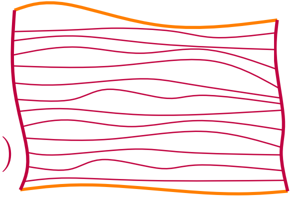
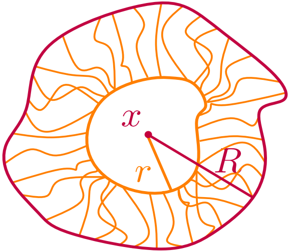
$\Gamma(Q)$



$\Gamma^*(Q)$





$\Gamma_r(x, R)$

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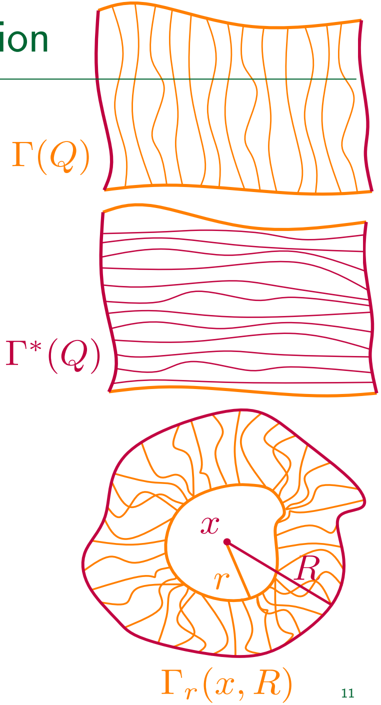
- Lower bound in (1) always holds for $\kappa^{-1} = (\pi/4)^2$ (Rajala–Romney, Eriksson-Bique–Poggi-Corradini).
- Upper bound in (1) implies reciprocality, whereas condition (2) does not imply reciprocality (Ntalampekos–Romney).



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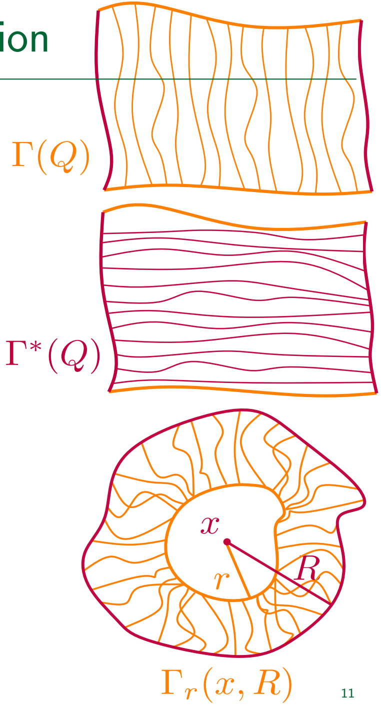
Definition: A metric surface X is *reciprocal* if $\exists\, \kappa \geq 1$ such that for every quadrilateral $Q \subset X$

$$\operatorname{mod}(\Gamma(Q)) \cdot \operatorname{mod}(\Gamma^*(Q)) \leq \kappa.$$

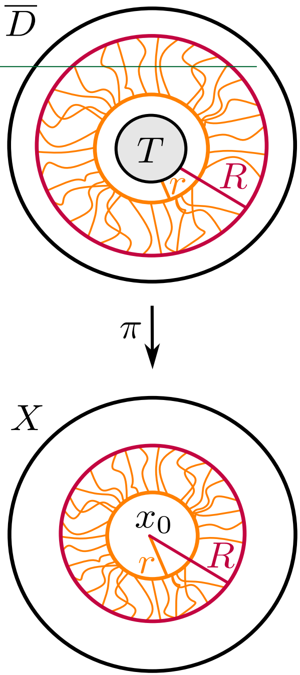
- In general, reciprocality condition is difficult to verify.
- There exist plenty of metric surfaces that are not reciprocal.



Quasiconformal uniformization

Example: Let $T := \overline{B}(0, \varepsilon)$ for $0 < \varepsilon < 1$ and $X := \overline{D}/T$.

- The natural projection $\pi \colon \overline{D} \to X$ is local isometry on $\overline{D} \setminus T$ and maps T to the point $x_0 := \pi(T)$.



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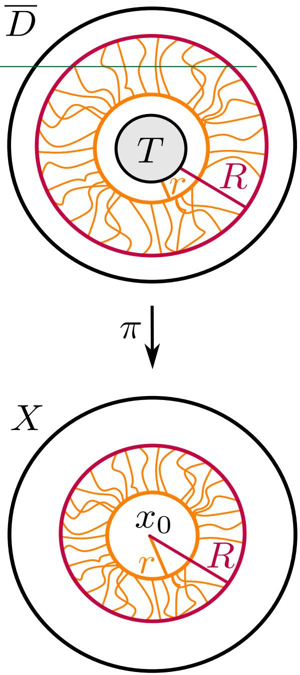

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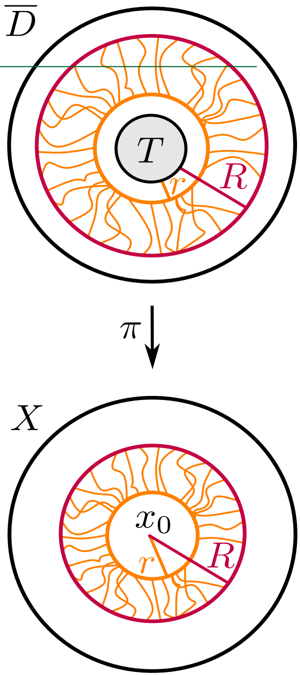
$\Rightarrow$ Non-existence of geometrically quasiconformal parametrization!



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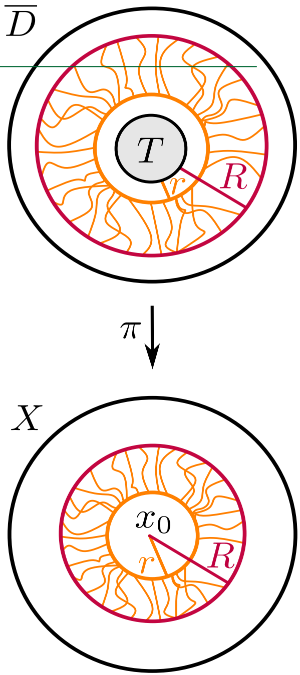
Question: Can we construct a good parametrization of a general metric surface X?



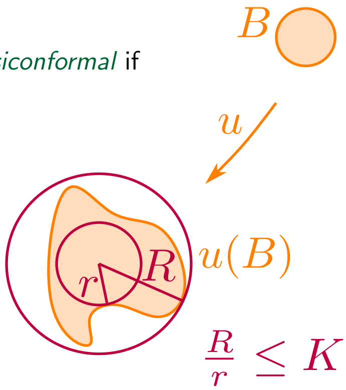
Weakly quasiconformal uniformization

Let $X \approx M$ be a compact metric surface of finite $\mathcal{H}^2$.

Definition: A continuous, surjective map $u\colon M \to X$ is *weakly quasiconformal* if

- u is a uniform limit of homeomorphisms $M \to X$, and
- there exists $K \geq 1$ s.th. for every family Γ of curves in M

$$\mathrm{mod}(\Gamma) \leq K \cdot \mathrm{mod}(u \circ \Gamma).$$



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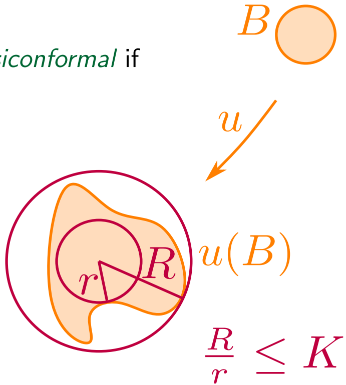

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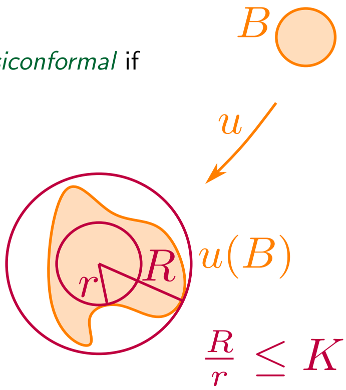

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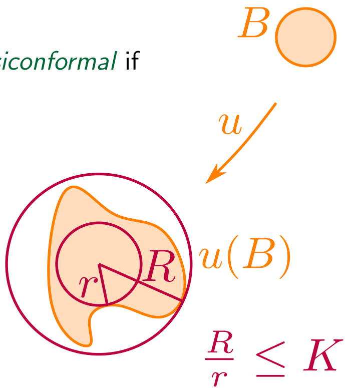

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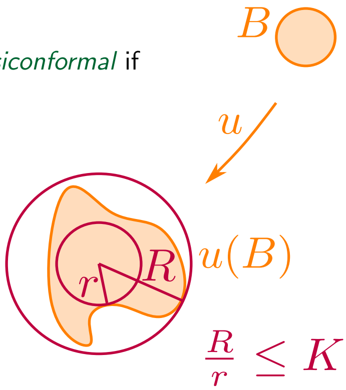

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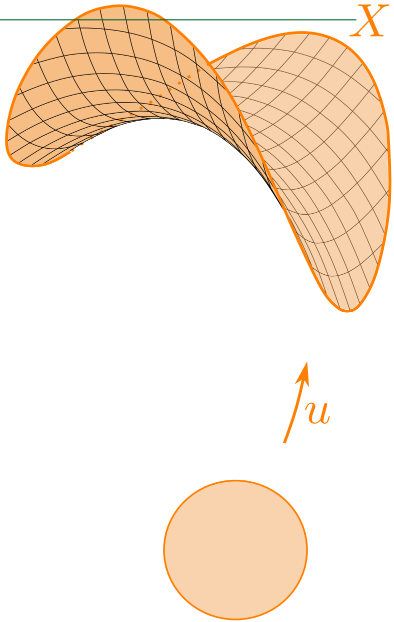
- X reciprocal: u is quasiconformal homeomorphism. $\Rightarrow$ Recover Theorem of Rajala.

Weakly quasiconformal uniformization

Theorem: Let X be a locally geodesic metric surface homeomorphic to $\overline{D}$. If $\mathcal{H}^2(X) < \infty$ and $\ell(\partial X) < \infty$, then there exists a weakly quasiconformal map $u\colon \overline{D} \to X$.



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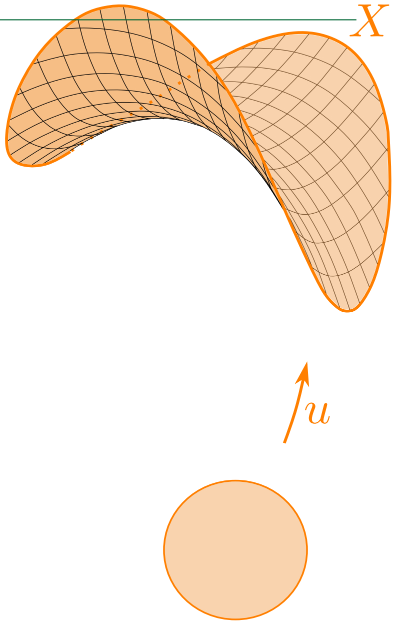
Strategy of proof (M.–Wenger):

Idea: *Use solution of Plateau's problem to parametrize X.*

Plateau's problem: Given a rectifiable Jordan curve γ in X. Find a surface having minimal area among all surfaces spanning γ.



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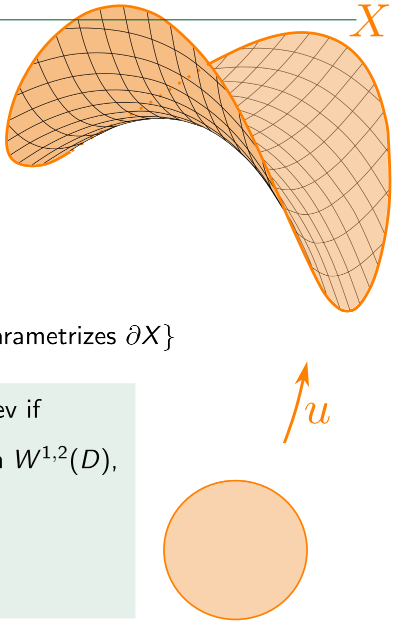

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$$\Lambda(X) := \{v\colon D \to X \text{ Sobolev: } \mathrm{tr}(v)\colon S^1 \to X \text{ almost parametrizes } \partial X\}$$

Metric space valued Sobolev maps: A map $v\colon D \to X$ is Sobolev if

- postcomposition with distance function $d_x(\cdot) = d(\cdot, x)$ is in $W^{1,2}(D)$,
- $\exists\ h \in L^2(D)$ s.th. $|\nabla(d_x \circ u)| \leq h$ a.e. on D.

Reshetnyak energy: $E_+^2(u) := \inf \left\{ \|h\|_{L^2(D)}^2 : h \text{ as above} \right\}$.

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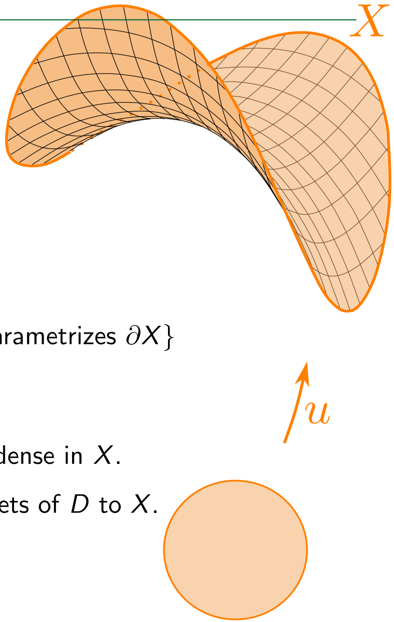

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1. Show that $\Lambda(X) \neq \emptyset$. $\rightarrow$ highly non-trivial
   - X might contain a purely 2-unrectifiable part that is dense in X.
   - In general, $\exists$ only few Lipschitz maps from open subsets of D to X.

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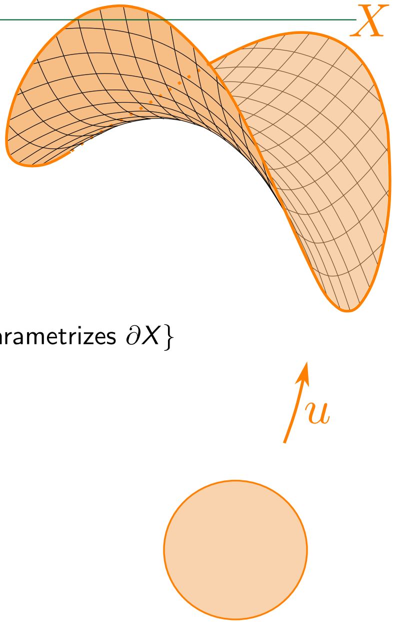

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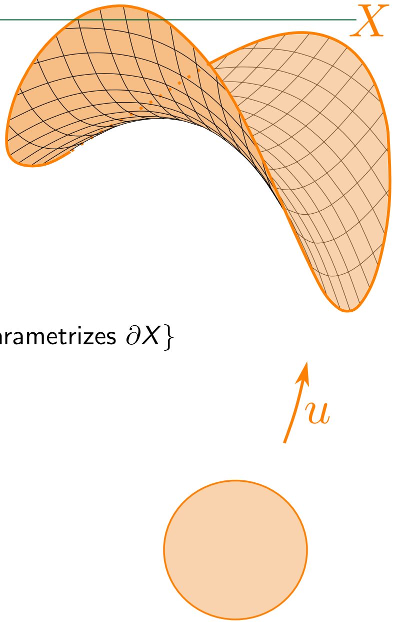

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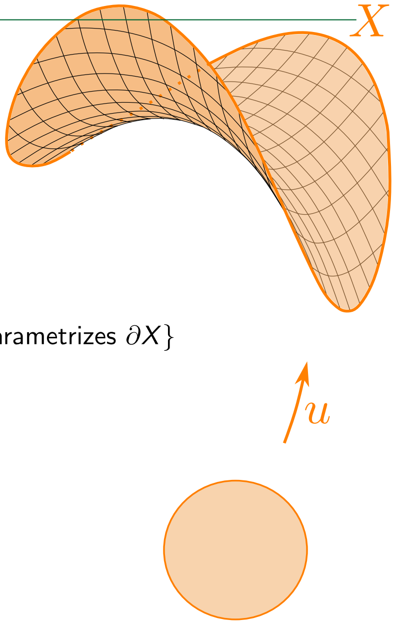

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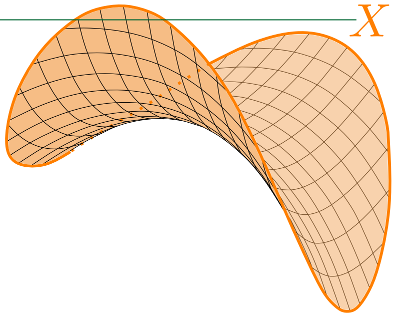

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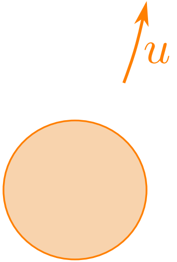

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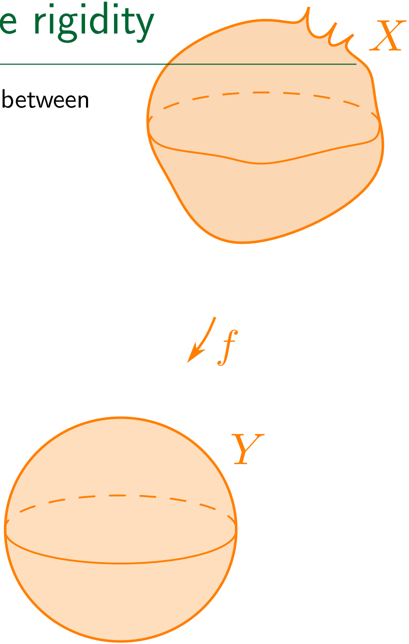
4. $\bar{u}$ is uniform limit of homeomorphisms.
5. $\bar{u}$ has desired distortion property.

Application: Lipschitz-volume rigidity

Question: Let $f: X \to Y$ be a 1-Lipschitz and surjective map between metric spaces that have the same volume. Is f an isometry?



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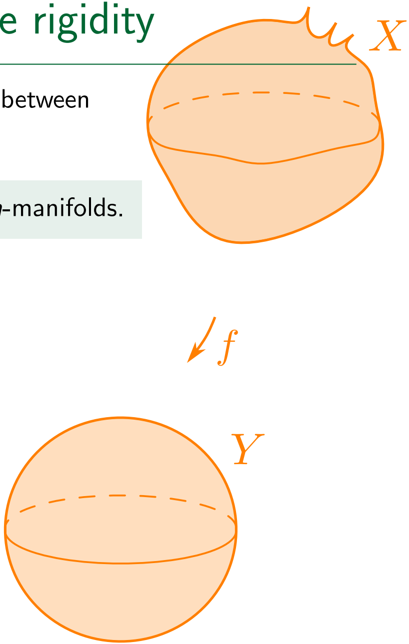

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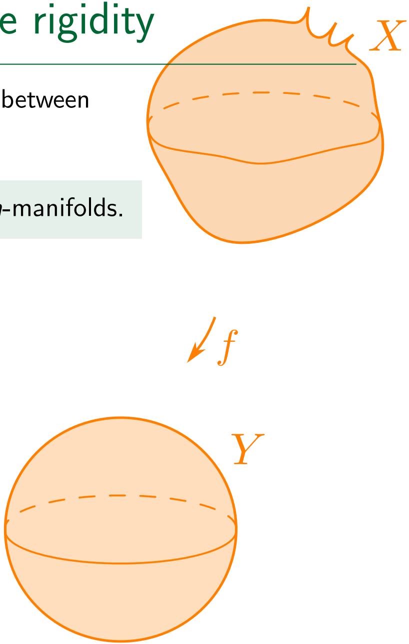
Theorem [M.–Ntalampekos 2024]: Let X be a closed metric surface and Y a closed Riemannian surface with $\mathcal{H}^2(X) = \mathcal{H}^2(Y)$. Then every 1-Lipschitz and surjective map $f : X \to Y$ is an isometry.

- Proof highly depends on existence of weakly quasiconformal parametrization of X.
- Intermediate results depending on regularity of Y.



Outline