MLMC for SPDEs

May 2022

MLMC for parabolic SPDE

Topics discussed:

- Assumptions for convergence of parabolic SPDE, Spectral Galerkin and Galerkin FEM more generally.
- MLMC: coupling of noise
- Antithetic MLMC for parabolic SPDE
- MIMC for SPDE

Early contributions on MLMC for SPDE

- Barth, Lang 2012, and Barth, Lang, Schwab 2013 Euler–Maruyama and Milstein numerical integration for parabolic and hyperbolic SPDE.
- Barth, Schwab and Sukys 2016: multilevel Monte Carlo simulation of statistical solutions to the navier–stokes equations (randomness only in initial condition?)
- Mishra, Schwab, Sukys: MLMC for hyperbolic pde
- Reisinger and Wang 2016 and 2021, MIMC for the Zakai equation in 1D and 2D.

Parabolic SPDE problem setting

$$\mathrm{d}U(t) = \left[AU + f(U)\right]\mathrm{d}t + G(U)\mathrm{d}W(t) \qquad (t,x) \in [0,T] \times D$$

$$U(t=0) \in L^2(\Omega, H) \quad \text{and} \quad \mathcal{F}_0 - \text{measurable.}$$

- For $H = L^2(D)$, we seek solution $U : [0,T] \times \Omega \to H$.
- Linear operator $A : \mathcal{D}(A) \subset H \to H$ with spectral decomposition

$$-Ae_j = \lambda_j e_j \qquad \text{with} \quad 0 < \lambda_1 \leq \lambda_2 \leq \dots$$

  and (e_j) complete orthonormal basis on H.
- $Q-$Wiener process

$$W(t,x) := \sum_{j=1}^{\infty} \sqrt{q_j}\phi_j(x) W^{(j)}(t)$$

  where (q_j, ϕ_j) are eigenpairs of linear operator $Q \in L_1^+(H)$ with a (ϕ_j) complete and orthonormal.
- $f : H \to H$ and $G : H \to \mathcal{L}_{HS}(Q^{1/2}(H), H)$ bounded and uniformly Lipschitz.

Assumptions on previous slide imply existence of a unique mild solution

$$U(t) = e^{At}U(0) + \int_0^t e^{A(t-s)} f(U(s))ds + \int_0^t e^{A(t-s)} G(U(s))dW(s) \; \forall t \in [0,$$

Numerical approx: For $d = 1$ and sufficiently smooth $U(0)$, Euler–Maruyama integration with a "stable rational approximation" of semigroup $\exp(At)$ yields rates:

$$\|\bar{U}^{\Delta x, \Delta t}(T) - U(T)\|_{L^2(\Omega,H)} \lesssim \sqrt{\Delta t} + \Delta x \qquad \text{(Barth et al. 2013)}$$

MLMC for same method

$$\|E_{MLMC}[U(T)] - \mathbb{E}[U(T)]\|_{L^2(\Omega,H)} \lesssim \epsilon \qquad \text{at cost} \quad \mathcal{O}(\epsilon^{-3}).$$

Coupling: in driving noise on shared subspace and U_0.

Questions:

- Can one extend antithetic coupling (Giles and Szpruch 2014) from SDE to above SPDE to achieve Milstein – double – convergence rate in time?

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- For improved stability, can tamed integrators be helpful for solving SPDE with MLMC?

- How to couple noise for pairwise solutions when not solving with Galerkin/spectral Galerkin method?

Spectral approach – additive noise

$$dU = \ (\Delta U + f(U(t,x)))\ dt + dW \quad \text{for} \quad (x,t) \in [0,T] \times D,$$
$$+ \text{Dirichlet initial and periodic boundary conditions.} \tag{1}$$

In addition to earlier assumptions, we assume that $A = \Delta$ and Q shares eigenbasis (e_j) with $H = \overline{\text{span}(e_j)}$.
Let

$$U^{\ell_1,\ell_2}(T,x) := \sum_{j=1}^{N_\ell} \bar{u}_j^{\ell_1,\ell_2}(T) e_j(x)$$

That is, solution on subspace using exponential Euler method

$$H^{N_{\ell_1}} = (e_j)_{j=1}^{N_{\ell_1}} \quad \text{using} \quad \Delta t_{\ell_2} = 2^{-\ell_2} \Delta_0.$$

Convergence rate (up to log terms):

$$\| U^{\ell_1+1,\ell_2+1} - U^{\ell_1,\ell_2} \|_{L^2(\Omega;H)} \lesssim N_{\ell_1}^{-2} + J_{\ell_2}^{-2}.$$

(Maybe Milstein method for SPDE can perform similarly?)

Multi-index Monte Carlo

Consider

$$E_{MIMC}[U(T)] := \sum_{(\ell_1,\ell_2)\in\mathcal{I}} E_{M_{\ell_1,\ell_2}}[U^{\ell_1,\ell_2} - U^{\ell_1-1,\ell_2} - U^{\ell_1,\ell_2-1} + U^{\ell_1-1,\ell_2-1}]$$

where $\mathcal{I} \subset \mathbb{N}_0^2$, and $M_{\ell_1,\ell_2} \geq 1$ for all $(\ell_1, \ell_2) \in \mathcal{I}$.

The MIMC parameters:

Rough understanding:

- "shape of" the index set $\mathcal{I}$ is determined by the weak rate (α_1, α_2)
- and num of samples M_{ℓ_1,ℓ_2} is determined by the variance decay rates (β_1, β_2) and cost rates (γ_1, γ_2).

Assumptions for MLMC

The reaction term $f : H \to H$ is twice continuously differentiable, where its derivatives satisfy the following

$$\|f'(x) - f'(y)\|_H \leq L\|x - y\|_H, \quad \text{and more}$$

and

$$\|A^{-1}f''(x)(v, w)\|_H \leq L\|(-A)^{-1/2}v\|_H\|(-A)^{-1/2}w\|_H,$$

for all $v, w \in H$.

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Question: What kind of extensions are needed for MIMC?

Crucial convergence rate MLMC

For MLMC we needed a rates

$$\mathbb{E}\| U^{\ell+1,\ell+1} - U^{\ell,\ell}\|_H^2 \lesssim N_\ell^{-2} + J_\ell^{-2}.$$

Setting $J_\ell \eqsim N_\ell \eqsim 2^\ell$ leads to (up to log factors) near optimal setting:

(i) $\left\|\mathbb{E}\left[U^{\ell,\ell}(T,\cdot) - U(T,\cdot)\right]\right\|_H \lesssim 2^{-\ell}$.

(ii) $V_\ell := \mathbb{E}\left[\| U^{\ell,\ell}(T,\cdot) - U^{\ell-1,\ell-1}(T,\cdot)\|_H^2\right] \lesssim 2^{-2\ell}$.

(iii) $C_\ell := \mathrm{Cost}(U^{\ell,\ell}) \eqsim 2^{2\ell}$.

Convergence rates for MIMC

For MIMC, the hope is to obtain multiplicative rates

$$\mathbb{E}[\| U^{\ell_1+1,\ell_2+1} - U^{\ell_1+1,\ell_2} - U^{\ell_1,\ell_2+1} + U^{\ell_1,\ell_2} \|_H^2] \lesssim N_{\ell_1}^{-2} \times J_{\ell_2}^{-2}$$

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Questions:

- What regularity assumptions must be imposed on f to achieve this? Third derivatives?
- Can it also be achieved for slowly varying multiplicative noise?
- Performance gains in $d > 1$?

Numerical rate test

Consider problem

$$dU_t = (\epsilon(\Delta - I)U_t + f(U_t))\, dt + dW_t \quad \text{for} \quad t \in [0, T = 0.5],$$

with $f(U) = U$ and white noise W $U_0 \in L^2(\Omega, H^{1/2})$. Numerical experiments with $(N_{\ell_1}, J_{\ell_2}) \eqsim (2^{\ell_1}, 2^{\ell_2})$ and $\epsilon = 0.00005$ yields the following output for $H = L^2(0,1)$: