Introduction to topological order

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Higher Structures and Field Theory, 2020/08

Condensed matter physics and higher category

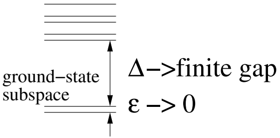
- **Condensed matter systems**:
  defined by microscopic theoretical **lattice models**
  probed by macroscopic experimental measurements
- **Concepts in condensed matter systems**
  defined by microscopic lattice models
  defined by macroscopic properties

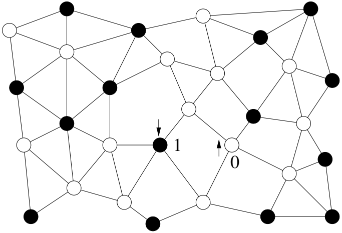
- **Superconductivity**: (micro) electron-pair condensation.
  (macro) zero resistance, vortex quantization

- **Concepts in mathematics** (in some areas)
  defined by **topological invariants** = macroscopic properties
- We have a microscopic definition of **gapped phases** in condensed matter. **A full macroscopic characterization of nd (n+1D) gapped phases → unitary fusion n-category**
- *We have a microscopic definition of* **gapless phases** *in condensed matter.* **A full macroscopic characterization of nd (n+1D) gapless quantum phases → ???**

A many-body quantum system (a lattice model)

- A **quantum system** is described by $(\mathcal{V}_N, H_N)$
  $\mathcal{V}_N$: a Hilbert space with a tensor decomposition
  $\mathcal{V}_N = \otimes_{i=1}^N \mathcal{V}_i$, where $\mathcal{V}_i$ has a finite dimension.
  H_N: a **local Hamiltonian** (hermitian operator) acting on $\mathcal{V}_N$:
  $H_N = \sum_i O_i + \sum_{ij} O_{\langle ij \rangle} + \cdots$.
  O_i hermitian operator acts on $\mathcal{V}_i$,
  O_{ij} hermitian operator acts on $\mathcal{V}_i \otimes \mathcal{V}_j$





- A **gapped quantum system** (a concept for $N \to \infty$ limit) $=$ a sequence of pairs, $\{(\mathcal{V}_{N_1}, H_{N_1}); (\mathcal{V}_{N_2}, H_{N_2}); (\mathcal{V}_{N_3}, H_{N_3}); \cdots\}$, where each H_N has gapped eigenvalue spectrum: $\Delta_N \to \Delta_\infty$, $0 < \Delta_\infty < \infty$ and $\varepsilon_N \to 0$, as $N \to \infty$
  $\to$ **ground-state subspace** $\mathcal{V}_{\text{grnd}}$ (= gapped state in physics)

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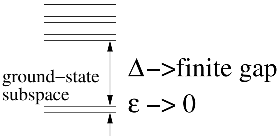

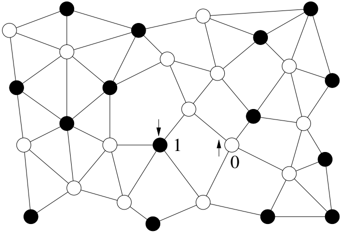

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Examples of gapped systems and gapped states

- **The trivial state** in 1d (space dim) *ie* 1+1D (spacetime dim)
  $\mathcal{V}_N = \mathbb{C}_2^{\otimes N}$, $\mathbb{C}_2 = \text{span}_{\mathbb{C}}\{|\uparrow\rangle, |\downarrow\rangle\}$. • • • • • • • •
  $H_N = \sum_i O_i = -\sum_i Z_i$, where $Z_i|\uparrow\rangle_i = |\uparrow\rangle_i$, $Z_i|\downarrow\rangle_i = -|\downarrow\rangle_i$
  $\to$ 1-dim. ground-state subspace $= \text{span}_{\mathbb{C}}\{|\cdots\uparrow\uparrow\uparrow\cdots\rangle\}$,
  where $|\cdots\uparrow\uparrow\uparrow\cdots\rangle = |\uparrow\rangle^{\otimes N}$ is a **product state**:

- **Ising model: symmetry breaking state**
  $\mathcal{V}_N = \mathbb{C}_2^{\otimes N}$, $\mathbb{C}_2 = \{|\uparrow\rangle, |\downarrow\rangle\}$. $H_N = \sum_i O_{i,i+1} = -\sum_i Z_i Z_{i+1}$
  $\to$ 2-dim. ground-state subspace $=$
  $\text{span}_{\mathbb{C}}\{|\cdots\uparrow\uparrow\uparrow\cdots\rangle, |\cdots\downarrow\downarrow\downarrow\cdots\rangle\}$

- H_N **has a $\mathbb{Z}_2$ on-site symmetry** generated by $U = \otimes_i X_i$
  $X_i|\uparrow\rangle_i = |\downarrow\rangle_i$, $X_i|\downarrow\rangle_i = -|\uparrow\rangle_i$: $UH_NU^{-1} = H_N$

  Symmetry breaking state: A basis of ground-state subspace $|\cdot\uparrow\uparrow\uparrow\cdot\rangle \pm |\cdot\downarrow\downarrow\downarrow\cdot\rangle$, that is **symmetric** ($U|\Psi\rangle = e^{i\theta}|\Psi\rangle$) but not product states. Another basis, $|\cdot\uparrow\uparrow\uparrow\cdot\rangle$, $|\cdot\downarrow\downarrow\downarrow\cdot\rangle$, that are product states but not symmetric.

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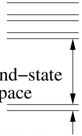
Gapped phases of many-body quantum systems



- Two gapped systems, *ie* two sequences $\{H_N|_{N\to\infty}\}$ and $\{H'_N|_{N\to\infty}\}$, are equivalent if H_N can smoothly deform into H'_N without closing the gap Δ. The resulting equivalent classes are **gapped quantum phases of matter**.
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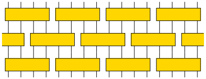
$|\text{SRE state}\rangle =$  $|\text{product state}\rangle$

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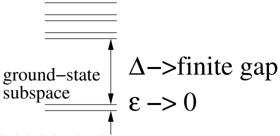

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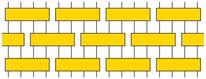
$|\text{SRE state}\rangle =$  $|\text{product state}\rangle$

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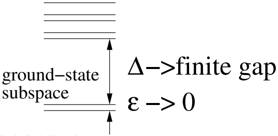

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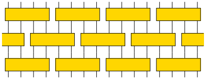
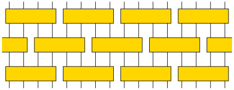
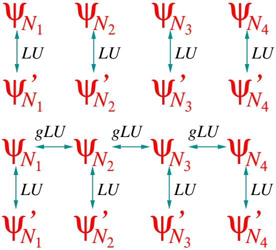
$|\text{SRE state}\rangle =$  $|\text{product state}\rangle$

More careful discussion of local unitary equivalence

- A **gapped quantum phase**: an equivalence class of **gapped quantum systems**: Chen Gu Wen, arXiv:1004.3835

Def: $\{H_{N_i}\} \sim \{H'_{N_i}\}$, if their ground-state subspaces satisfy $\Psi'_{N_i} = U_{\text{LU}}\Psi_{N_i}$, where U_{LU} is a **local unitary** transformation:

$U_{\text{LU}} =$ 



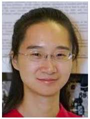
- **A gapped quantum liquid phase:**

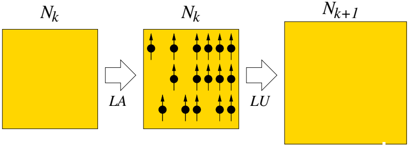
Zeng Wen, arXiv:1406.5090

$$\Psi_{N_{i+1}} \xrightarrow{\text{local addition}} \Psi_{N_i} \otimes |\uparrow\rangle^{\otimes(N_{i+1}-N_i)}$$

Generalized local unitary (gLU) trans,

- **Trivial phase** and **symmetry breaking phases** are examples of **Gapped liquid phases**





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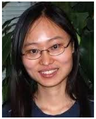

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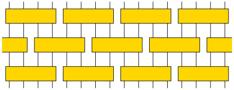
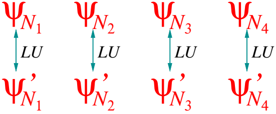
$U_{\text{LU}} =$ 



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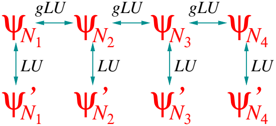

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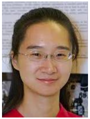
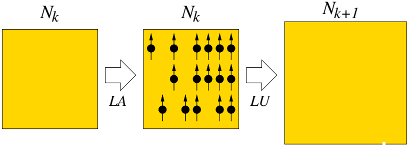
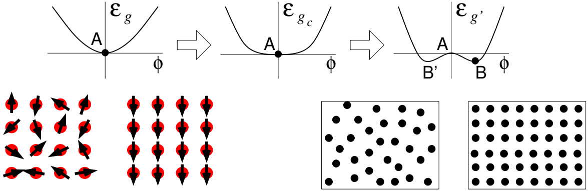




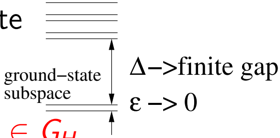
Symmetry breaking phase: quantum point of view

In most textbooks, symmetry breaking phase is explained using a classical point of view.



The Hamitonian H_N has a symmetry G_H: $U_g H_N U_g^{-1} = H_N$, where U_g form a representation of a group $g \in G_H$.

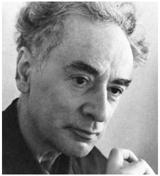
- **Symmetry breaking phase**: The ground-state subspace has a SRE basis, *ie* each basis vector is **local unitary equivalent** to a product state. Such a basis is not symmetric under $U_g \in G_H$. But the basis may be symmetric under the transformations in a subgroup $U_g \in G_\Psi \subset G_H$.



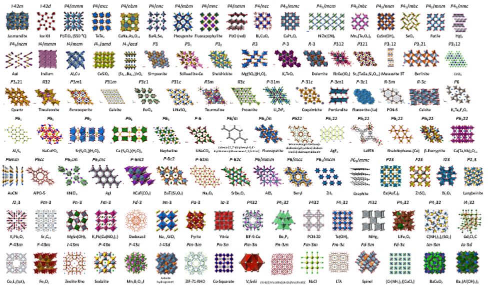
Classify phases of quantum matter ($T = 0$ phases)

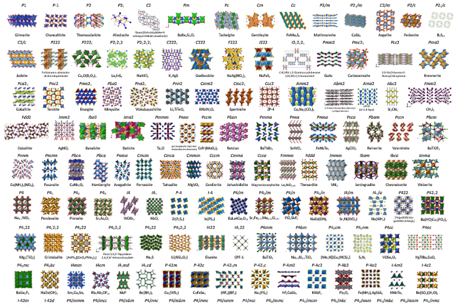
For a long time, we thought that Landau symmetry breaking classify all phases of matter

- **Symm. breaking phases are characterized by order parameters and classified by a pair** $G_\Psi \subset G_H$
  - G_H = symmetry group of the system.
  - G_Ψ = symmetry group of the ground states.



- **230 crystals** from group theory





Topological orders in quantum Hall effect

- Quantum Hall states $R_{xy} = V_y/I_x = \frac{m}{n}\frac{2\pi\hbar}{e^2}$

  vonKlitzing Dorda Pepper, PRL **45** 494 (1980)
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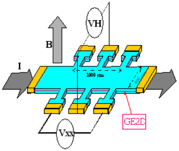
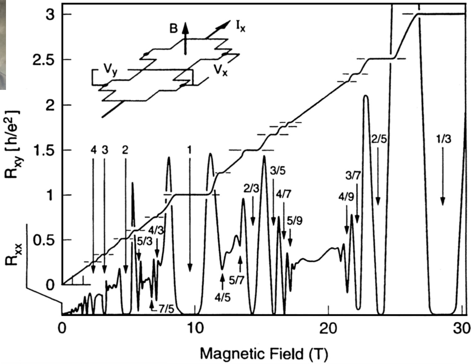


- FQH states have different phases even when there is no symm. ($G_H = 1$) and no symm. breaking. ($G_\Psi = G_H$)
- FQH liquids must contain a new kind of order, named as **topological order**

  Wen, PRB **40** 7387 (89); IJMP **4** 239 (90)



- New equivalent classes of $\{H_N\}$ beyond symm. breaking phase

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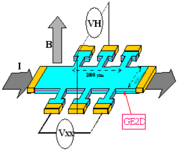

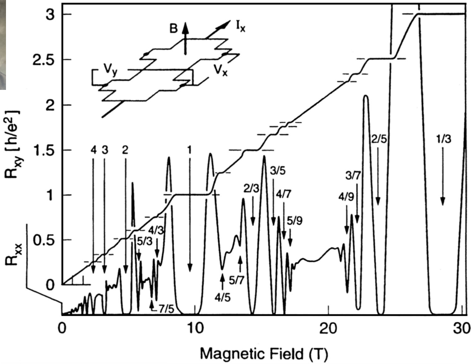

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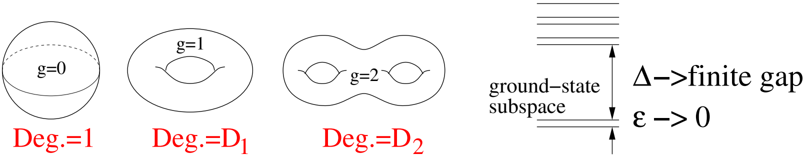
Macroscopic characterization of topological order

- New equivalent classes → new **topological invariants**.
  How to extract new topological invariants beyond symmetry breaking from complicated many-body state
  $|\Psi\rangle = \sum_{x_1,\cdots,x_{10^{20}}} \Psi(x_1,\cdots,x_{10^{20}})|x_1,\cdots,x_{10^{20}}\rangle$

  Put the gapped system on space with various topologies, and measure the ground state degeneracy.

  Wen PRB **40** 7387 (89)

  New topological invariant → Notion of **topological order**



Haldane PRL **51** 605 (83); Tao-Wu, PRB **30** 1097 (84)

Why ground state degeneracy is a topological invariant?

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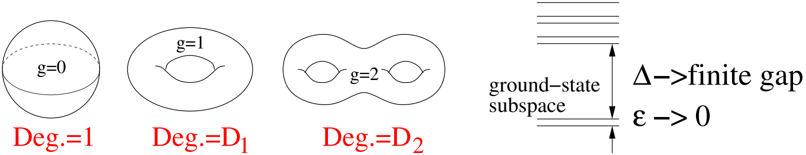

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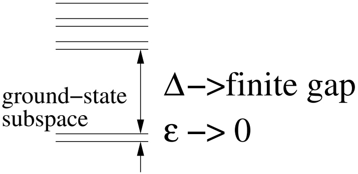
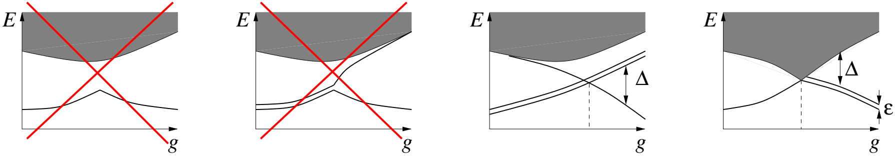
Why ground state degeneracy is a topological invariant?

The ground state degeneracy is topological

- The ground state degeneracies, in $N \to \infty$ limit, are robust against any local perturbations that can break any symmetries. The ground state degeneracies have nothing to do with symmetry. We call such a degeneracy as **topological degeneracy** Wen Niu PRB **41** 9377 (90)

- The ground state degeneracies can only vary by some large changes of Hamiltonian $\to$ gap-closing phase transition.





The ground state degeneracy is topological

- The ground state degeneracies, in $N \to \infty$ limit, are robust against any local perturbations that can break any symmetries. The ground state degeneracies have nothing to do with symmetry. We call such a degeneracy as **topological degeneracy** Wen Niu PRB **41** 9377 (90)

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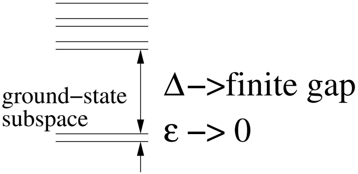
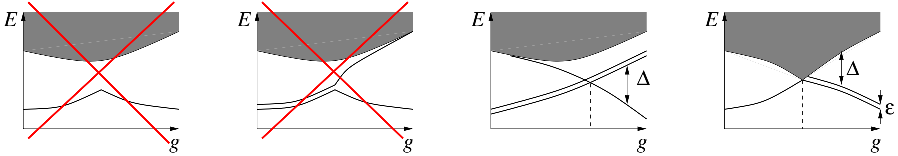




Topological invariants that fully define topo. orders

The ground state degeneracy only partially characterize topological order, not fully define it.

We conjectured that nd (*ie* $n+1$D) topological order can be completely defined via the following topological property: Wen IJMPB 4, 239 (90); Keski-Vakkuri Wen IJMPB 7, 4227 (93)

- **Vector bundle on the moduli space**

i. Consider a closed 2-dim space Σ_g w/ metrics g_{ij}.

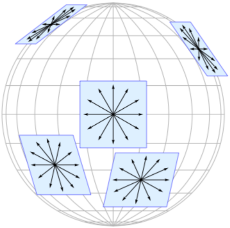
ii. Different diffeomorphic equivalent classes of metrics g_{ij} form the moduli space $\mathcal{M}_{\Sigma_g}$.

iii. The moduli space is the space of Hamiltonians $H(g_{ij})$.
We jumped here: discrete lattice → continuous manifold
The emergence of continuous geometry from discrete algebra

iv. The ground subspace $\mathcal{V}_{\text{grnd}}(g_{ij})$ (an n-dim vector space) of $H(g_{ij})$ depends on the diffeomorphic equivalent classes of the spacial metrics g_{ij} → a vector bundle over $\mathcal{M}_{\Sigma_g}$ with fiber $\mathcal{V}_{\text{grnd}}(g_{ij})$.

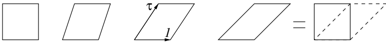
Topological invariants that fully define topo. orders

Vector bundle on the moduli space
is a $U(n)$ bundle with $SU(n)$ flat connection (due to the topological degeneracy).



Tangent bundle on a 2-sphere

- Local $U(1)$ curvature $\to$ gravitational Chern-Simons term $e^{-S_{eff}} = e^{i\frac{2\pi c}{24}\int_{M^2\times S^1}\omega_3}$
  $\to$ chiral central charge c
  $\to$ quantized thermal Hall conductance
- Flat $SU(n)$ connection: $\pi_1(\mathcal{M}_{\text{torus}}) = SL(2,\mathbb{Z})$
  90° rotation $|\Psi_\alpha\rangle \to |\Psi'_\alpha\rangle = S_{\alpha\beta}|\Psi_\beta\rangle$
  Dehn twist: $|\Psi_\alpha\rangle \to |\Psi'_\alpha\rangle = T_{\alpha\beta}|\Psi_\beta\rangle$



$S, T \to$ a proj. rep. of $SL(2,\mathbb{Z})$: Wen IJMPB 4, 239 (90)
$S^2 = (ST)^3 e^{-2\pi i \frac{c}{8}} = C,\ C^2 = 1$ Keski-Vakkuri Wen IJMPB 7, 4227 (93)
X.-D. Wen & X.-G. Wen arXiv:1908.10381

Conjecture: **The vector bundles on all $\mathcal{M}_{\Sigma_g}$ (*ie* the data (S,T,c), ...) completely characterize the topo. orders**
(S,T,c) *for torus almost fully characterize 2+1D topo. order*

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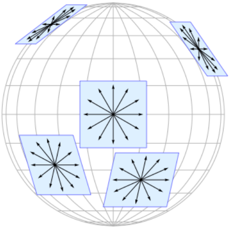

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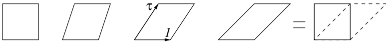

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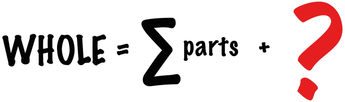
The microscopic origin of topological degeneracy

- For a highly entangled many-body quantum systems: **knowing every parts still cannot determine the whole**

- In other words, there are different "wholes", that their every local parts are identical.



- Local Hamiltonians can only see the parts → those different "wholes" (the whole quantum states) have the same energy.

- What is a "whole"?, what is "part"?
  whole = many-body wave function $|\Psi\rangle = \Psi(m_1, m_2, \cdots, m_N)$
  where m_i label states on site-i
  part = entanglement density matrix:

$$\rho_{\text{site-1,2}} = \mathrm{Tr}_{\text{site-3},\cdots,N}|\Psi\rangle\langle\Psi|, \quad \langle H_{1,2}\rangle = \mathrm{Tr}(H_{1,2}\rho_{\text{site-1,2}})$$

$$\rho_{m_1,m_2;m_1',m_2'} = \sum_{m_3,\cdots,m_N} \Psi^*(m_1, m_2, m_3, \cdots, m_N)\Psi(m_1', m_2', m_3, \cdots, m_N)$$

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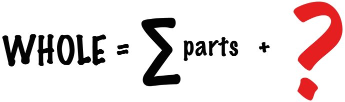

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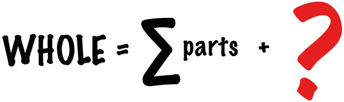


- Those kinds of many-body quantum systems have

topological entanglement entropy

Kitaev-Preskill hep-th/0510092

Levin Wen cond-mat/0510613

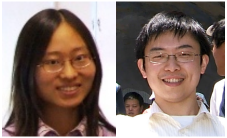
and **long range quantum entanglement**

Chen Gu Wen arXiv:1004.3835

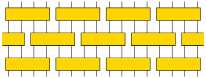
Long range entanglement $\rightarrow$ Topo. degeneracy

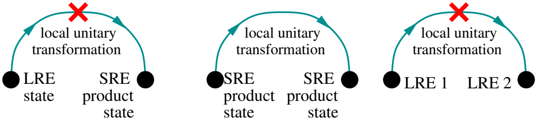
What is long-range entanglement?

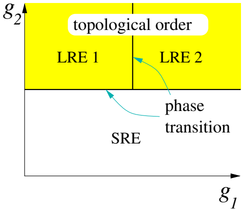
Chen Gu Wen arXiv:1004.3835



- Define **long range entanglement** via local unitary (LU) transformations (*ie* **local quantum circuit**)

$|\text{LRE}\rangle \neq$  $|\text{product state}\rangle = |\text{SRE}\rangle$





- All SRE states belong to the same trivial phase
- LRE states can belong to many different phases
  = different **patterns of long-range entanglements**
  = different **topological orders** Wen PRB **40** 7387 (89)

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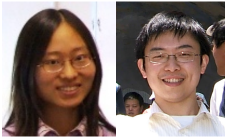

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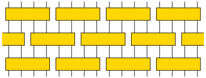
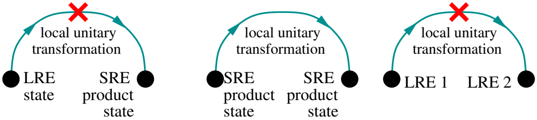
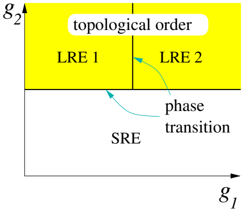
$|\text{LRE}\rangle \neq$  $|\text{product state}\rangle = |\text{SRE}\rangle$





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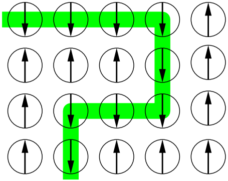
How to make long range entanglement?

- Short-range-entanglement (SRT) $\sim$ product state $|\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\rangle$
- To make topological order, need to sum over many different product states. But summing over everything with equal weight $\sum_{\text{all spin config.}} |\uparrow\downarrow ..\rangle = (|\uparrow\rangle + |\downarrow\rangle)^{\otimes N}$ $\to$ product state



- Sum over everything with phase factors
  $\sum_{\text{all spin config.}} \prod_{i<j} (z_i^\uparrow - z_j^\uparrow)^m |\uparrow\downarrow ..\rangle$
  $\to$ **chiral spin liquid** or **FQH state**.
- Sum over a subset of spin configurations:

  $|\Phi^{\mathbb{Z}_2}_{\text{loops}}\rangle = \sum \Big|$  $\Big\rangle$

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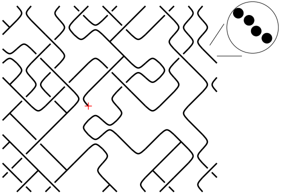

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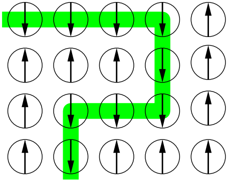

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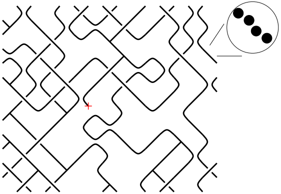

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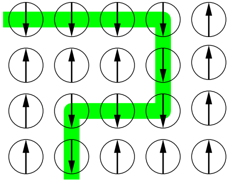

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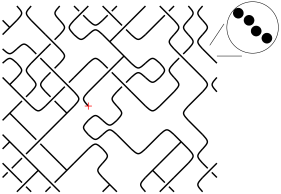

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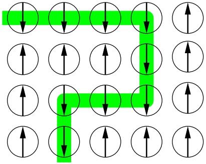
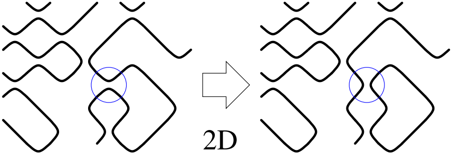
Local dance rule → global dance pattern





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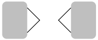
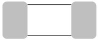
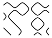
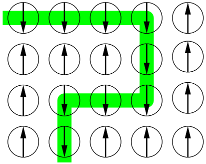
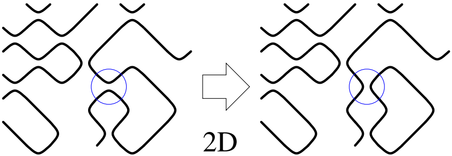
  (2) $\Phi_{\text{str}}\Big($  $\Big) = \Phi_{\text{str}}\Big($  $\Big)$, $\Phi_{\text{str}}\Big($  $\Big) = \Phi_{\text{str}}\Big($  $\Big)$
  → Global wave function of loops $\Phi_{\text{str}}\Big($  $\Big) = 1$

- There is a Hamiltonian H (the toric code model):
  (1) Open ends cost energy
  (2) string can hop and reconnect freely.
  The ground state of H gives rise to the above string lqiuid wave function.

Kitaev quant-ph/9707021

Local dance rule → global dance pattern





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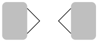
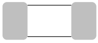
  (2) Φ_{str} () $= \Phi_{str}$ () , Φ_{str} () $= \Phi_{str}$ ()
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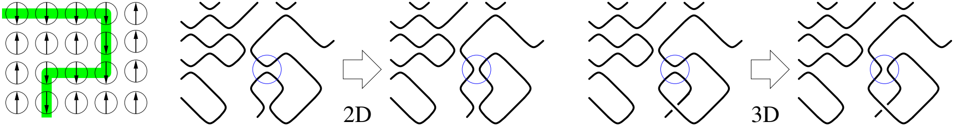

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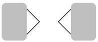
  (2) Φ_{str} () $= \Phi_{\text{str}}$ () , Φ_{str} () $= -\Phi_{\text{str}}$ ()
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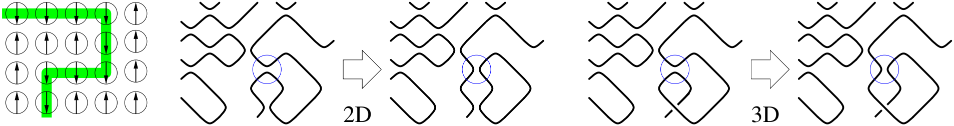

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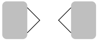
  (2) Φ_{str} () $=$ Φ_{str} () , Φ_{str} () $= -\Phi_{str}$ ()
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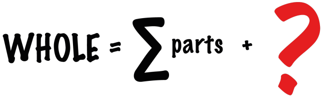
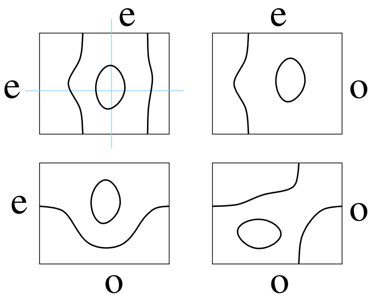
Knowing all the parts ≠ knowing the whole

- **Quantum entanglement** →



- 4 locally indistinguishable states on torus for both liquids → **topo. order**

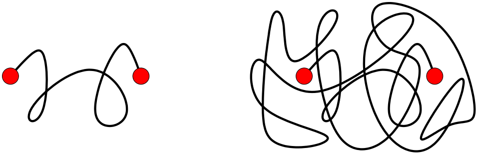
- Ground state degeneracy cannot distinguish them.



$D^{\text{tor}}=4$

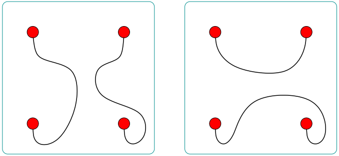
Topological excitations

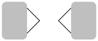
- **Ends of strings** behave like point objects.
- They cannot be created alone → **topological**



- Let us fix 4 ends of string on a sphere S^2. *How many locally indistinguishable states are there?*

- There are 2 sectors → 2 states (?)



- In fact, there is only 1 sector → 1 state, due to the string reconnection fluctuations $\Phi_{\text{str}}\Big($  $\Big) = \pm\Phi_{\text{str}}\Big($  $\Big)$.

- In general, fixed $2N$ ends of string → 1 state. Each end of string has no degeneracy → no internal degrees of freedom.

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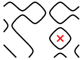
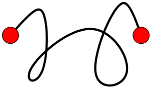
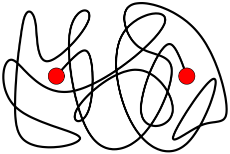
$|m\rangle = \sum(-)^{\#\text{ of loops around }\times}\Big|$  $\Big\rangle$

Topological excitations

- **Ends of strings** behave like point objects.
- They cannot be created alone → **topological**





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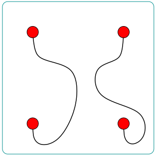

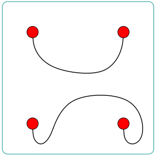

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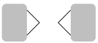
- In fact, there is only 1 sector → 1 state, due to the string reconnection fluctuations $\Phi_{\text{str}}\Big($  $\Big) = \pm\Phi_{\text{str}}\Big($  $\Big)$.

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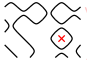
$|m\rangle = \sum(-)^{\#\text{ of loops around }\times}\Big|$  $\Big\rangle$

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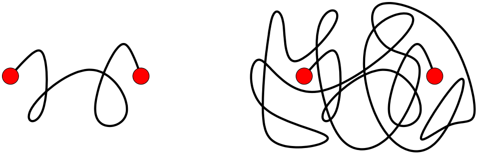

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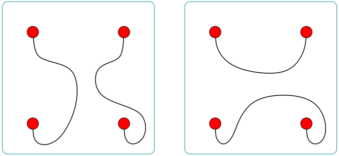

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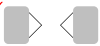
- In fact, there is only 1 sector → 1 state, due to the string reconnection fluctuations $\Phi_{\text{str}}\Big($  $\Big) = \pm\Phi_{\text{str}}\Big($  $\Big)$.

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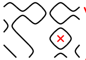
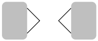
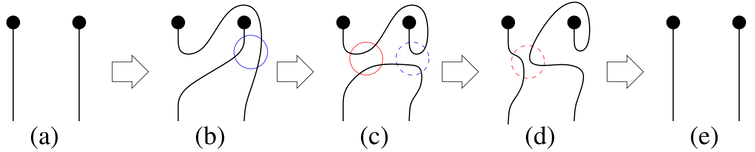
$|m\rangle = \sum(-)^{\#\text{ of loops around }\times}\Big|$  $\Big\rangle$

Emergence of fractional spin (topological spin)

- Ends of strings are point-like. Are they bosons or fermions? *Two ends = a single string = a boson, but each end can still be a fermion.* Fidkowski Freedman Nayak Walker Wang cond-mat/0610583
- $\Phi_{\text{str}}\Big($  $\Big) = 1$ string liquid $\Phi_{\text{str}}\Big($  $\Big) = \Phi_{\text{str}}\Big($  $\Big)$
- End of string wave function: $|\text{end}\rangle =$  $+ c$  $+ c$  $+ \cdots$
  The string near the end is totally fixed, since the end is determined by a trapping Hamiltonian δH which can be chosen to fix the string. The string alway from the end is not fixed, since they are determined by the bluk Hamiltonian H which gives rise to a string liquid.
- 360° rotation:  $\to$  and  $=$  $\to$ : $R_{360^\circ} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- We find two types of topological exitations
  (1) $|e\rangle =$  $+$  spin 0. (2) $|f\rangle =$  $-$  spin $1/2$.

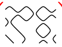
Spin-statistics theorem: Emergence of Fermi statistics



- (a) $\to$ (b) $=$ exchange two string-ends.
- (d) $\to$ (e) $=$ 360° rotation of a string-end.
- Amplitude (a) $=$ Amplitude (e)
- Exchange two string-ends plus a 360° rotation of one of the string-end generate no phase.

$\to$ **Spin-statistics theorem**

$\mathbb{Z}_2$ topological order and its physical properties

Φ_{str} () $= 1$ string liquid has $\mathbb{Z}_2$-topological order.

- 4 **types** of topological excitations: *(f is a fermion)*

  (1) $|e\rangle =$  $+$  spin 0. (2) $|f = e \otimes m\rangle =$  $-$  spin $1/2$.

  (3) $|m\rangle = \times -$  spin 0. (4) $|1\rangle = \times +$  spin 0.
- The type-1 excitation is the tirivial excitation, that can be created by local operators.
  The type-e, type-m, and type-f excitations are non-tirivial excitation, that cannot be created by local operators.
- 1, e, m are bosons and f is a fermion. e,m, and f have π mutual statistics between them.
- **Fusion rule**:

  $e \otimes e = 1$; $f \otimes f = 1$; $m \otimes m = 1$;
  $e \otimes m = f$; $f \otimes e = m$; $m \otimes f = e$;
  $1 \otimes e = e$; $1 \otimes m = m$; $1 \otimes f = f$;

Topo. order and topological quantum field theory

$\mathbb{Z}_2$ topologica order is described by $\mathbb{Z}_2$ gauge theory – a topological quantum field theory

Physical properties of $\mathbb{Z}_2$ gauge theory
= Physical properties of $\mathbb{Z}_2$ topological order

- $\mathbb{Z}_2$-charge $\to e$, $\mathbb{Z}_2$-vortex $\to m$, bound state $\to f$.

- $\mathbb{Z}_2$-charge (a representatiosn of $\mathbb{Z}_2$) and $\mathbb{Z}_2$-vortex (π-flux) as two bosonic point-like excitations.
- $\mathbb{Z}_2$-charge and $\mathbb{Z}_2$-vortex bound state $\to$ a fermion (f), since $\mathbb{Z}_2$-charge and $\mathbb{Z}_2$-vortex has a π mutual statistics between them (charge-1 around flux-π).
- $\mathbb{Z}_2$-charge, $\mathbb{Z}_2$-vortex, and their bound state has a π mutual statistics between them.
- $\mathbb{Z}_2$ gauge theory on torus also has 4 degenerate ground states

Emergence of fractional spin and semion statistics

$\Phi_{str}(\;) = (-)^{\#\text{ of loops}}$ string liquid. $\Phi_{str}(\;) = -\Phi_{str}(\;)$

- End of string wave function: $|\text{end}\rangle = \; + c \; - c \; + \cdots$
- 360° rotation: $\to$ and $= - \to -$: $R_{360^\circ} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
- **Types** of topological excitations: *($s_\pm$ are semions)*
  (1) $|s_+\rangle = \; + \mathrm{i}$ spin $\frac{1}{4}$. (2) $|s_- = s_+ \otimes m\rangle = \; - \mathrm{i}$ spin $-\frac{1}{4}$
  (3) $|m\rangle = \times - \otimes$ spin 0. (4) $|1\rangle = \times + \otimes$ spin 0.
- **double-semion topo. order** $= U^2(1)$ Chern-Simon gauge theory $L(a_\mu) = \frac{2}{4\pi} a_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda} - \frac{2}{4\pi} \tilde{a}_\mu \partial_\nu \tilde{a}_\lambda \epsilon^{\mu\nu\lambda}$
- Two string lqiuids $\to$ Two topological orders:
  $\mathbb{Z}_2$ **topo. order** Read Sachdev PRL 66, 1773 (91), Wen PRB 44, 2664 (91), Moessner Sondhi PRL 86 1881 (01) and **double-semion topo. order** Freedman etal cond-mat/0307511, Levin Wen cond-mat/0404617

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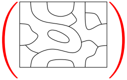
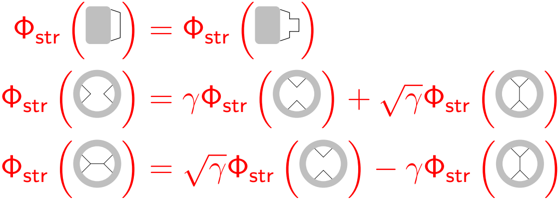
String-net liquid

Ground state:

- String-net liquid: allow three strings to join, but do not allow a string to end Φ_{str} 

Levin Wen cond-mat/0404617

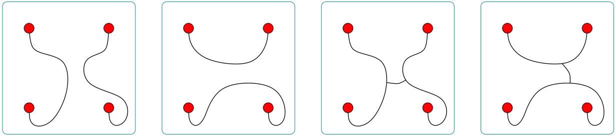
- The dancing rule :



$$\gamma = (\sqrt{5} - 1)/2$$

Topological excitations in string-net liquid

- **Topological excitations**:
  For fixed 4 ends of string-net on a sphere S^2, how many locally indistinguishable states are there? **four states?**



- In fact, there are only two linearly independent states.
  This can be obtain using fusion rule: $\phi \otimes \phi = 1 \oplus \phi$.

  $\phi \otimes \phi$ means bound state of two ϕ-particles (fusion).
  But what does $1 \oplus \phi$ *means?*

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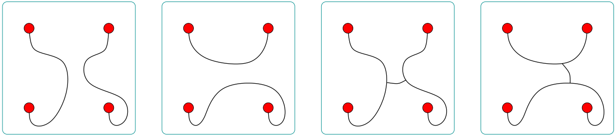

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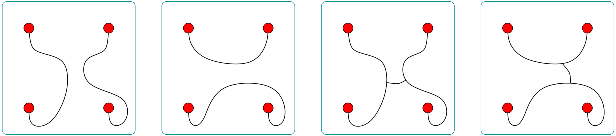

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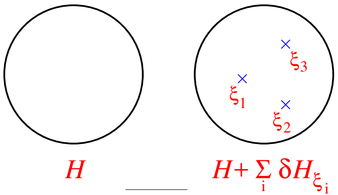
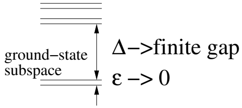
A general theory of topological excitations

- In a gapped system: $H = \sum_{\boldsymbol{x}} \hat{O}_{\boldsymbol{x}}$,
  excitations = δH_{ξ_i} gapped traps
  $H + \delta H_{\xi_1} + \delta H_{\xi_2} + \delta H_{\xi_3} \to$
  gapped ground space $\mathcal{V}_{\text{exc}}(\xi, \xi', \cdots)$

- Different excitations are labeled by different trap Hamiltonians δH_ξ





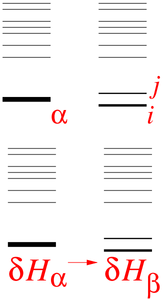
- **Topological types**: Two excitations, δH_ξ and $\delta \tilde{H}_{\tilde{\xi}}$, are equivalent if δH_ξ and $\delta \tilde{H}_{\tilde{\xi}}$ can deform into each other without closing the gap.
  The equivalent class of excitations $[\delta H_\xi] \equiv$ type-α.
- **Trivial type-1** if the corresponding equiv. class $[\delta H_\xi] \ni \delta H_\xi = 0$

- It can be created by local O_ξ: $\mathcal{V}_{\text{exc}}(\xi, \xi', \cdots) = O_\xi \mathcal{V}_{\text{exc}}(\xi', \cdots)$
- It has trivial double braiding (mutual statistics) with all excitations.

- **Non-trivial type-α** at ξ: $[\delta H_\xi] \not\ni \delta H_\xi = 0$

Simple/composite excitation and fusion category

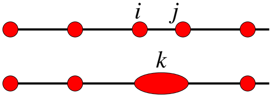
- **simple excitation** at ξ: The ground space $\mathcal{V}_{\text{exc}}^{\text{simple}}(\xi,\cdots)$ is robust against local perturbation near ξ $\rightarrow$ type i.
- **composite excitation** at ξ: The ground space $\mathcal{V}_{\text{exc}}(\xi,\cdots)$ (the degeneracy) can be splitted by local perturbation near ξ, *ie* contain accidental degeneracy $\rightarrow$ type $\alpha = i \oplus j$.



- **Excitations in 1d $\rightarrow$ Fusion cat. theory**

- Excitations δH_ξ = objects
- Morphism = deformation $\delta H_\alpha \rightarrow \delta H_\beta$: $\alpha \rightarrow i$
- The object type-i = isomorphism classes of excitations δH_ξ.
- In 1D and above,

$$i \otimes j = \underbrace{k \oplus \cdots \oplus k}_{N_k^{ij} \text{ copies}} \oplus \cdots = \oplus_k N_k^{ij} k$$



- **Fusion space**: $\mathcal{V}_{\text{exc}}(\xi_1,\xi_2,\cdots) = \mathcal{V}(i_1,i_2,\cdots)$

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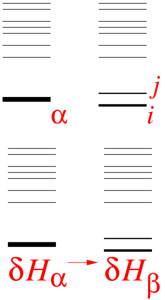

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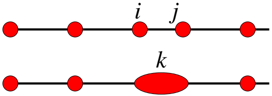


- **Fusion space**: $\mathcal{V}_{\text{exc}}(\xi_1, \xi_2, \cdots) = \mathcal{V}(i_1, i_2, \cdots)$

Consistent conditions on N^{ij}_k

Consider two ways to compute $i \otimes j \otimes k = \oplus_l N^{ijk}_l l$

$$(i \otimes j) \otimes k = \oplus_m N^{ij}_m \; m \otimes k = \oplus_{m,l} N^{ij}_m N^{mk}_l \; l$$
$$i \otimes (j \otimes k) = \oplus_n N^{jk}_n \; n \otimes k = \oplus_{n,l} N^{jk}_n N^{in}_l \; l$$

$\rightarrow$

$$\sum_{m,l} N^{ij}_m N^{mk}_l = \sum_{n,l} N^{jk}_n N^{in}_l$$
$$N^{\mathbf{1}i}_j = N^{i\mathbf{1}}_j = \delta_{ij}, \quad N^{i\bar{j}}_{\mathbf{1}} = \delta_{ij}.$$

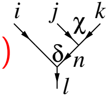
But N^{ij}_k is not all the data to describe the fusion of excitations. There is an additional data.

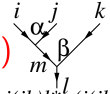
The F-symbol: $F^{ijk;m\alpha\beta}_{l;n\chi\delta}$

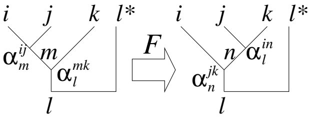
- Consider the fusion $i \otimes j \otimes k \to l \oplus \cdots \oplus l \to \mathcal{V}(i,j,k;\cdots) = \mathcal{V}(l;\cdots) \oplus \cdots \oplus \mathcal{V}(l;\cdots)$, but the direct sum $\oplus$ decomposition is not unique (like different choices of basis)

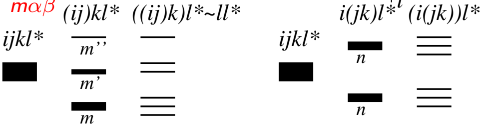
- $\mathcal{V}(i,j,k;\cdots) \to \oplus_{m,\alpha=1\cdots N^{ij}_m} \mathcal{V}_\alpha(m,k;\cdots)$
  $\to \oplus_{m,\alpha} \; \oplus_{\beta,l} \mathcal{V}_{\alpha;m,\beta}(l;\cdots) = \oplus_{m,\alpha;\beta,l} \mathcal{V}_{\alpha;m,\beta}(l;\cdots)$
- $\mathcal{V}(i,j,k;\cdots) \to \oplus_{n,\chi=1\cdots N^{jk}_n} \mathcal{V}_\chi(i,n;\cdots)$
  $\to \oplus_{n,\chi} \; \oplus_{\delta,l} \mathcal{V}_{\chi;n,\delta}(l;\cdots) = \oplus_{n,\chi;\delta,l} \mathcal{V}_{\chi;n,\delta}(l;\cdots)$
- $\mathcal{V}_{\alpha;m,\beta}(l;\cdots)$ and $\mathcal{V}_{\chi;n,\delta}(l;\cdots)$ like two sets of basis that span the same fusion space $\mathcal{V}(i,j,k;\cdots)$

- The F-symbol is a unitary matrix that relate the two basis

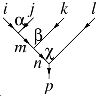
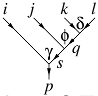
$\mathcal{V}_{\chi;n,\delta}(l;\cdots)$ 

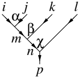
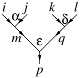
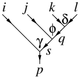
$= \sum_{m\alpha\beta} (F^{ijk}_l)^{m\alpha\beta}_{n\chi\delta} \mathcal{V}_{\alpha;m,\beta}(l;\cdots)$ 

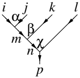
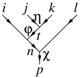
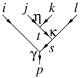


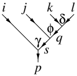


Consistent conditions for $F^{ijk;m\alpha\beta}_{l;n\chi\delta}$ and UFC

Two different ways of fusion  ,  (two sets of basis) are related via two different paths of F-moves:

 $= \sum_{q,\delta,\epsilon} F^{mkl;n\beta\chi}_{p;q\delta\epsilon}$  $= \sum_{q,\delta,\epsilon;s,\phi,\gamma} F^{mkl;n\beta\chi}_{p;q\delta\epsilon} F^{ijq;m\alpha\epsilon}_{p;s\phi\gamma}$  ,

 $= \sum_{t,\eta,\varphi} F^{ijk;m\alpha\beta}_{n;t\eta\varphi}$  $= \sum_{t,\eta,\varphi;s,\kappa,\gamma} F^{ijk;m\alpha\beta}_{n;t\eta\varphi} F^{itl;n\varphi\chi}_{p;s\kappa\gamma}$ 

$= \sum_{t,\eta,\kappa;\varphi;s,\kappa,\gamma;q,\delta,\phi} F^{ijk;m\alpha\beta}_{n;t\eta\varphi} F^{itl;n\varphi\chi}_{p;s\kappa\gamma} F^{jkl;t\eta\kappa}_{s;q\delta\phi}$  .

The two paths should lead to the same unitary trans.:

$$\sum_{t,\eta,\varphi,\kappa} F^{ijk;m\alpha\beta}_{n;t\eta\varphi} F^{itl;n\varphi\chi}_{p;s\kappa\gamma} F^{jkl;t\eta\kappa}_{s;q\delta\phi} = \sum_{\epsilon} F^{mkl;n\beta\chi}_{p;q\delta\epsilon} F^{ijq;m\alpha\epsilon}_{p;s\phi\gamma}$$

Such a set of non-linear algebraic equations is the famous pentagon identity. MacLane 63; Moore-Seiberg 89

$(N^{ij}_k, F^{ijk;m\alpha\beta}_{l;n\chi\delta}) \to$ Unitary fusion category $\to$ theory of 1d excitations

An example of UFC: Fibonacci fusion category

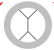
- A 1d topo. order described by a Fibonacci fusion category:

- Two types of topological excitations $\mathbf{1}, \phi$.
- Fusion rule N_k^{ij}: $\phi \otimes \phi = \mathbf{1} \oplus \phi$.
- F-symbol $F^{ijk;m\alpha\beta}_{l;n\chi\delta}$: $F^{\phi\phi\phi}_{\phi} = \begin{pmatrix} \gamma & \gamma^{1/2} \\ \gamma^{1/2} & -\gamma \end{pmatrix}, \ \gamma = \frac{\sqrt{5}-1}{2}$

$$F^{\phi\phi\phi}_{\mathbf{1}} = F^{\mathbf{1}\phi\phi}_{\phi} = F^{\phi\mathbf{1}\phi}_{\phi} = F^{\phi\phi\mathbf{1}}_{\phi} = \cdots = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Relation to the 2d string-net wave function

Φ_{str} () $= \gamma\Phi_{\text{str}}$ () $+ \sqrt{\gamma}\Phi_{\text{str}}$ ()

Φ_{str} () $= \sqrt{\gamma}\Phi_{\text{str}}$ () $- \gamma\Phi_{\text{str}}$ ()

Internal degrees of freedom – quantum dimension

- Let D_n be the number of locally indistinguishable states for n ϕ-particles on a sphere. The internal degrees of freedom of ϕ – **quantum dimension** – $d = \lim_{n\to\infty} D_n^{1/n}$

$$\underbrace{\phi \otimes \cdots \otimes \phi}_{n} = \underbrace{\mathbf{1} \oplus \cdots \oplus \mathbf{1}}_{D_n} \oplus \underbrace{\phi \oplus \cdots \oplus \phi}_{F_n}$$

$$D_n = \text{Dim}\big(\text{Hom}(\phi^{\otimes n}, \mathbf{1})\big), \quad F_n = \text{Dim}\big(\text{Hom}(\phi^{\otimes n}, \phi)\big),$$

$$\underbrace{\phi \otimes \cdots \otimes \phi}_{n} \otimes \phi = \underbrace{\mathbf{1} \oplus \cdots \oplus \mathbf{1}}_{F_n} \oplus \underbrace{\phi \oplus \cdots \oplus \phi}_{F_n + D_n}$$

$$D_{n+1} = F_n, \quad F_{n+1} = F_n + D_n = F_n + F_{n-1}, \quad D_1 = 0, \; F_1 = 1.$$

The internal degrees of freedom of ϕ is (spin-$\frac{1}{2}$ electron $d = 2$)

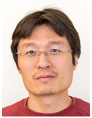
$$d = \lim_{n\to\infty} F_{n-1}^{1/n} = \frac{1+\sqrt{5}}{2} = 1.61803398874989\cdots.$$

Anomaly and the principle of remote detectablility

We say a UFC describes 1d excitations. But can we really find a 1d local lattice model such that its excitations are described by the UFC? Answer: **No**. This obstruction is called **anomaly**

How to tell if a theory for excitations is anomalous or not?

- **Remotely detectable = Realizable (anomaly-free)**
  Every non-trivial topological excitation *i* can be remotely detected by at least one topo. excitation *j* via remote operations (such as braiding) ↔ the topological order is realizable in the same dimension.



Levin arXiv:1301.7355, Kong Wen arXiv:1405.5858

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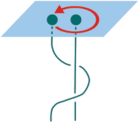

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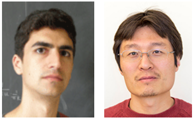

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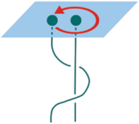

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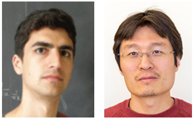

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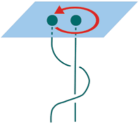

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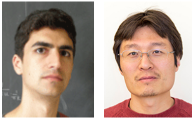

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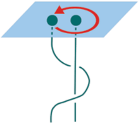

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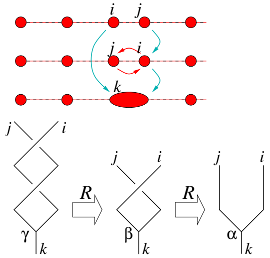
Theory of 2d excitations = braided fusion category

- Above 1D, particles can braid → unitary braided fusion category
- Braiding requires that $N^{ij}_k = N^{ji}_k$.
- Braiding → $R^{ij;\alpha}_{k;\beta}$

  mutual statistics = double braiding:

  $R^{ij;\alpha}_{k;\beta} R^{ji;\beta}_{k;\gamma} = \mathrm{e}^{\mathrm{i}\,\theta^{(k)}_{ij}} \delta_{\alpha\gamma}$

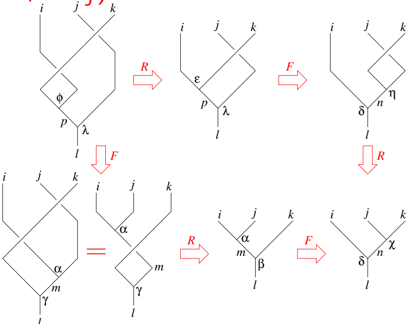
  topological spin s_i: $\theta^{(k)}_{ij} = 2\pi(s_k - s_i - s_j)$



- Hexagon identity:

  $R^{ik;\phi}_{p;\epsilon} F^{ikj;p\epsilon\lambda}_{l;n\eta\delta} R^{jk;\eta}_{n;\chi} = \sum_{m\alpha\beta} F^{kij;p\phi\lambda}_{l;m\alpha\gamma} R^{mk;\gamma}_{l;\beta} F^{ijk;m\alpha\beta}_{l;n\chi\delta}$

- **Theory of unitary braided fusion category (UBFC)** are fully characterized by those $(N^{ij}_k, F^{ijk;m\alpha\beta}_{l;n\gamma\lambda}, R^{ij;\alpha}_{k;\beta})$



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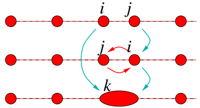

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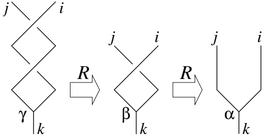

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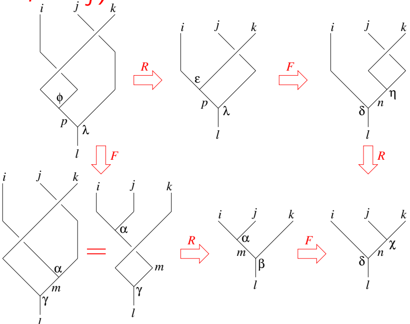

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Examples of UBFC (excitations in 2d topo. orders)

- **Anomalous (degenerate) UBFC**

- $i : (\mathbf{1}, e)$, $d_i : (1, 1)$, $s_i : (0, 0)$ (symm. fusion cat. $\mathcal{R}ep(\mathbb{Z}_2)$)

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- $i : (\mathbf{1}, s)$, $d_i : (1, 1)$, $s_i : (0, \frac{1}{4})$. ($\nu = \frac{1}{2}$ bosonic FQH state)
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$$Z_2(\mathcal{C}) \equiv \{i \mid \theta_{ij}^{(k)} = 0, \ \forall j, k\}$$

Remote detectable ↔ $Z_2(\mathcal{C}) = \{1\}$ (modular) ↔ Realizable

Excitations in an anomaly-free (realizable) 2d topological order are described by an unitary modular tenser category (UMTC)

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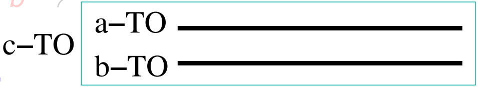
Do UMTC's classify 2d bosonic topo. orders?

- **No!** UMTC's classify $\frac{\{\text{2d bosonic topological orders}\}}{\{\text{2d bosonic invertible topological orders}\}}$
- Stacking two topological phases a, b give rise to a third topological phase $c = a \otimes b \to$ **The set of topological phases forms a monoid.**



- 1) A topo. order is **invertible** iff it has no non-trivial topo. excitations (but has a non-trivial domain wall (morphisms) to other topo. phases).
  2) A topo. order is invertible iff its **topo. partition function** are pure phases: $Z_{\text{top}}(M^n) \in U(1) \to$ classify inv. topo. orders

H-type invertible topo. order	$1+1D$	$2+1D$	$3+1D$	$4+1D$	$5+1D$	$6+1D$
Boson:	0	$\mathbb{Z}_{E_8}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z} \oplus \mathbb{Z}$
Fermion:	$\mathbb{Z}_{2\,p\text{-wave}}$	$\mathbb{Z}_{p+ip}$	0	0	0	$\mathbb{Z} \oplus \mathbb{Z}$

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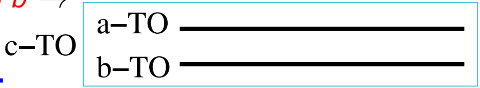

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Invertible topo. order (no fractionalized excitation)

- **2+1D:** $Z_{\text{top}}(M^3) = e^{i\frac{2\pi c}{24}\int_{M^3}\omega_3(g_{\mu\nu})}$ where ω_3 is the grav. CS term: $\mathrm{d}\omega_3 = p_1$ and p_1 is the first Pontryagin class.
- The quantization of the topo. term: $c = 8 \times \text{int.} \to \mathbb{Z}$-class:
  $\int_M \omega_3(g_{\mu\nu}) = \int_{N,\partial N=M} p_1 = \int_{N',\partial N'=M} p_1 \text{ mod } 3$,
  since $\int_{N_{\text{closed}}} p_1 = 0 \text{ mod } 3$.
- **4+1D:** $Z_{\text{top}}(M^5) = e^{i\pi\int_{M^5} w_2 w_3}$ where w_i is the i^{th} Stiefel-Whitney class $\to \mathbb{Z}_2$-class. We find $\int_{M^5} w_2 w_3 = 1$ when $M^5 = \mathbb{C}P^2 \rtimes_\varphi S^1$ and $\varphi: \mathbb{C}P^2 \to (\mathbb{C}P^2)^*$
- **6+1D:** Two independent gravitational Chern-Simons terms:
  $Z_{\text{top}}(M^7) = e^{2\pi i\int_{M^7}\left[k_1\frac{\tilde\omega_7 - 2\omega_7}{5} + k_2\frac{-2\tilde\omega_7+5\omega_7}{9}\right]}$
  where $\mathrm{d}\omega_7 = p_2$, $\mathrm{d}\tilde\omega_7 = p_1 p_1 \to \mathbb{Z}\oplus\mathbb{Z}$-class (k_1, k_2).
- **Topological order = UMTC + extra info (such as edge)**
  UMTC = Topo.-orders/invertible-topo.-orders

UMTC: 2+1D bosonic topo. orders mod invertibles

$\zeta_n^m = \frac{\sin(\pi(m+1)/(n+2))}{\sin(\pi/(n+2))}$ Rowell Stong Wang arXiv:0712.1377; Wen arXiv:1506.05768

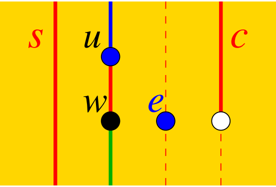
N_c^B	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$ wave func.	N_c^B	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$ wave func.
1_1^B	1	0			
2_1^B	$1, 1$	$0, \frac{1}{4}$ semion $\prod(z_i - z_j)^2$	2_{-1}^B	$1, 1$	$0, -\frac{1}{4}$ $\prod(z_i^* - z_j^*)^2$
$2_{14/5}^B$	$1, \zeta_3^1$	$0, \frac{2}{5}$ chiral Fibonacci TO	$2_{-14/5}^B$	$1, \zeta_3^1$	$0, -\frac{2}{5}$ anti-chiral Fib.
3_2^B	$1, 1, 1$	$0, \frac{1}{3}, \frac{1}{3}$ (221) double-layer	3_{-2}^B	$1, 1, 1$	$0, -\frac{1}{3}, -\frac{1}{3}$
$3_{8/7}^B$	$1, \zeta_5^1, \zeta_5^2$	$0, -\frac{1}{7}, \frac{2}{7}$	$3_{-8/7}^B$	$1, \zeta_5^1, \zeta_5^2$	$0, \frac{1}{7}, -\frac{2}{7}$
$3_{1/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{1}{16}$ Ising TO	$3_{-1/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{1}{16}$
$3_{3/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{3}{16}$ $\mathcal{S}(220)$, Ψ_{Pfaffian}	$3_{-3/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{3}{16}$
$3_{5/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{5}{16}$ $\Psi_{\nu=2}^2$ $SU(2)_2^f$	$3_{-5/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{5}{16}$
$3_{7/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{7}{16}$	$3_{-7/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{7}{16}$
$4_0^{B,a}$	$1, 1, 1, 1$	$0, 0, 0, \frac{1}{2}$ $(1, e, m, f)$ $\mathbb{Z}_2$-gauge	4_4^B	$1, 1, 1, 1$	$0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$
4_1^B	$1, 1, 1, 1$	$0, \frac{1}{8}, \frac{1}{8}, \frac{1}{2}$ $\prod(z_i - z_j)^4$	4_{-1}^B	$1, 1, 1, 1$	$0, -\frac{1}{8}, -\frac{1}{8}, \frac{1}{2}$
4_2^B	$1, 1, 1, 1$	$0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}$ (220) double-layer	4_{-2}^B	$1, 1, 1, 1$	$0, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{2}$
4_3^B	$1, 1, 1, 1$	$0, \frac{3}{8}, \frac{3}{8}, \frac{1}{2}$	4_{-3}^B	$1, 1, 1, 1$	$0, -\frac{3}{8}, -\frac{3}{8}, \frac{1}{2}$
$4_0^{B,b}$	$1, 1, 1, 1$	$0, 0, \frac{1}{4}, -\frac{1}{4}$ double semion	$4_{9/5}^B$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, -\frac{1}{4}, \frac{3}{20}, \frac{2}{5}$
$4_{-9/5}^B$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{4}, -\frac{3}{20}, -\frac{2}{5}$	$4_{19/5}^B$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{4}, -\frac{7}{20}, \frac{2}{5}$
$4_{-19/5}^B$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, -\frac{1}{4}, \frac{7}{20}, -\frac{2}{5}$ $\Psi_{\nu=3}^2$ $SU(2)_3^f$	$4_0^{B,c}$	$1, \zeta_3^1, \zeta_3^1, \zeta_3^1\zeta_3^1$	$0, \frac{2}{5}, -\frac{2}{5}, 0$ Fibonacci^2
$4_{12/5}^B$	$1, \zeta_3^1, \zeta_3^1, \zeta_3^1\zeta_3^1$	$0, -\frac{2}{5}, -\frac{2}{5}, \frac{1}{5}$	$4_{-12/5}^B$	$1, \zeta_3^1, \zeta_3^1, \zeta_3^1\zeta_3^1$	$0, \frac{2}{5}, \frac{2}{5}, -\frac{1}{5}$
$4_{10/3}^B$	$1, \zeta_7^1, \zeta_7^2, \zeta_7^3$	$0, \frac{1}{3}, \frac{2}{9}, -\frac{1}{3}$	$4_{-10/3}^B$	$1, \zeta_7^1, \zeta_7^2, \zeta_7^3$	$0, -\frac{1}{3}, -\frac{2}{9}, \frac{1}{3}$
5_0^B	$1, 1, 1, 1, 1$	$0, \frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}$ (223) DL	5_4^B	$1, 1, 1, 1, 1$	$0, \frac{2}{5}, \frac{2}{5}, -\frac{2}{5}, -\frac{2}{5}$
$5_2^{B,a}$	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, \frac{1}{8}, -\frac{3}{8}, \frac{1}{3}$	$5_2^{B,b}$	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, -\frac{1}{8}, \frac{3}{8}, \frac{1}{3}$
$5_{-2}^{B,b}$	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, \frac{1}{8}, -\frac{3}{8}, -\frac{1}{3}$	$5_{-2}^{B,a}$	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, -\frac{1}{8}, \frac{3}{8}, -\frac{1}{3}$
$5_{16/11}^B$	$1, \zeta_9^1, \zeta_9^2, \zeta_9^3, \zeta_9^4$	$0, -\frac{2}{11}, \frac{2}{11}, \frac{1}{11}, -\frac{5}{11}$	$5_{-16/11}^B$	$1, \zeta_9^1, \zeta_9^2, \zeta_9^3, \zeta_9^4$	$0, \frac{2}{11}, -\frac{2}{11}, -\frac{1}{11}, \frac{5}{11}$
5^B [illegible]	[illegible]	[illegible]	5^B [illegible]	[illegible]	[illegible]

Classify nd topological orders via excitations

- Excitations in an nd topo. orders are described by a fusion n-category

  objects = codim-1 excitations
  1-morphisms = codim-2 excitations
  (n-1)-morphisms = point excitations

  An example of fusion 2-category:

  s → object (string excitation)
  u → 1-morphisms (domain wall between strings)
  e → 1-morphisms (domain wall between trivial string = point excitations)
  c → string connecting trivial string via a domain wall (condensation excitation or descendent excitation)

- Vertical and horizontal fusions → braiding of particules

Which fusion n-cats correspond to topo. orders?

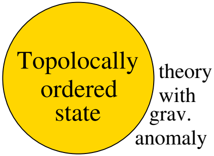
- **Realizable topo. orders $\xrightarrow{\eta}$ unitary fusion n-categories**
  Ker(η) = invertible topological orders.
  Img(η) = **anomaly-free** unitary fusion n-categories.
  A generic **unitray fusion n-category** may not realizable by any nd lattice models, and are called **anomalous**.
  Unitary defined in Kong Wen Zheng arXiv:1502.01690

- **Anomaly-free** fusion n-categories = ???
  Define **anomaly-free** *macroscopically (ie mathematically), instead of microscopically via realizable by lattice models.*

- We have defined **Anomaly-free** via the E_2-center $Z_2(\mathcal{C}) = n\text{Vec}$ (*ie* via mutual statistics). But this approach is hard to understand for higher categories.

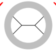
Try to define anomaly via boundary-bulk relation

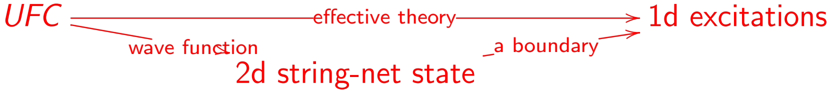
- A UFC describes 1d topological excitations
  → 1+1D locally consistent effective theory.
- It is not realizable → not globally consistent
  → having **gravitational anomaly**



- A 1d UFC (locally consistent) can always be realized as a gapped boundry of a 2d topolgocal order (a UMTC)
  - The Fibonacci fusion category $\phi \otimes \phi = \mathbf{1} \oplus \phi$ describe the 1d excitations at a gapped boundary of 2d string-net state (UMTC $i: (\mathbf{1}, \phi, \bar{\phi}, \phi\bar{\phi})$, $d_i: (1, \gamma, \gamma, \gamma^2)$, $s_i: (0, \frac{2}{5}, -\frac{2}{5}, 0)$):

Φ_{str} () $= \gamma\Phi_{\text{str}}$ () $+ \sqrt{\gamma}\Phi_{\text{str}}$ ()

Φ_{str} () $= \sqrt{\gamma}\Phi_{\text{str}}$ () $- \gamma\Phi_{\text{str}}$ ()



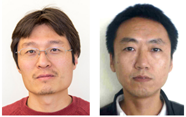
- Non-trivial bulk topo. order → grav. anomaly at boundary

Generalization to higher dimensions

Up to invertible topologioca orders

- **Potencially anomalous nd topological orders** = boundary of $(n+1)$d topological orders = fusion n-catgeories.
- **Anomaly-free nd topological orders** = boundary of $(n+1)$d trivial product state = realizable by nd lattice models = special fusion n-catgeories. *But which ones?*

Kong Wen arXiv:1405.5858; Kong Wen Zheng arXiv:1502.01690



- **Holographic principle of topological order**: **The boundary uniquely determines the bulk.** A potentially anomalous topological order (a fusion n-category $\mathcal{C}^n$) determines a unique bulk topological order (a braided fusion n-category $\mathcal{M}^n$):

$$Z_1(\mathcal{C}^n) = \mathcal{M}^n$$

Z_1 is the E_1-center: $Z_1(\mathcal{C}^n) = \mathcal{M}^n$ a braided fusion n-category.

- The bulk topological order $\mathcal{M}^n$ is anomaly-free

$$\mathcal{M}^{n+1} = \Sigma\mathcal{M}^n;\ Z_1(\mathcal{M}^{n+1}) = (n+1)\text{Vec}$$

Generalization to higher dimensions

Up to invertible topologioca orders

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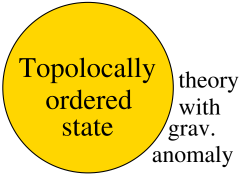
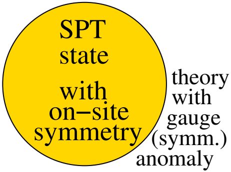
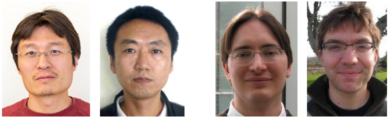
$$\mathcal{M}^{n+1} = \Sigma\mathcal{M}^n; \; Z_1(\mathcal{M}^{n+1}) = (n+1)\text{Vec}$$

Anomaly and holographic principle → Classification

- **Gravitational anomally** Kong Wen arXiv:1405.5858; Kong Wen Zheng
  = topological order in one higher dimension arXiv:1502.01690
- **Symmetry (t' Hooft) anomally** Wen arXiv:1303.1803
  =SPT order in one higher dimension







Anomaly-free (realizable) nd topological orders (up to invertibles) are classified by unitary fusion n-categories $\mathcal{C}^n$ that satisfy $Z_1(\mathcal{C}^n) = n\mathrm{Vec}$ and include all condansation excitations. ($n\mathrm{Vec}$ = trivial braided fusion n-category.)

Kong Wen arXiv:1405.5858; Kong Wen Zheng arXiv:1502.01690
Gaiotto Johnson-Freyd arXiv:1905.09566; Johnson-Freyd 2003.06663

Graviational anomaly: an old point of view

- The action of a classical field theory

$$S(\phi, v_\mu) = \int \mathrm{d}^n x \sqrt{\det(g_{\mu\nu})}\mathcal{L}(\phi, v_\mu; g_{\mu\nu})$$

diffeomorphism invariance $x^\mu \to \tilde{x}^\mu$

- But for the path integral that define quantum theory, the partition function

$$Z = \int D[\phi] D[v_\mu] \mathrm{e}^{-S(\phi, v_\mu)}$$

is not invariant under the diffeomorphism transformation due to the Jacobian for the change of integration measure
$\to$ **invertible graviational anomaly**

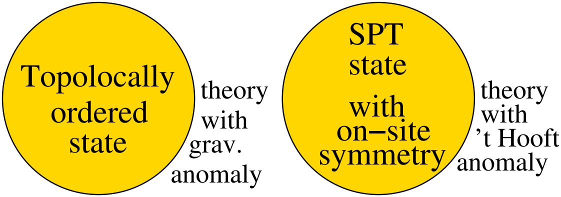
- Jacobian $=$ non-zero complex number $\to$ The anomalies are **invertible**.

Anomaly: a modern point of view → non-invertible

Anomaly-free = realizable by lattice model in the same dim
Anomalous = realizable by a boundary of a gapped lattice model in one higher dimension.

- A quantum field theory with gravitational anomaly cannot be realized as the low energy effective theory of a lattice model in the same dimension. Wen arXiv:1303.1803; Kong Wen arXiv:1405.5858 Fiorenza Valentino arXiv:1409.5723; Monnier arXiv:1410.7442

But can be realized as the low energy effective theory of a boundary of a lattice model in one-higher dimension.



- **Gravitational anomaly** = **Topological order** in one higher dimension → **non-invertible** gravitational anomaly
- **Symmetry ('t Hooft) anomaly** = **SPT order** in one higher dimension → **invertible** symmetry anomaly

Try to characterize/classify gapless CFTs via non-invertible gravitational anomalies

Unlike **invertible gravitational anomaly**, the **non-invertible gravitational anomaly** (*ie* the topological order in one higher dimension, also called **categorical symmetry**) contain a lot of information, that can be used to characterize (or even classify) gapless conformal field theories (CFT) (*ie* with linear dispertion relation $\omega = v|\boldsymbol{k}|$).

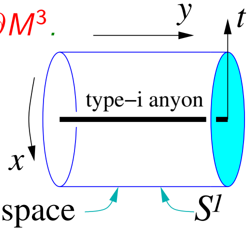
CFTs are characterized (or even classified) by their maximal emergent categorical symmetries

CFTs are characterized (or even classified) by their maximal emergent non-invertible gravitational anomalies

Understand degenerate ground states on torus

- Remember that 2+1D topological order is characterized by **degenerate ground states** on torus and the **modular matrices** S, T that generate the representations of the mapping class group of the torus.

Consider a spacetime evolution M^3, $T^2 = \partial M^3$.



- The Euclidean spacetime evolution produce a ground state on the torus $T^2 = \partial M^3$
- Embeding the worldline of different types of anyon gives rise to different degenerate ground states $|\Psi_i\rangle$ on torus.

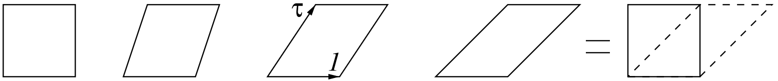
So the degenerate ground states are labeled by anyon types i.

- Under the modular transformations S, T they transform as

$$|\Psi_i\rangle \to S_{ij}|\Psi_j\rangle, \qquad |\Psi_i\rangle \to T_{ij}|\Psi_j\rangle$$

A 1+1D non-invertible anomaly (=1+1D categorical symmetry = 2+1D topo. order) is described by S, T

How to understand various 1+1D boundaries of a 2+1D topological order? Ji & Wen arXiv:1905.13279

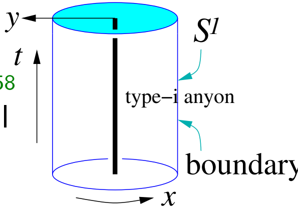


- The partition function of a 1+1D lattice model (or boundary of 2+1D trivial gapped state) dependent on the shape of the spacetime: $Z(\tau) = \mathrm{Tr} e^{-(\mathrm{Im}\tau)H + \mathrm{i}(\mathrm{Re}\tau)P}$, which is **modular invariant** $Z(\tau) = Z(\tau+1)$, $Z(\tau) = Z(-\frac{1}{\tau})$

- *modular invariance → 1+1D anomaly free*

- Boundary of 2d topo. order → 1d theory w/ non-inv. anomaly. Wen 1303.1803; Kong Wen 1405.5858

- 1d non-invertible anomalous theory has several partition functions $Z_i(\tau)$ labeled by the anyon types i of the 2+1D bulk topo. order, and is



- **modular covariant** $Z_i(\tau+1) = T_{ij} Z_j(\tau)$, $Z_i(-\frac{1}{\tau}) = S_{ij} Z_j(\tau)$

S, T-matrices = the 2+1D bulk topo. order = 1+1D anomaly

Gapped boundaries of 2+1D topological order

- The partition functions for 1+1D gapped state are constant integer $Z(\tau) = Z \in \mathbb{Z}$. The gapped boundaries have partition functions that satisfy $Z_i = T_{ij} Z_j,\ Z_i = S_{ij} Z_j,\ Z_{\mathbf{1}} = 1.$

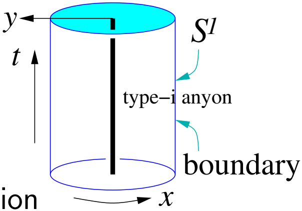

Lan Wang Wen arXiv:1408.6514

- For $\mathbb{Z}_2$ topological order,

$$T_{Z_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad S_{Z_2} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

we find two solutions

$$\begin{pmatrix} Z_1 \\ Z_e \\ Z_m \\ Z_f \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}_{e\text{-cond}}, \quad \begin{pmatrix} Z_1 \\ Z_e \\ Z_m \\ Z_f \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}_{m\text{-cond}}$$



→ two kinds of boundaries from e-condensation and m-condensation.

View topo. order as **categorical symmetry**: New name $\to$ new understanding and new results

- **2+1D topological order = 1+1D non-invertible gravitational anomaly** can be viewed as symmetry, which is called **1+1D categorical symmetry** (due to the conervation of 2+1D excitations as described by their fusion rule).
  For example: the 2+1D $\mathbb{Z}_2$ topological order (with excitations $\mathbf{1}, e, m, f$) corresponds to categorycal symmetry $\mathbb{Z}_2^{(e)} \vee \mathbb{Z}_2^{(m)}$ from mod-2 conservation of e and m.

- Gapped boundaries spontaneously break part of the categorical symmetry.
  e-condensed boundary: $\mathbb{Z}_2^{(e)} \vee \mathbb{Z}_2^{(m)} \to \mathbb{Z}_2^{(m)}$.
  m-condensed boundary: $\mathbb{Z}_2^{(e)} \vee \mathbb{Z}_2^{(m)} \to \mathbb{Z}_2^{(e)}$

Gapless boundaries of 2+1D topological order

- *What is the gapless 1+1D CFT with a given non-invertible gravitational anomaly (=1+1D categorical symmetry = 2+1D topological order)?* Ji Wen arXiv:1912.13492

- For example: The *e*-condensed gapped boundary and the *m*-condensed gapped boundary are separated by a gapless critical point, which is nothing but the 1+1D $\mathbb{Z}_2^{(e)}$(or $\mathbb{Z}_2^{(m)}$) symmetry breaking critical point (the CFT of Ising model). The critical point has no *e* condensation nor *m* condensation, and thus has the full $\mathbb{Z}_2^{(e)} \vee \mathbb{Z}_2^{(m)}$ categorical symmetry.
- The 2+1D $\mathbb{Z}_2$ topological order (*ie* the 1+1D $\mathbb{Z}_2^{(e)} \vee \mathbb{Z}_2^{(m)}$ categorical symmetry) determines the 1+1D CFT, hinting **categorical symmetry may be used to classify CFTs.**

2+1D Z_2 topological order (*ie* 1+1D $\mathbb{Z}_2^{(e)} \vee \mathbb{Z}_2^{(m)}$ categorical symmetry) can determine 1+1D CFTs

- The 2+1D Z_2 topological order (*ie* the $\mathbb{Z}_2^{(e)} \vee \mathbb{Z}_2^{(m)}$ categorical symmetry) has four types of excitations $\mathbf{1}, e, m, f$ and is characterized by

$$T_{Z_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad S_{Z_2} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

- Its gapless boundary has 4-component partition function $Z_1(\tau)$, $Z_e(\tau)$, $Z_m(\tau)$, and $Z_f(\tau)$ that satisfy

$$Z_i(\tau+1) = T_{ij} Z_j(\tau), \quad Z_i(-1/\tau) = S_{ij} Z_j(\tau),$$

where $i, j = 1, e, m, f$.

- The above equations have many possible solutions with no condensation (*ie* $Z_i \neq 0$) and τ-dependence (thus gapless).

Categorical symmetries → CFTs
Non-invertible gravitational anomalies → CFTs

- **Ising CFT (minimal model $(4,3)$)**: $c = \bar{c} = \frac{1}{2}$

$$\begin{pmatrix} Z_1(\tau,\bar\tau) \\ Z_e(\tau,\bar\tau) \\ Z_m(\tau,\bar\tau) \\ Z_f(\tau,\bar\tau) \end{pmatrix} = \begin{pmatrix} |\chi_0^{\text{Is}}(\tau)|^2 + |\chi_{\frac{1}{2}}^{\text{Is}}(\tau)|^2 \\ |\chi_{\frac{1}{16}}^{\text{Is}}(\tau)|^2 \\ |\chi_{\frac{1}{16}}^{\text{Is}}(\tau)|^2 \\ \chi_0^{\text{Is}}(\tau)\bar\chi_{\frac{1}{2}}^{\text{Is}}(\bar\tau) + \chi_{\frac{1}{2}}^{\text{Is}}(\tau)\bar\chi_0^{\text{Is}}(\bar\tau) \end{pmatrix},$$

- **Minimal model $(5,4)$ CFT**: $c = \bar{c} = \frac{7}{10}$

$$\begin{pmatrix} Z_1 \\ Z_e \\ Z_m \\ Z_f \end{pmatrix} = \begin{pmatrix} |\chi_0^{m4}|^2 + |\chi_{\frac{1}{10}}^{m4}|^2 + |\chi_{\frac{3}{5}}^{m4}|^2 + |\chi_{\frac{3}{2}}^{m4}|^2 \\ |\chi_{\frac{7}{16}}^{m4}|^2 + |\chi_{\frac{3}{80}}^{m4}|^2 \\ |\chi_{\frac{7}{16}}^{m4}|^2 + |\chi_{\frac{3}{80}}^{m4}|^2 \\ \chi_0^{m4}\bar\chi_{\frac{3}{2}}^{m4} + \chi_{\frac{1}{10}}^{m4}\bar\chi_{\frac{3}{5}}^{m4} + \chi_{\frac{3}{5}}^{m4}\bar\chi_{\frac{1}{10}}^{m4} + \chi_{\frac{3}{2}}^{m4}\bar\chi_0^{m4} \end{pmatrix}$$

- The correspondence is not 1-to-1. We can improve it by considering CFTs with minimal number of excitations.

A categorical symm. $\mathbb{Z}_2^{(e)} \vee \mathbb{Z}_2^{(m)}$ → the canonical minimal Ising CFT

The canonical gapless boundary of topo. order

- **A $n+1$d gapped topological order has one (or more) canonical gapless CFT boundaries**, that
  (1) has no condensation of bulk excitations, and
  (2) has minimal amount of boundary excitations.

\- Topological Wick rotation: Kong Zheng arXiv:1905.04924; 1912.01760
2+1D topo. orders (UMTCs) classify 1+1D CFTs


CFTs → Categorical symmetries
CFTs → Non-invertible gravitational anomalies

- The **Ising CFT (minimal model (4, 3))**: $c = \bar{c} = \frac{1}{2}$

$$\begin{pmatrix} Z_1(\tau,\bar\tau) \\ Z_e(\tau,\bar\tau) \\ Z_m(\tau,\bar\tau) \\ Z_f(\tau,\bar\tau) \end{pmatrix} = \begin{pmatrix} |\chi^{\text{Is}}_0(\tau)|^2 + |\chi^{\text{Is}}_{\frac{1}{2}}(\tau)|^2 \\ |\chi^{\text{Is}}_{\frac{1}{16}}(\tau)|^2 \\ |\chi^{\text{Is}}_{\frac{1}{16}}(\tau)|^2 \\ \chi^{\text{Is}}_0(\tau)\bar\chi^{\text{Is}}_{\frac{1}{2}}(\bar\tau) + \chi^{\text{Is}}_{\frac{1}{2}}(\tau)\bar\chi^{\text{Is}}_0(\bar\tau) \end{pmatrix},$$

  is a boundary of 2+1D $\mathbb{Z}_2$ topological order with 4 anyons.
- The Ising CFT actually have a larger emergent categorical symmetry $\text{UMTC}_{\text{Ising}} \otimes \overline{\text{UMTC}}_{\text{Ising}}$ with nine anyons (*ie* can be a boundary of the 2+1D double Ising topological order with more topological excitations or more total quantum dim). The nine component partition function is given by

$$Z_{ij}(\tau) = \chi^{\text{Is}}_i(\tau)\bar\chi^{\text{Is}}_j(\bar\tau), \qquad i,j = 0,\ 1/2,\ 1/16.$$

- A Ising CFT → the canonical maximal categorical symm. $\text{UMTC}_{\text{Ising}} \otimes \overline{\text{UMTC}}_{\text{Ising}}$

Ji Wen arXiv:1912.13492

- The **minimal model** $(5,4)$ **CFT**: $c = \bar{c} = \frac{7}{10}$

$$\begin{pmatrix} Z_1 \\ Z_e \\ Z_m \\ Z_f \end{pmatrix} = \begin{pmatrix} |\chi_0^{m4}|^2 + |\chi_{\frac{1}{10}}^{m4}|^2 + |\chi_{\frac{3}{5}}^{m4}|^2 + |\chi_{\frac{3}{2}}^{m4}|^2 \\ |\chi_{\frac{7}{16}}^{m4}|^2 + |\chi_{\frac{3}{80}}^{m4}|^2 \\ |\chi_{\frac{7}{16}}^{m4}|^2 + |\chi_{\frac{3}{80}}^{m4}|^2 \\ \chi_0^{m4}\bar{\chi}_{\frac{3}{2}}^{m4} + \chi_{\frac{1}{10}}^{m4}\bar{\chi}_{\frac{3}{5}}^{m4} + \chi_{\frac{3}{5}}^{m4}\bar{\chi}_{\frac{1}{10}}^{m4} + \chi_{\frac{3}{2}}^{m4}\bar{\chi}_0^{m4} \end{pmatrix}$$

is a boundary of 2+1D $\mathbb{Z}_2$ topological order with 4 anyons.

- The $(5,4)$ CFT actually have a larger emergent categorical symmetry: it is a boundary of 2+1D topo. order $(2^B_{-14/5} \otimes 3^B_{7/2}) \otimes (2^B_{14/5} \otimes 3^B_{-7/2})$. ($2^B_{14/5} \sim G(2)|_1$ CS theory)

  The minimal model $(5,4)$ CFT has the maximal emergent categorical symmetry (maximal non-invertible gravitational anomaly) given by $(2^B_{-14/5} \otimes 3^B_{7/2}) \otimes (2^B_{14/5} \otimes 3^B_{-7/2})$.

- **1+1D rational CFTs $\overset{\text{1-to-1}}{\longleftrightarrow}$ Maximal emergent 1+1D categorical symmetries**

Are *n*d gapless CFTs "classified" by their maximal emergent categorical symmetry?

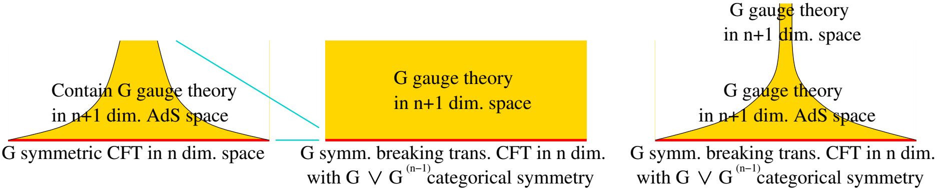
- The CFT at $n > 1$d spontaneous G-symmetry breaking transition point has a $G \vee G^{(n-1)}$ categorical symmetry (*ie* is a boundary of $n+1$d topological order of G-gauge theory, where G is finite. Ji Wen arXiv:1912.13492

- Such a critical point has a 0-symmetry G, and has an **algebraic $(n-1)$-symmetry** $G^{(n-1)}$.

Kong Lan Wen Zhang Zheng arXiv:2003.08898; arXiv:2005.14178

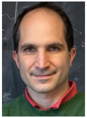
The relation between the CFT and its categorical symmetry (*ie* topological order in one higher dimension) is similar to the AdS/CFT duality.

Categorical symmetry and AdS/CFT duality



- **AdS/CFT duality**: Maldacena hep-th/9711200; Witten hep-th/9802150

  (1) A CFT with G-symmetry has a AdS bulk that contains G-gauge theory.

  (2) AdS bulk that contains G-gauge theory (and gravity) has a boundary CFT that contain a G-symmetry.

- **A more detailed proposal**: Ji Wen arXiv:1912.13492

  Pure G-gauge theory (w/ charge fluc. & gravity) in $(n+1)$d AdS space ∼ a particular CFT that appears at the nd spontaneous G-symmetry breaking transition, not other CFT's with G-symmetry.