Beyond Algebraic Models of Stringy Spacetime

Tristan Hübsch

@ ESI: 2026.07.06

Departments of Physics & Astronomy and Mathematics, Howard University, Washington DC
Department of Physics, Faculty of Natural Sciences, Novi Sad University, Serbia
Department of Mathematics, University of Maryland, College Park, MD

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Beyond Algebraic Stringy Spacetime

Playbill

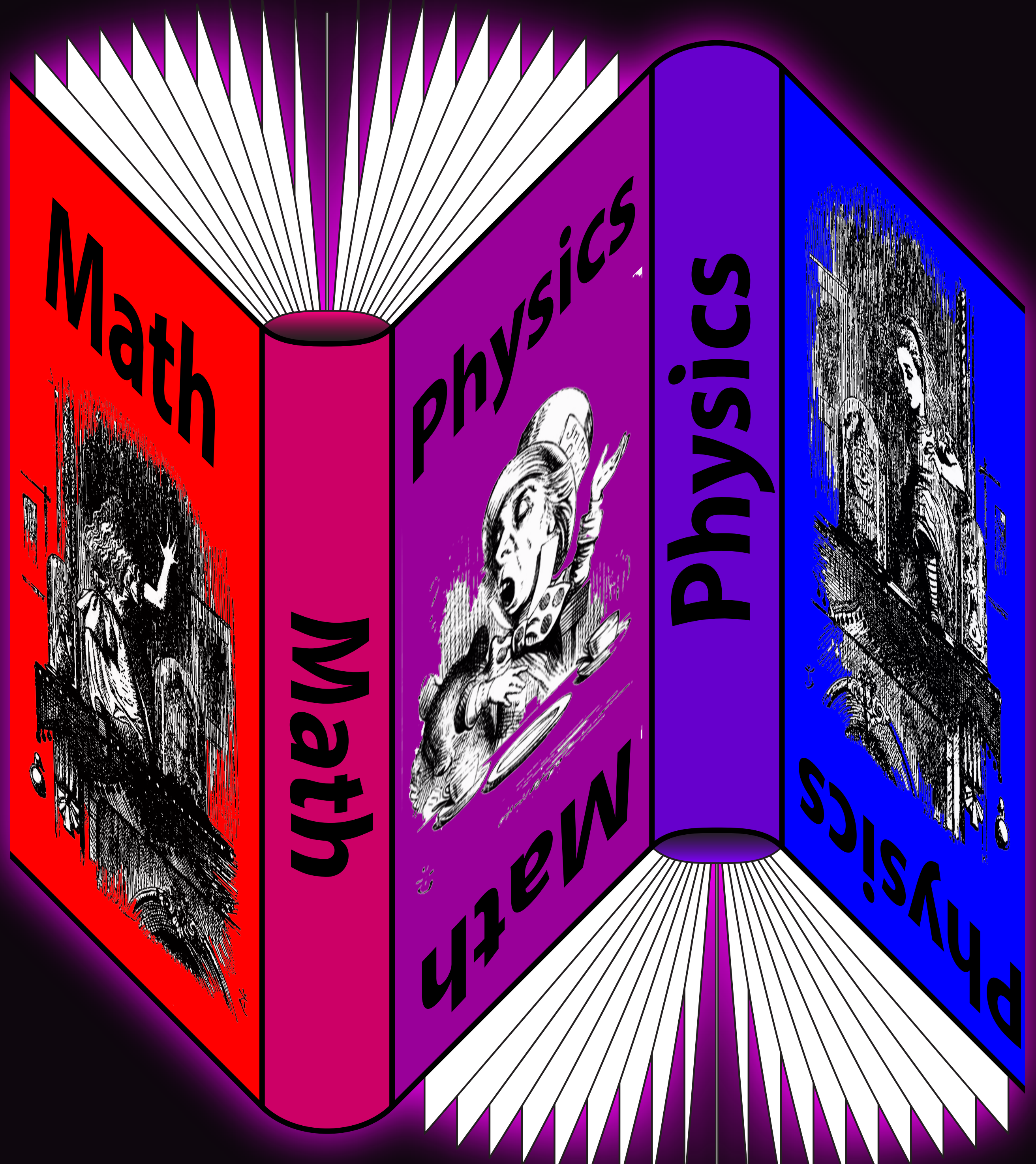
Prehistoric Prelude

Fractional Fugue

Meromorphic March

Mirror Minuet

Conjectural Coda

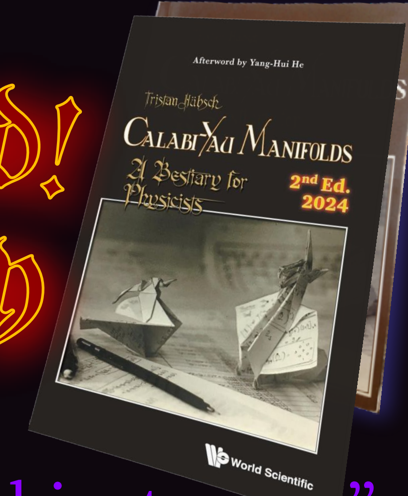


* "It doesn't matter what it's called,
...as long as it has substance."
— S.-T. Yau

worldsheet SCFTs (super)string compactification

What-Where-How?

2 nd! B-day



→ Who, When (We, for ⅓ century)

nice "ambient space" $= \prod_i \mathbb{P}^{n_i}$, $\mathbb{P}^{n_i}_{\vec{w}}$, toric…

- Complete Intersection: $X = \big(\cap_i \{f_i(z) = 0\}\big) \subset A$
- Constrained subspace: $\mathfrak{X}_i = \{f_i(z) = 0\} \subset A$

Tian-Yau: $\{\text{Fano}\}_c \smallsetminus \{\text{CY}\}_c = \{\text{CY}\}_{nc}$

Also: $\{\mathscr{K}^*_{X_c}\} = \{\text{CY}\}_{nc}$

- Functions: $\mathscr{O}_X(d) \ni \phi_X(z) \simeq \big[\,\phi_A(z) \pmod{f_i(z)}\,\big]$

generated by the defining system

- *Calculus:* $T^*_X(d) \ni \mathrm{d}z_X \simeq \big[\,\mathrm{d}z_A \pmod{\mathrm{d}f_i(z)}\,\big]$—*"adjunction theorem"*
- *Transversality:* $\{\wedge_i\, \mathrm{d}f_i = 0\} \cap \{f_i = 0\} \not\subset A$
- Anomaly-free: $\mathrm{d}^{\dim X} z_X \overset{!}{\doteq} \mathscr{O}_X(0) \;\Leftrightarrow\; \deg[\mathrm{d}^{\dim A} z_A] \doteq \deg[\mathrm{d}^K f_i]$
- Massless fields: $H^{p,q}(X) = H^q(X, \wedge^p T^*_X)$, also $H^q(X, \mathrm{End}\, T_X)$
- "Gauge" (for "gauge" (for "gauge" (for…))) **equivalence** classes
- Bott-Borel-Weil: $\mathbb{P}^n = \dfrac{U(n+1)}{U(n)\times U(1)} \Rightarrow \boxed{\phi^{a_1\cdots}_{b_1\cdots}(z_X) \sim U(n+1)\text{-tensor } \textbf{expressions}}$

  & Eastwood

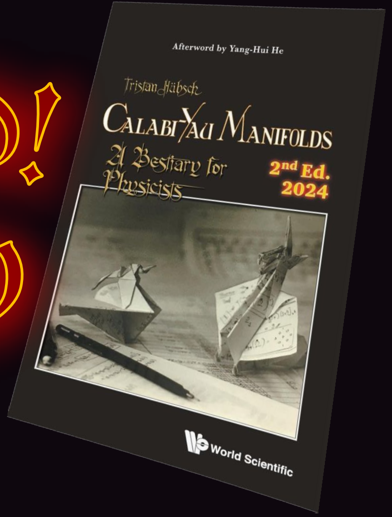
+ `Macaulay2, SAGE, Magma,...` *(new tricks/old dogs…)*

→ *sequel: "old dogs strike back"*

worldsheet SCFT ground-states compactification

What-Where-*How*?

2 nd! B-day



→ Who, When (We, for ⅓ century)

- Mirror models: $f(x) := \sum_{j=1}^{n} a_j \big(\prod_{i=1}^{n} x_i^{e_{ij}}\big)$, s.t. $\det\big[\mathbb{E} = [\![e_{ij}]\!]\big] \neq 0$
- Then, $\mathrm{col}[\mathbb{E}^{-1}] \mapsto \mathcal{S}_{diag}[f(x)] = Q \times G$: $Q \supset H$; $Q \circlearrowleft \mathscr{H}_{1,1}$ & $G \circlearrowleft \mathscr{H}_{2,1}$
  - Now, ${}^{\mathsf{T}}\!f(y) := \sum_{i=1}^{n} b_i \big(\prod_{j=1}^{n} y_j^{e_{ji}}\big)$ $Q = G'$ & $G = Q'$
  - $\mathrm{row}[\mathbb{E}^{-1}] \mapsto {}^{\mathsf{T}}\mathcal{S}_{diag}[{}^{\mathsf{T}}\!f(x)] = Q' \times G'$: $Q' \supset H'$; $Q' \circlearrowleft {}^{\mathsf{T}}\mathscr{H}_{1,1}$ & $G' \circlearrowleft {}^{\mathsf{T}}\mathscr{H}_{2,1}$
  - So $\mathrm{Bl}\big[\{f^{-1}(0) \subset X\}/H\big] \xleftrightarrow{\text{mm}} \mathrm{Bl}\big[\{{}^{\mathsf{T}}\!f^{-1}(0) \subset Y\}/H'\big]$
- Complete deformation family, $f(x) := \sum_{j=1}^{m} a_j \big(\prod_{i=1}^{n'} x_i^{\mathfrak{E}_{ij}}\big)$, BIG
  - specifies combinatorially many mirrors by "reduction":
  - set: some $a_j, b_i \to 0$ & some $x_i, y_j \to 0$ or 1: $\mathrm{Red}[\mathfrak{E}_{ij}]$ IPP simplices
  - ➠ combinatorial data ~ toric spec's

Mirror Generating "Machine"

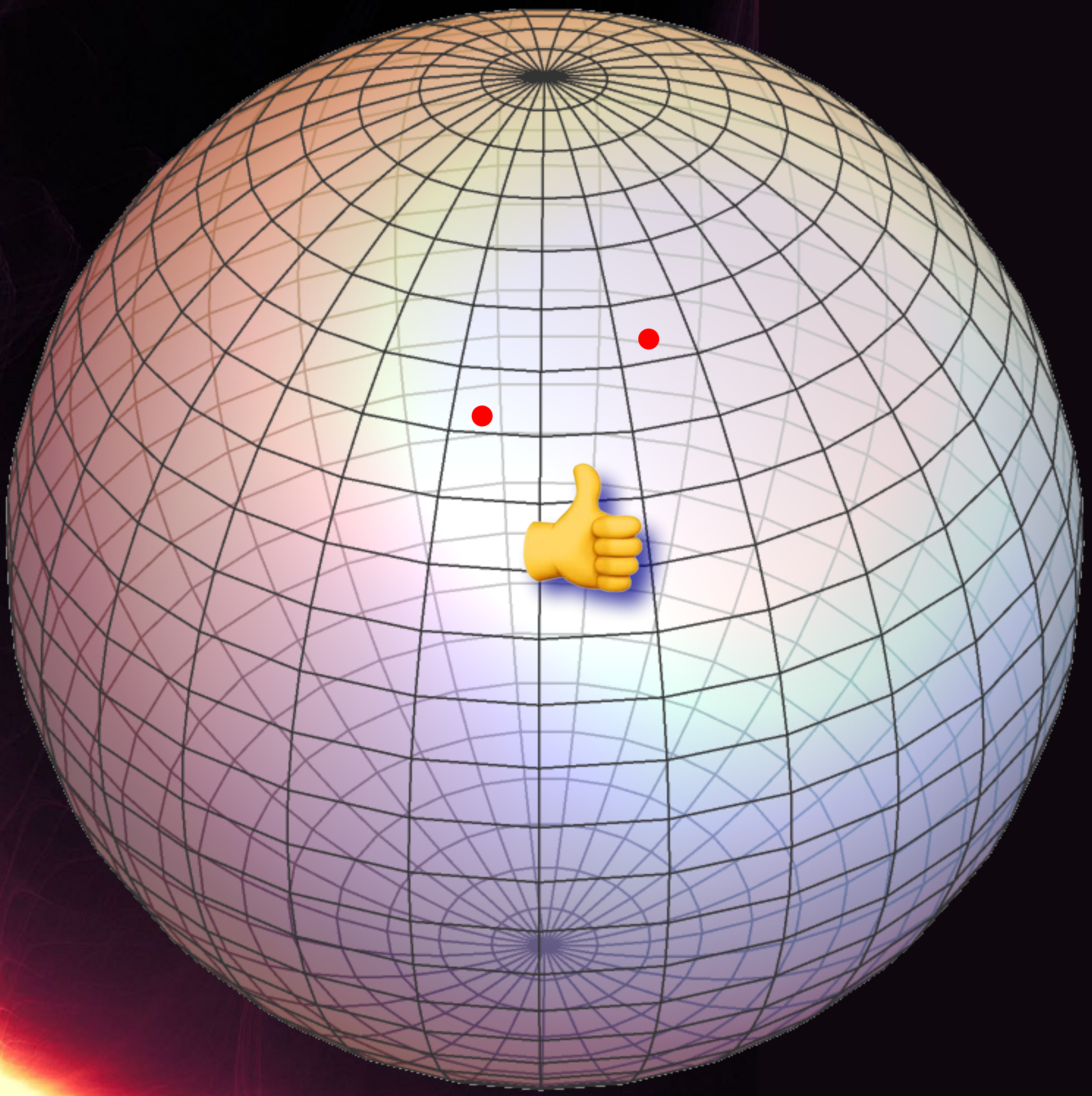
→ *sequel: "old dogs strike back"*

How Hard Can it Be?

Spiritus Movens

Why: CY ⊂ Some "Nice" Ambient Space

- Reduce to 0 dimensions: $\mathbb{P}^4[5] \to \mathbb{P}^3[4] \to \mathbb{P}^2[3] \to \mathbb{P}^1[2]$



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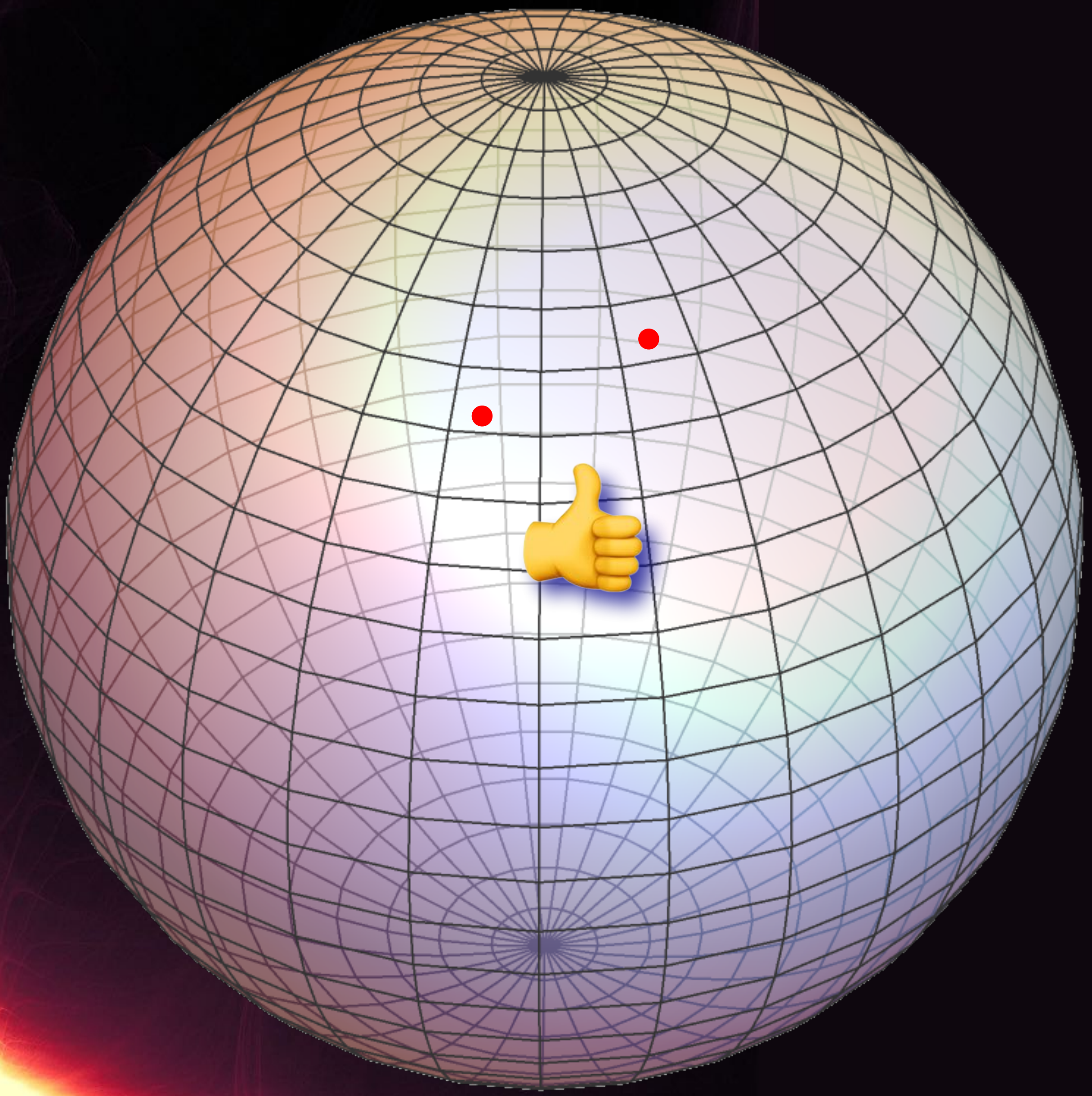

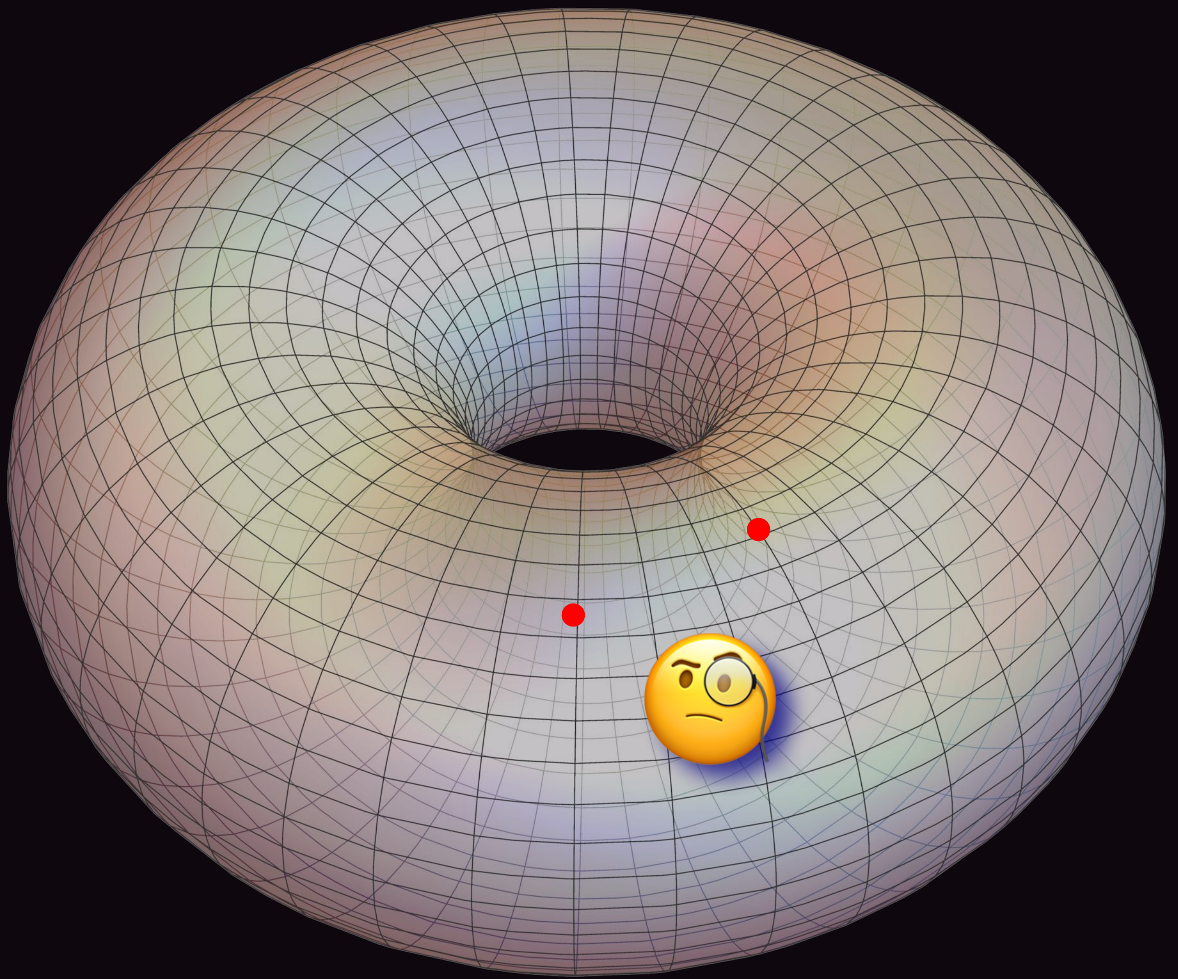

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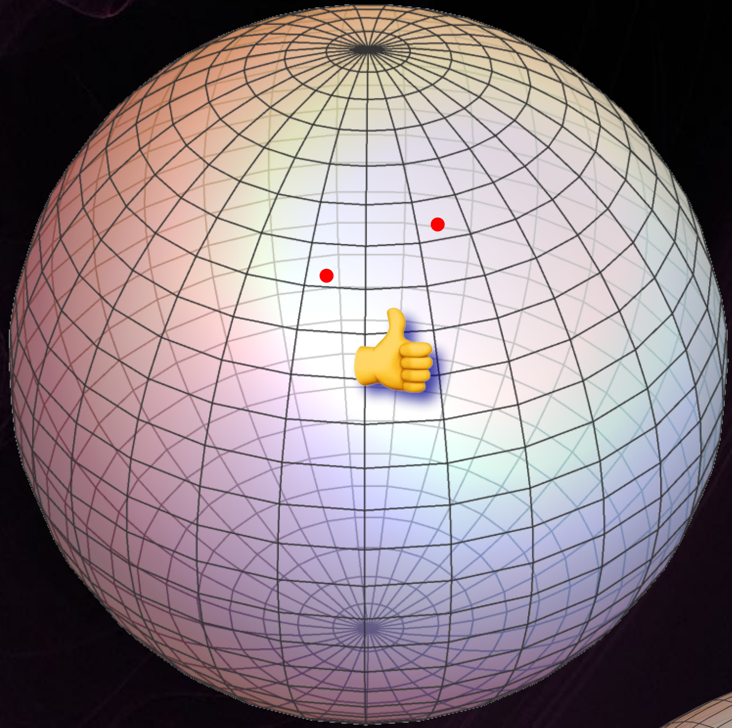
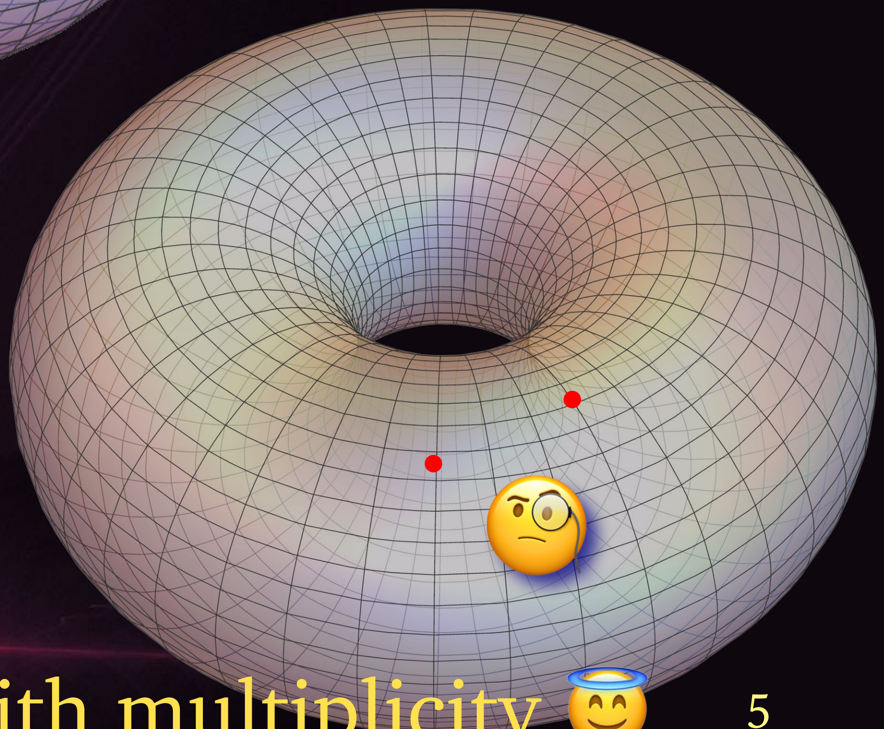
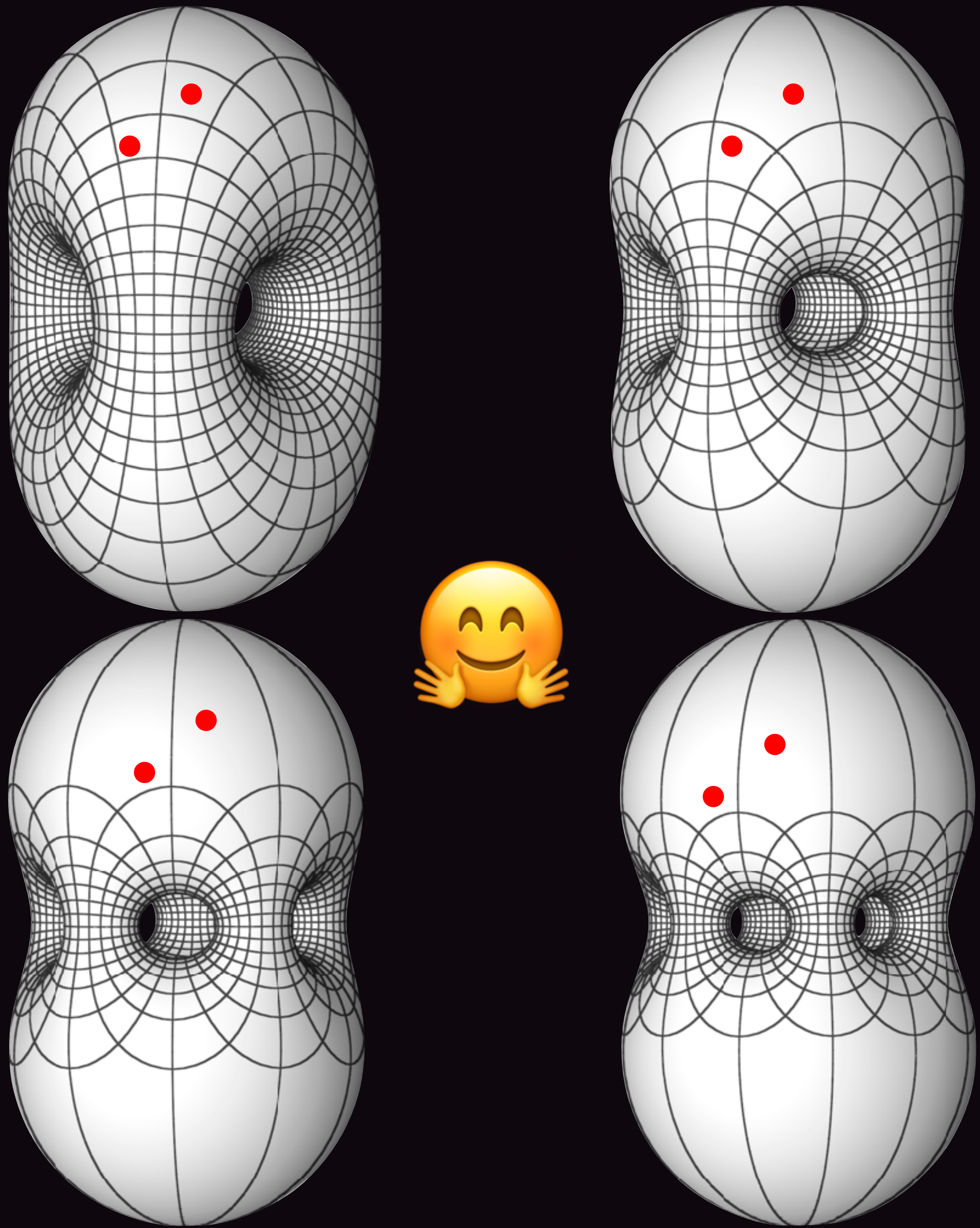


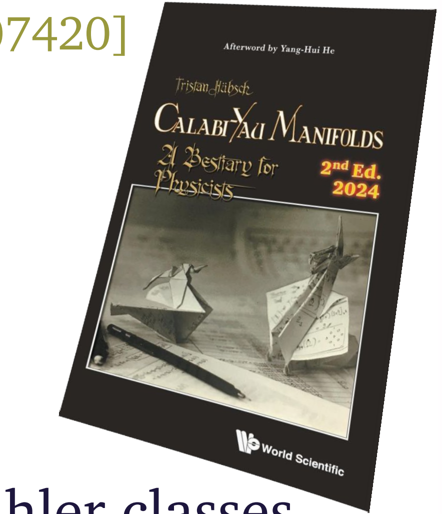




$\mathscr{X} \in A[c_1]$

Counting with multiplicity 😇

[arXiv:1606.07420]



a Decade Ago…

Classical Constructions

- E.g: $X_m \in \left[\boxed{\begin{matrix}\mathbb{P}^4 \\ \mathbb{P}^1\end{matrix}\left\|\begin{matrix}1 \\ m\end{matrix}\right.}\ \begin{matrix}4 \\ 2-m\end{matrix}\right]^{(2,86)}_{-168}$

smooth $_{\mathbb{R}}$ models

$b_2 = 2 = h^{1,1}$ dim. space of Kähler classes

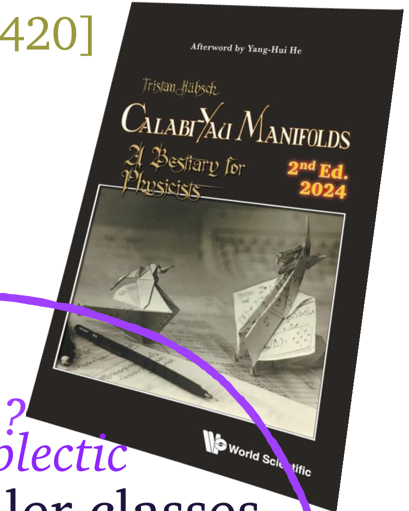
$\frac{1}{2}b_3 - 1 = 86 = h^{2,1}$ dim. space of cpx structures

$-168 = \chi = 2(h^{1,1} - h^{2,1})$ the Euler #

  - Zero-set of $\boxed{p(x,y)=0,}$ $\deg[p] = \binom{1}{m}$ & $q(x,y)=0$, $\deg[q] = \binom{4}{2-m}$
  - Generic $\{p=0\} \cap \{q=0\}$ smooth; $\deg_{\mathbb{P}^n}[p] + \deg_{\mathbb{P}^n}[q] = n+1 \Rightarrow \boxed{c_1 = 0}$

- Sequentially: $X_m \xrightarrow{q=0} \left(F_m^{(4)} \xrightarrow{p=0} \mathbb{P}^4 \times \mathbb{P}^1\right)$

[arXiv:1606.07420]



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smooth$_{\mathbb{R}}$ models

special? symplectic

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  - Then: $C^{(c)}_{4-k}[(aJ_1+bJ_2)^k] = \widetilde{C}_k(\underline{4b+ma})$ = *diffeomorphism* invariants
    - e.g., $\int_{F_m} c_1 c_2[(aJ_1+bJ_2)] = \int_{X_m} c_2[(aJ_1+bJ_2)]$, but also $\int_{F_m} c_2[(aJ_1+bJ_2)^2]$, …
  - So, $F_m^{(4)} \approx_{\mathbb{R}} F^{(4)}_{m\,(\mathrm{mod}\,4)}$ & $X_m \approx_{\mathbb{R}} X_{m\,(\mathrm{mod}\,4)}$: 4 *diffeomorphism types*

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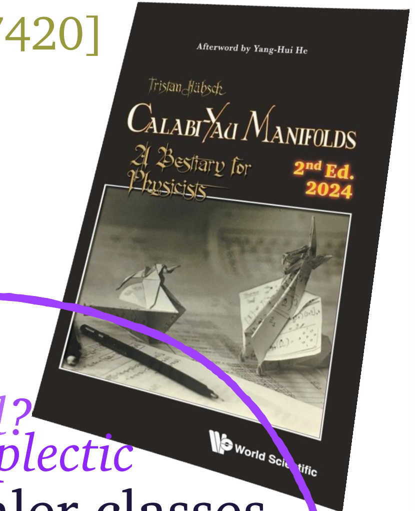

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- …so, $m = 0,\ 1,\ 2, \cdots$

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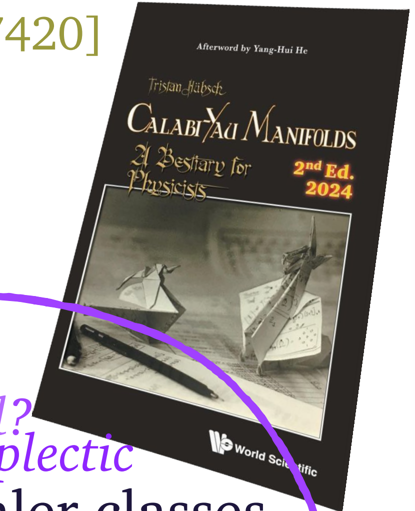

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smooth $_{\mathbb{R}}$ *models* — *special? symplectic*

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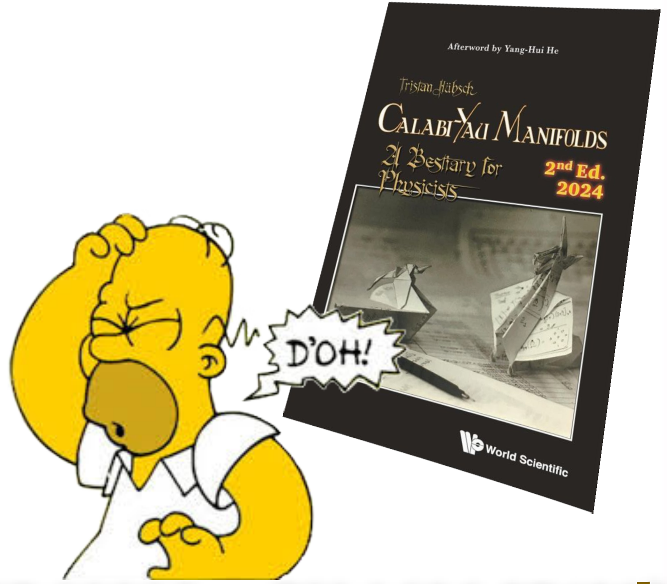
- …so, $m = 0,\ 1,\ 2,\ 3 \ \Rightarrow\ \deg[q] = \binom{4}{-1}$?!

Fractional Fugue

Why Haven't We Thought of This Before?



- $\deg[q] = \binom{4}{-1}$ holomorphic sections?! [AAGGL:1507.03235 + BH:1606.07420]
  - Not *everywhere* on $\mathbb{P}^4 \times \mathbb{P}^1$ — (simple poles)
  - but yes on $F_3^{(4)} \subset \mathbb{P}^4 \times \mathbb{P}^1$ — $\geqslant 105$ of 'em!

$$X_m \in \left[\begin{array}{c||c|c} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & m & 2-m \end{array}\right]^{(2,86)}_{-168} \quad \text{for } m=3$$

- How? On $F_3^{(4)}$, $q(x,y) \simeq q(x,y) + \lambda \cdot p(x,y) \leftarrow$ *equivalence class!*

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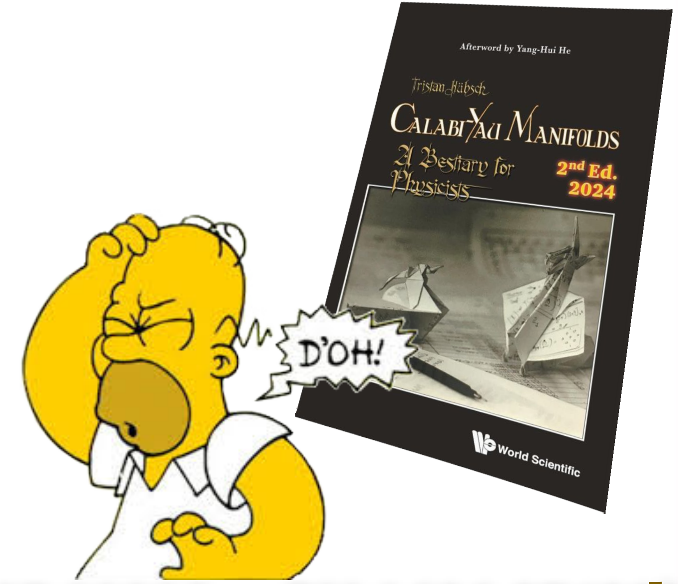

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  - [Hirzebruch, 1951] $\Rightarrow$ $p = x_0 {y_0}^3 + x_1 {y_1}^3$ & $q = c(x)\left(\frac{x_0 y_0}{y_1{}^2} - \frac{x_1 y_1}{y_0{}^2}\right)$ $\deg[c] = \binom{3}{0}$
  - So, $q_0 = \left[q(x,y) + \frac{\lambda\, c(x)}{(y_0 y_1)^2} p(x,y)\right]_{\lambda \to -1} = c(x)\left(-2 \frac{x_1 y_1}{y_0{}^2}\right)$ where $y_0 \neq 0$
  - & $q_1 = \left[q(x,y) + \frac{\lambda\, c(x)}{(y_0 y_1)^2} p(x,y)\right]_{\lambda \to +1} = c(x)\left(2 \frac{x_0 y_0}{y_1{}^2}\right)$ where $y_1 \neq 0$
  - & $q_1(x,y) - q_0(x,y) = 2 \frac{c(x)}{(y_0 y_1)^2} p(x,y) = 0$ on $F_3 := \{p(x,y) = 0\}$

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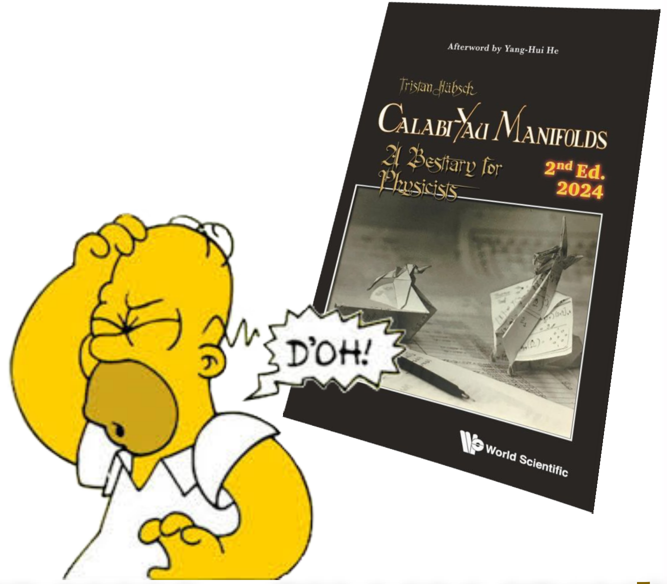

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[+ GvG:1708.00517]

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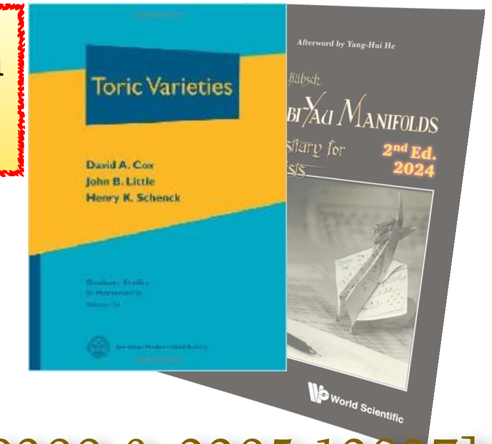
  - & $q_1(x,y) - q_0(x,y) = 2\frac{c(x)}{(y_0 y_1)^2} p(x,y) = 0$ on $F_3 := \{p(x,y) = 0\}$

Wu-Yang monopole

- [GvG, 1708.00517] *scheme-th*. "*generalized* complete intersections"

Reverse-engineered: Mayer-Vietoris sequence & "patching" of the two charts

$p_0{}^{-1}(0) \cap \mathfrak{s}^{-1}(0)$ is smooth
$\mathrm{d}p_0(x,y) \wedge \mathrm{d}\mathfrak{s}(x,y) \neq 0$



Fractional Fugue

...in well-tempered counterpoint

[BH:1606.07420, 1611.10300 & 2205.12827]
+more

- For $\left\{0 \stackrel{!}{=} \boxed{\underbrace{x_0 y_0{}^m + x_1 y_1{}^m}_{:=p_0(x,y)}} + \sum_{a=2}^{n} \sum_{\ell=1}^{m-1} \cancel{\epsilon_{a,\ell}}\, 0\, x_a\, y_0{}^{m-\ell} y_1{}^{\ell} \right\} = F^{(n)}_{m;\cancel{\epsilon}\,0} \in \left[\begin{smallmatrix}\mathbb{P}^n \\ \mathbb{P}^1\end{smallmatrix} \middle\| \begin{smallmatrix}1 \\ m\end{smallmatrix}\right]$

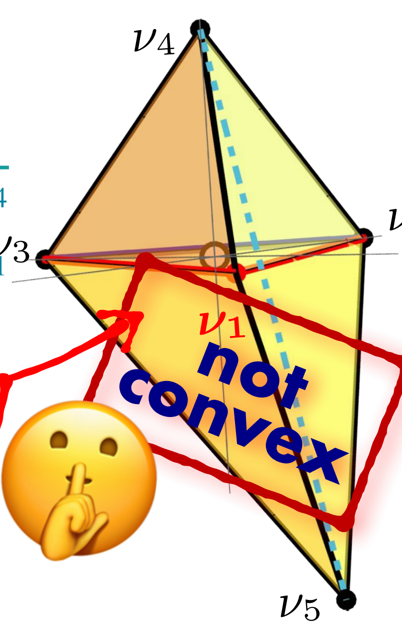
  even $p_0(x,y)$ is transverse, $p_0^{-1}(0)$ is smooth

  - The central ($\epsilon = 0$) member of the family is a Hirzebruch scroll F_m :
  - Directrix: $\boxed{\mathfrak{s}(x,y) := \left(\frac{x_0}{y_1{}^m} - \frac{x_1}{y_0{}^m}\right) + \frac{\lambda}{(y_0 y_1)^m}[x_0 y_0{}^m + x_1 y_1{}^m]}$ degree $\binom{1}{-m}$

- On $F^{(n)}_m$: $p_0(x,y) = x_0 y_0{}^m + x_1 y_1{}^m = 0 \Rightarrow x_0 = -x_1 (y_1/y_0)^m$, @ $y_0 \neq 0$

  - So, $x_1 \to X_1 = \mathfrak{s}(x,y)$ & $(X_i,\ i=2,\cdots) = (x_2, \cdots, x_n; y_0, y_1)$
  - Key: $\det\left[\frac{\partial(p_0(x,y), \mathfrak{s}(x,y), x_2, \cdots; y_0, y_1)}{\partial(x_0, x_1, x_2, \cdots; y_0, y_1)}\right] = \text{const.}$
  - $\mathbb{P}^4 \times \mathbb{P}^1$ bi-degree $\longrightarrow$ toric $(\mathbb{C}^\times)^2$-action:

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow \mathbb{P}^4$
$-m$	0	0	0	1	1	$\leftarrow \mathbb{P}^1$



REM*

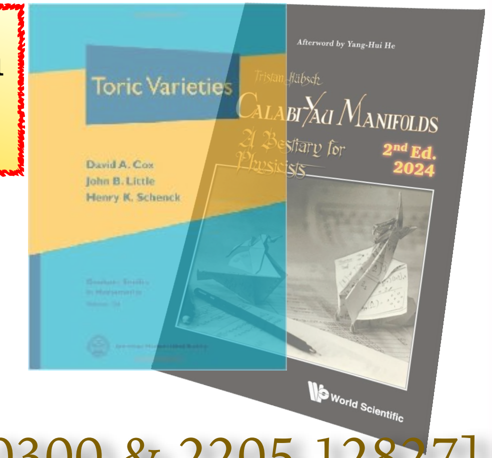
- *What about the rest of the $\epsilon_{a,\ell}$-deformation family?*
  (wherein smooth models are all diffeomorphic to each other)

*Reverse-Engineered (Toric) Model

Fractional Fugue

…with a meandering motif

$p_0{}^{-1}(0) \cap \mathfrak{s}^{-1}(0)$ **is smooth**
$\mathrm{d}p_0(x,y) \wedge \mathrm{d}\mathfrak{s}(x,y) \neq 0$



[BH:1606.07420, 1611.10300 & 2205.12827]
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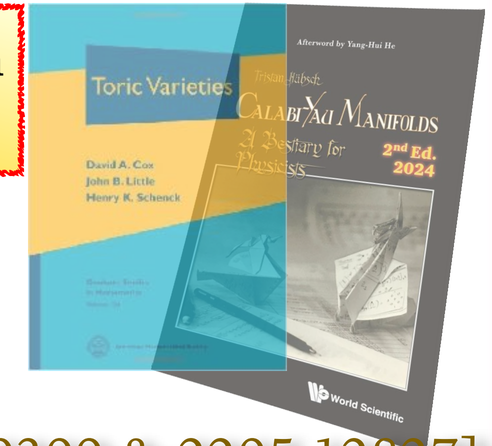
- **Construction 2.1** *Given a degree-*$\binom{1}{m}$ *hypersurface* $\{p_{\vec\epsilon}(x,y)\,0\} \subset \mathbb{P}^n \times \mathbb{P}^1$ *as in* (2.2), *construct*

$$\deg = \binom{1}{m-r_0-r_1}: \quad \mathfrak{s}_{\vec\epsilon}(x,y;\lambda) := \operatorname*{Flip}_{y_0}\left[\frac{1}{y_0{}^{r_0}\,y_1{}^{r_1}}\,p_{\vec\epsilon}(x,y)\right] \pmod{p_{\vec\epsilon}(x,y)}, \qquad \left[\begin{array}{c||c} \mathbb{P}^n & 1 \\ \mathbb{P}^1 & m \end{array}\right]$$

progressively decreasing $r_0+r_1=2m, 2m-1, \cdots$, *and keeping only Laurent polynomials containing both* y_0*- and* y_1*-denominators but no* y_0, y_1*-mixed ones. The "*Flip_{y_i}*" operator changes the relative sign of the rational monomials with* y_i*-denominators. For algebraically independent such sections, restrict to a subset with maximally negative degrees that are not overall* (y_0, y_1)*-multiples of each other.*

Fractional Fugue



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Construction 2.1 *Given a degree-$\binom{1}{m}$ hypersurface $\{p_{\vec{\epsilon}}(x,y)\,0\} \subset \mathbb{P}^n \times \mathbb{P}^1$ as in (2.2), construct*

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$m=2$

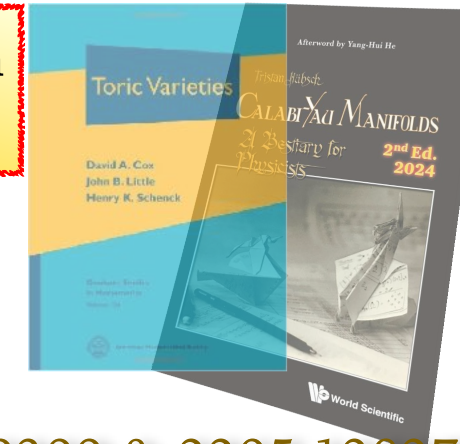
E.g.: `p0 =` $x_0 y_0^2 + x_1 y_1^2$`; ep[α_] := Table[`$\frac{1}{y_0^{\alpha-i}\, y_1^{i}}$`, {i, 0, α}]; Expand /@ (p0 {ep[5], ep[4], ep[3]})`

① $$\left\{\left\{\frac{x_0}{y_0^3} + \frac{x_1 y_1^2}{y_0^5}, \frac{x_0}{y_0^2 y_1} + \frac{x_1 y_1}{y_0^4}, \frac{x_1}{y_0^3} + \frac{x_0}{y_0 y_1^2}, \frac{x_0}{y_1^3} + \frac{x_1}{y_0^2 y_1}, \frac{x_0 y_0}{y_1^4} + \frac{x_1}{y_0 y_1^2}, \frac{x_0 y_0^2}{y_1^5} + \frac{x_1}{y_1^3}\right\},\right.$$

② $$\left\{\frac{x_0}{y_0^2} + \frac{x_1 y_1^2}{y_0^4}, \frac{x_0}{y_0 y_1} + \frac{x_1 y_1}{y_0^3}, \boxed{\frac{x_1}{y_0^2} + \frac{x_0}{y_1^2}}, \frac{x_0 y_0}{y_1^3} + \frac{x_1}{y_0 y_1}, \frac{x_0 y_0^2}{y_1^4} + \frac{x_1}{y_1^2}\right\}, \left.\left\{\frac{x_0}{y_0} + \frac{x_1 y_1^2}{y_0^3}, \frac{x_0}{y_1} + \frac{x_1 y_1}{y_0^2}, \frac{x_1}{y_0} + \frac{x_0 y_0}{y_1^2}, \frac{x_0 y_0^2}{y_1^3} + \frac{x_1}{y_1}\right\}\right\}$$

Fractional Fugue

$\boxed{\begin{array}{l} p_0{}^{-1}(0)\cap\mathfrak{s}^{-1}(0)\ \ \textbf{is smooth} \\ \mathrm{d}p_0(x,y)\wedge\mathrm{d}\mathfrak{s}(x,y)\neq 0 \end{array}}$



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+more

Construction 2.1 *Given a degree-$\binom{1}{m}$ hypersurface $\{p_{\vec\epsilon}(x,y)\,0\}\subset\mathbb{P}^n\times\mathbb{P}^1$ as in (2.2), construct*

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progressively decreasing $r_0+r_1=2m,2m{-}1,\cdots,$ and keeping only Laurent polynomials con-
② *taining* $\boxed{\textit{both } y_0\textit{- and } y_1\textit{-denominators but no } y_0,y_1\textit{-mixed ones.}}$ *The "Flip_{y_i}" operator changes the relative sign of the rational monomials with y_i-denominators.* ③ $\boxed{\textit{For algebraically independent such sections, restrict to a subset with maximally negative degrees that are not overall } (y_0,y_1)\textit{-multiples of each other.}}$

E.g.: $m=2$

```
p0 = x₀ y₀² + x₁ y₁²; ep[α_] := Table[1/(y₀^(α-i) y₁^i), {i, 0, α}]; Expand /@ (p0 {ep[5], ep[4], ep[3]})
```

① $$\Big\{\Big\{\cancel{\frac{x_0}{y_0^3}+\frac{x_1y_1^2}{y_0^5}},\ \cancel{\frac{x_0}{y_0^2y_1}}+\frac{x_1y_1}{y_0^4},\ \frac{x_1}{y_0^3}+\cancel{\frac{x_0}{y_0y_1^2}},\ \frac{x_0}{y_1^3}+\cancel{\frac{x_1}{y_0^2y_1}},\ \frac{x_0y_0}{y_1^4}+\cancel{\frac{x_1}{y_0y_1^2}},\ \cancel{\frac{x_0y_0^2}{y_1^5}+\frac{x_1}{y_1^3}}\Big\},\quad \boxed{\cdot\,y_1,\quad\cdot\,y_0}\quad ③$$

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finds $\mathfrak{s}(x,y)=\left(\frac{x_0}{y_1{}^2}-\frac{x_1}{y_0{}^2}\right)\ \mathrm{mod}\left(x_0y_0{}^2+x_1y_1{}^2\right)$; $\deg=\binom{1}{-2}$, $[\mathfrak{s}^{-1}(0)]=[J_1]-2[J_2]$.

Fractional Fugue

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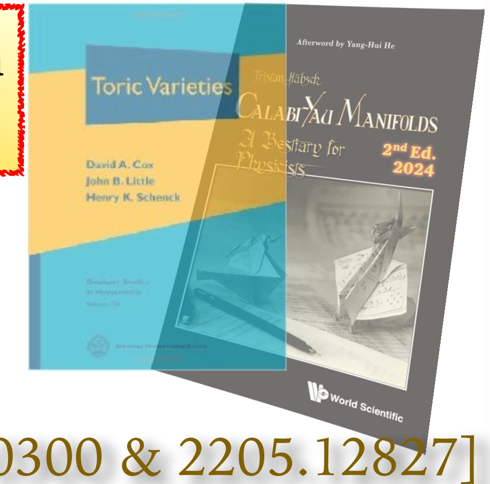

[BH:1606.07420, 1611.10300 & 2205.12827]
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```
p0 = x0 y0^2 + x1 y1^2; ep[α_] := Table[1/(y0^(α-i) y1^i), {i, 0, α}]; Expand /@ (p0 {ep[5], ep[4], ep[3]})
```

①
$$\Big\{\Big\{\cancel{\tfrac{x_0}{y_0^3}+\tfrac{x_1y_1^2}{y_0^5}},\ \cancel{\tfrac{x_0}{y_0^2y_1}}+\tfrac{x_1y_1}{y_0^4},\ \tfrac{x_1}{y_0^3}+\cancel{\tfrac{x_0}{y_0y_1^2}},\ \tfrac{x_0}{y_1^3}+\cancel{\tfrac{x_1}{y_0^2y_1}},\ \tfrac{x_0y_0}{y_1^4}+\cancel{\tfrac{x_1}{y_0y_1^2}},\ \cancel{\tfrac{x_0y_0^2}{y_1^5}+\tfrac{x_1}{y_1^3}}\Big\},\ \boxed{\cdot\,y_1,\quad\cdot\,y_0}$$

③

②
$$\Big\{\cancel{\tfrac{x_0}{y_0^2}+\tfrac{x_1y_1^2}{y_0^4}},\ \cancel{\tfrac{x_0}{y_0y_1}}+\tfrac{x_1y_1}{y_0^3},\ \boxed{\tfrac{x_1}{y_0^2}+\tfrac{x_0}{y_1^2}},\ \tfrac{x_0y_0}{y_1^3}+\cancel{\tfrac{x_1}{y_0y_1}},\ \cancel{\tfrac{x_0y_0^2}{y_1^4}+\tfrac{x_1}{y_1^2}}\Big\},\ \Big\{\cancel{\tfrac{x_0}{y_0}+\tfrac{x_1y_1^2}{y_0^3}},\ \boxed{\tfrac{x_0}{y_1}+\tfrac{x_1y_1}{y_0^2}},\ \boxed{\tfrac{x_1}{y_0}+\tfrac{x_0y_0}{y_1^2}},\ \cancel{\tfrac{x_0y_0^2}{y_1^3}+\tfrac{x_1}{y_1}}\Big\}\Big\}$$

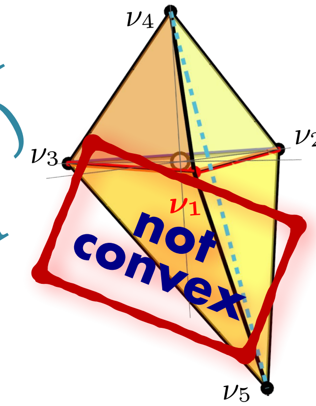
- finds $\boxed{\mathfrak{s}(x,y)=\Big(\frac{x_0}{y_1{}^2}-\frac{x_1}{y_0{}^2}\Big)\ \mathrm{mod}\big(x_0y_0{}^2+x_1y_1{}^2\big)}$; $\deg=\binom{1}{-2}$, $[\mathfrak{s}^{-1}(0)]=[J_1]-2[J_2]$.

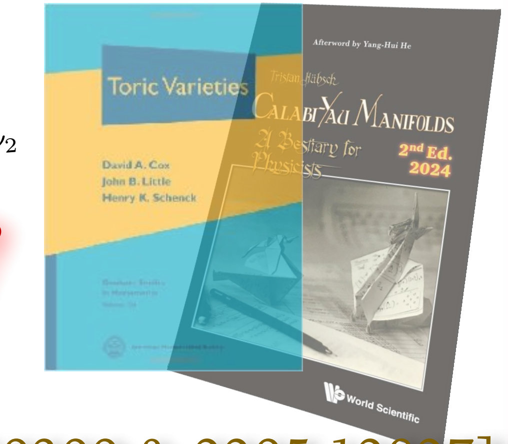
THE **exceptional section**

THE **exceptional curve $[S]^2=-1$ in $F_2^{(2)}$**

Fractional Fugue

central $(\epsilon_{a,\ell}=0)$ model



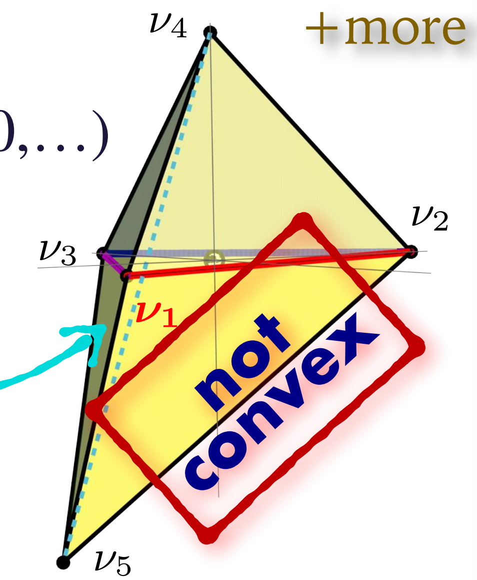


…in well-tempered counterpoint

[BH:1606.07420, 1611.10300 & 2205.12827] +more

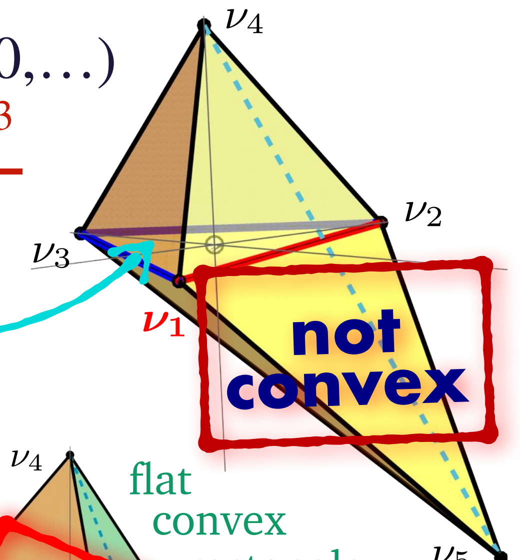
- Deform: $p_1(x,y)=x_0y_0{}^5+x_1y_1{}^5+x_2y_0y_1{}^4$ toric $F^{(n)}_{(4,1,0,\dots)}$
  - Now: $\mathfrak{s}_{1,1}(x,y)=\frac{x_0y_0}{y_1{}^5}+\frac{x_2}{y_1{}^4}-\frac{x_1}{y_1{}^4}$ & $\mathfrak{s}_{1,2}(x,y)=\frac{x_0}{y_1}-\frac{x_2}{y_0}-\frac{x_1y_1{}^4}{y_0{}^5}$
  - & $\det\left[\frac{\partial(p_1,\mathfrak{s}_{1,1},\mathfrak{s}_{1,2},x_3,\cdots;y_0,y_1)}{\partial(x_0,x_1,x_2,x_3,\cdots;y_0,y_1)}\right]=\text{const.}$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow\mathbb{P}^4$
-4	-1	0	0	1	1	$\leftarrow\mathbb{P}^1$



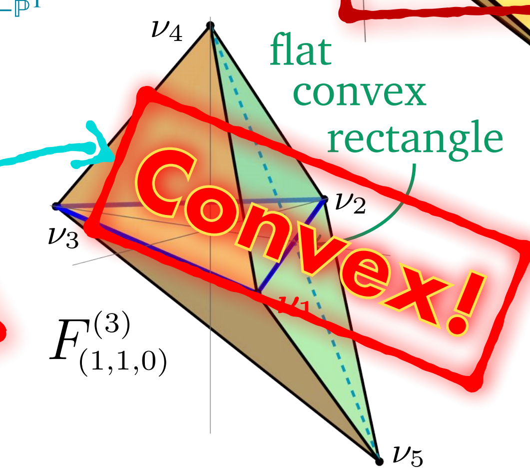
- Deform: $p_2(x,y)=x_0y_0{}^5+x_1y_1{}^5+x_2y_0{}^2y_1{}^3$ toric $F^{(n)}_{(3,2,0,\dots)}$
  - Now: $\mathfrak{s}_{2,1}(x,y)=\frac{x_0y_0{}^2}{y_1{}^5}+\frac{x_2}{y_1{}^3}-\frac{x_1}{y_1{}^3}$ & $\mathfrak{s}_{2,2}(x,y)=\frac{x_0}{y_1{}^2}-\frac{x_2}{y_0{}^2}-\frac{x_1y_1{}^3}{y_0{}^5}$
  - & $\det\left[\frac{\partial(p_2,\mathfrak{s}_{2,1},\mathfrak{s}_{2,2},x_3,\cdots;y_0,y_1)}{\partial(x_0,x_1,x_2,x_3,\cdots;y_0,y_1)}\right]=\text{const.}$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow\mathbb{P}^4$
-3	-2	0	0	1	1	$\leftarrow\mathbb{P}^1$



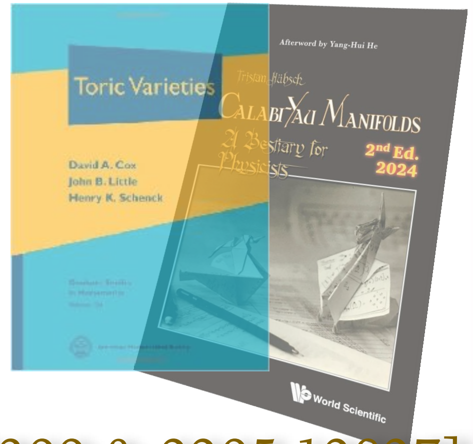
- … and $p_3(x,y)=x_0y_0{}^5+x_1y_1{}^5+x_2y_0{}^2y_1{}^3+x_3y_0{}^3y_1{}^2$
  - $\to$ toric $F^{(n)}_{(2,2,1,\cdots)}$ for $n=3$, $\boxed{F^{(3)}_{(2,2,1)}\approx F^{(3)}_{(1,1,0)}}$ [~Veronese]

Fano!



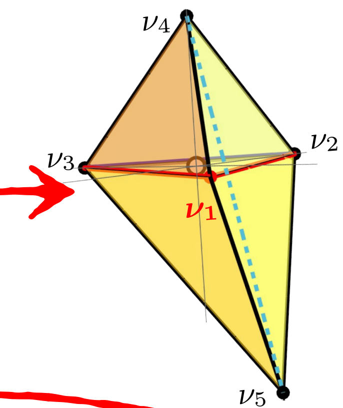
Fractional Fugue

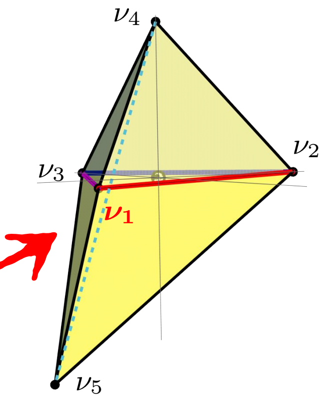
…in well-tempered counterpoint

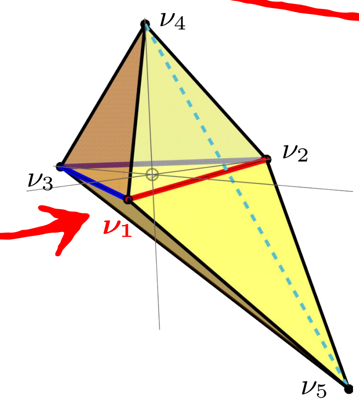


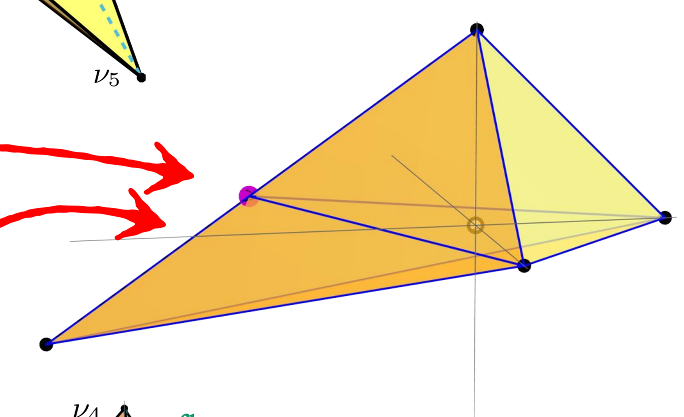
[BH:1606.07420, 1611.10300 & 2205.12827]
2605.06998+more

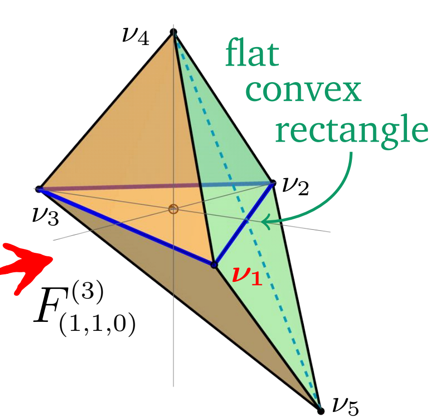
Deg.	Biprojective Defining Section	Name, GLSM Charges
	$p_0(x,y) = x_0\,y_0^5 + x_1\,y_1^5$	$F_5^{(3)}$
$\binom{1}{-5}$:	$\mathfrak{s}_0(x,y) = \Big[\Big(\frac{x_0}{y_1^5} - \frac{x_1}{y_0^5}\Big)\Big/p_0(x,y)\Big]$	$\begin{array}{ccccc} X_1 & X_2 & X_3 & X_4 & X_5 \\ \hline 1 & 1 & 1 & 0 & 0 \\ -5 & 0 & 0 & 1 & 1 \end{array}$
	$p_1(x,y) = p_0(x,y) + x_2\,y_0^4 y_1$	$F_{4,1}^{(3)}$
$\binom{1}{-2}$:	$\mathfrak{s}_{1a}(x,y) = \Big[\Big(\frac{x_0\,y_0}{y_1^5} - \frac{x_1}{y_0^4} + \frac{x_2}{y_1^4}\Big)\Big/p_1(x,y)\Big]$	$\begin{array}{ccccc} X_1 & X_2 & X_3 & X_4 & X_5 \\ \hline 1 & 1 & 1 & 0 & 0 \\ -4 & -1 & 0 & 1 & 1 \end{array}$
$\binom{1}{-2}$:	$\mathfrak{s}_{1b}(x,y) = \Big[\Big(\frac{x_0}{y_1} - \frac{x_1\,y_1^4}{y_0^5} - \frac{x_2}{y_0}\Big)\Big/p_1(x,y)\Big]$	
	$p_2(x,y) = p_0(x,y) + x_2\,y_0^3\,y_1^2$	$F_{3,2}^{(3)}$
$\binom{1}{-3}$:	$\mathfrak{s}_{2a}(x,y) = \Big[\Big(\frac{x_0\,y_0^2}{y_1^5} - \frac{x_1}{y_0^3} + \frac{x_2 y_0}{y_1^4} + \frac{x_3}{y_1^3}\Big)\Big/p_3(x,y)\Big]$	$\begin{array}{ccccc} X_1 & X_2 & X_3 & X_4 & X_5 \\ \hline 1 & 1 & 1 & 0 & 0 \\ -3 & -2 & 0 & 1 & 1 \end{array}$
$\binom{1}{-1}$:	$\mathfrak{s}_{2b}(x,y) = \Big[\Big(\frac{x_0}{y_1^2} - \frac{x_1 y_1^3}{y_0^5} - \frac{x_2}{y_0^2}\Big)\Big/p_2(x,y)\Big]$	
	$p_3(x,y) = p_0(x,y) + x_2 y_0^4 y_1^1 + x_3 y_0^1 y_1^4$	$F_{3,1,1a}^{(3)} \approx_{\mathbb{R}} F_2^{(3)}$
$\binom{1}{-3}$:	$\mathfrak{s}_{3a}(x,y) = \Big[\Big(\frac{x_0 y_0}{y_1^4} - \frac{x_1 y_1}{y_0^4} + \frac{x_2}{y_1^3} - \frac{x_3}{y_0^3}\Big)\Big/p_3(x,y)\Big]$	
$\binom{1}{-1}$:	$\mathfrak{s}_{3b}(x,y) = \Big[\Big(\frac{x_0}{y_1} - \frac{x_1 y_1^4}{y_0^5} - \frac{x_2}{y_0} - \frac{x_3 y_1^3}{y_0^4}\Big)\Big/p_3(x,y)\Big]$	$\begin{array}{ccccc} X_1 & X_2 & X_3 & X_4 & X_5 \\ \hline 1 & 1 & 1 & 0 & 0 \\ -3 & -1 & -1 & 1 & 1 \end{array}$
$\binom{1}{-1}$:	$\mathfrak{s}_{3c}(x,y) = \Big[\Big(\frac{x_0 y_0^4}{y_1^5} - \frac{x_1}{y_0} + \frac{x_2 y_0^3}{y_1^4} + \frac{x_3}{y_1}\Big)\Big/p_3(x,y)\Big]$	
	$p_4(x,y) = p_0(x,y) + x_2\,y_0^4 y_1 + x_3\,y_0^3\,y_1^2$	$F_{3,1,1b}^{(3)} \approx_{\mathbb{R}} F_2^{(3)}$
$\binom{1}{-3}$:	$\mathfrak{s}_{4a}(x,y) = \Big[\Big(\frac{x_0\,y_0^2}{y_1^5} - \frac{x_1}{y_0^3} + \frac{x_2 y_0}{y_1^4} + \frac{x_3}{y_1^3}\Big)\Big/p_4(x,y)\Big]$	$\begin{array}{ccccc} X_1 & X_2 & X_3 & X_4 & X_5 \\ \hline 1 & 1 & 1 & 0 & 0 \\ -3 & -1 & -1 & 1 & 1 \end{array}$
$\binom{1}{-1}$:	$\mathfrak{s}_{4b}(x,y) = \Big[\Big(\frac{x_0}{y_1} - \frac{x_1 y_1^4}{y_0^5} - \frac{x_2}{y_0} - \frac{x_3 y_1}{y_0^2}\Big)\Big/p_4(x,y)\Big]$	
$\binom{1}{-1}$:	$\mathfrak{s}_{4c}(x,y) = \Big[\Big(\frac{x_0 y_0}{y_1^2} - \frac{x_1 y_1^3}{y_0^4} + \frac{x_2}{y_1} - \frac{x_3}{y_0}\Big)\Big/p_4(x,y)\Big]$	equivalent to $p_3(x,y)$
	$p_5(x,y) = p_0(x,y) + x_2 y_0^3 y_1^2 + x_3 y_0^2 y_1^3$	$F_{2,2,1}^{(3)} \approx_{\mathbb{R}} F_{1,1}^{(3)}$
$\binom{1}{-2}$:	$\mathfrak{s}_{5a}(x,y) = \Big[\Big(\frac{x_0}{y_1^2} - \frac{x_1 y_1^3}{y_0^5} - \frac{x_2}{y_0^2} - \frac{x_3 y_1}{y_0^3}\Big)\Big/p_5(x,y)\Big]$	
$\binom{1}{-2}$:	$\mathfrak{s}_{5b}(x,y) = \Big[\Big(\frac{x_0 y_0^3}{y_1^5} - \frac{x_1}{y_0^2} + \frac{x_2 y_0}{y_1^3} + \frac{x_3}{y_1^2}\Big)\Big/p_5(x,y)\Big]$	$\begin{array}{ccccc} X_1 & X_2 & X_3 & X_4 & X_5 \\ \hline 1 & 1 & 1 & 0 & 0 \\ -2 & -2 & -1 & 1 & 1 \end{array}$
$\binom{1}{-1}$:	$\mathfrak{s}_{5c}(x,y) = \Big[\Big(\frac{x_0 y_0^2}{y_1^3} - \frac{x_1 y_1^2}{y_0^3} + \frac{x_2}{y_1} - \frac{x_3}{y_0}\Big)\Big/p_5(x,y)\Big]$	

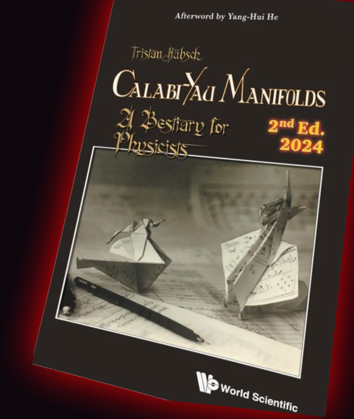










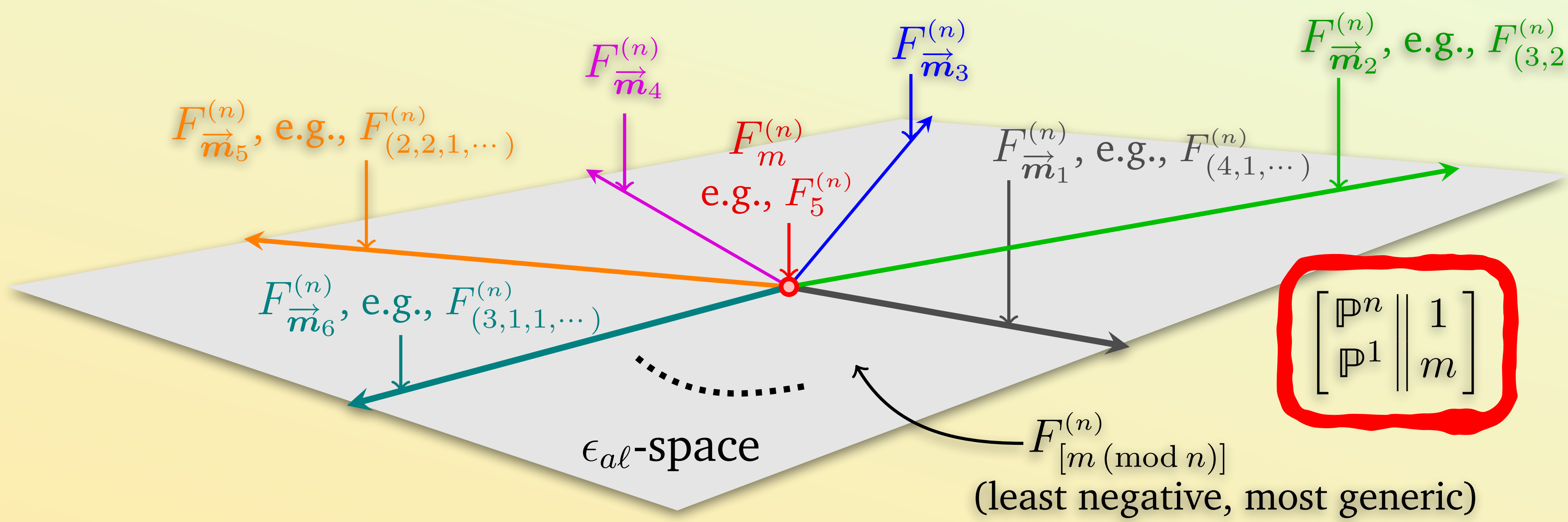


Fractional Fugue

…in well-tempered counterpoint

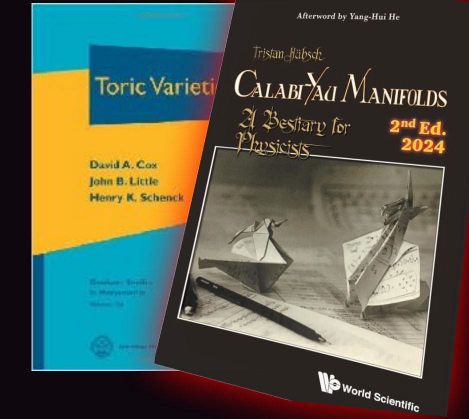
[BH:1606.07420, 1611.10300 & 2205.12827] +more

- A deformation *family picture*:



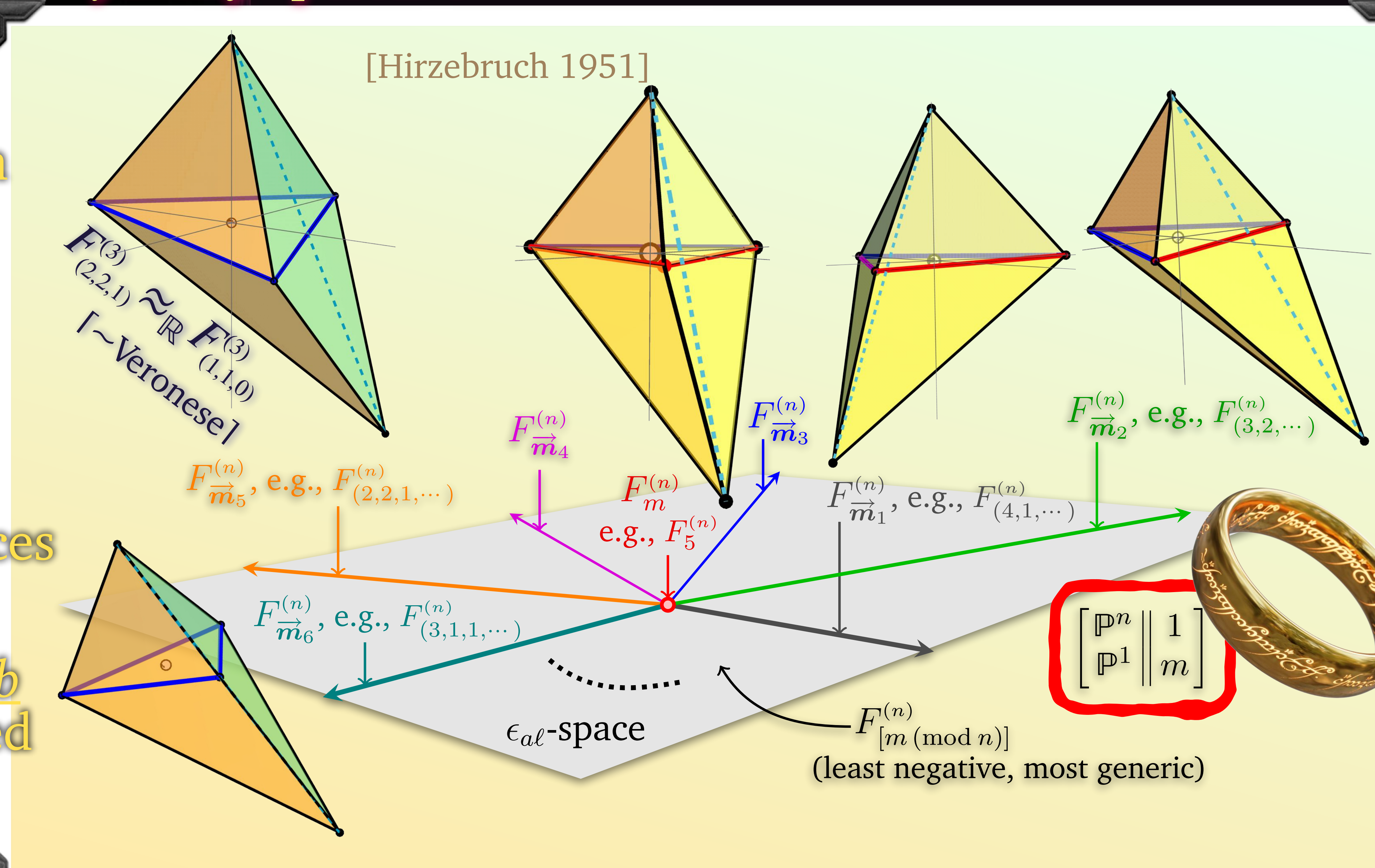
Fractional Fugue

…in well-tempered counterpoint



[BH:1606.07420, 1611.10300 & 2205.12827]
+more

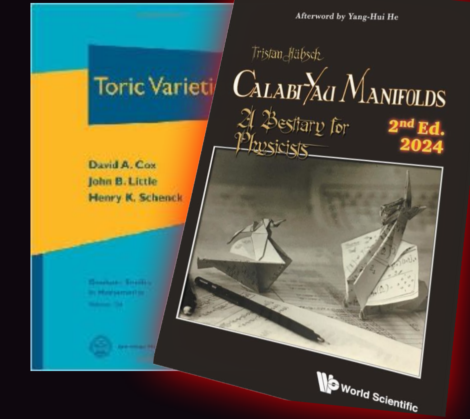
- A deformation *family picture*:



- One complete intersection
- to "rule them all" (toric varieties)
- …and CY hypersurfaces therein
- …and a <u>web</u> of connected GLSMs

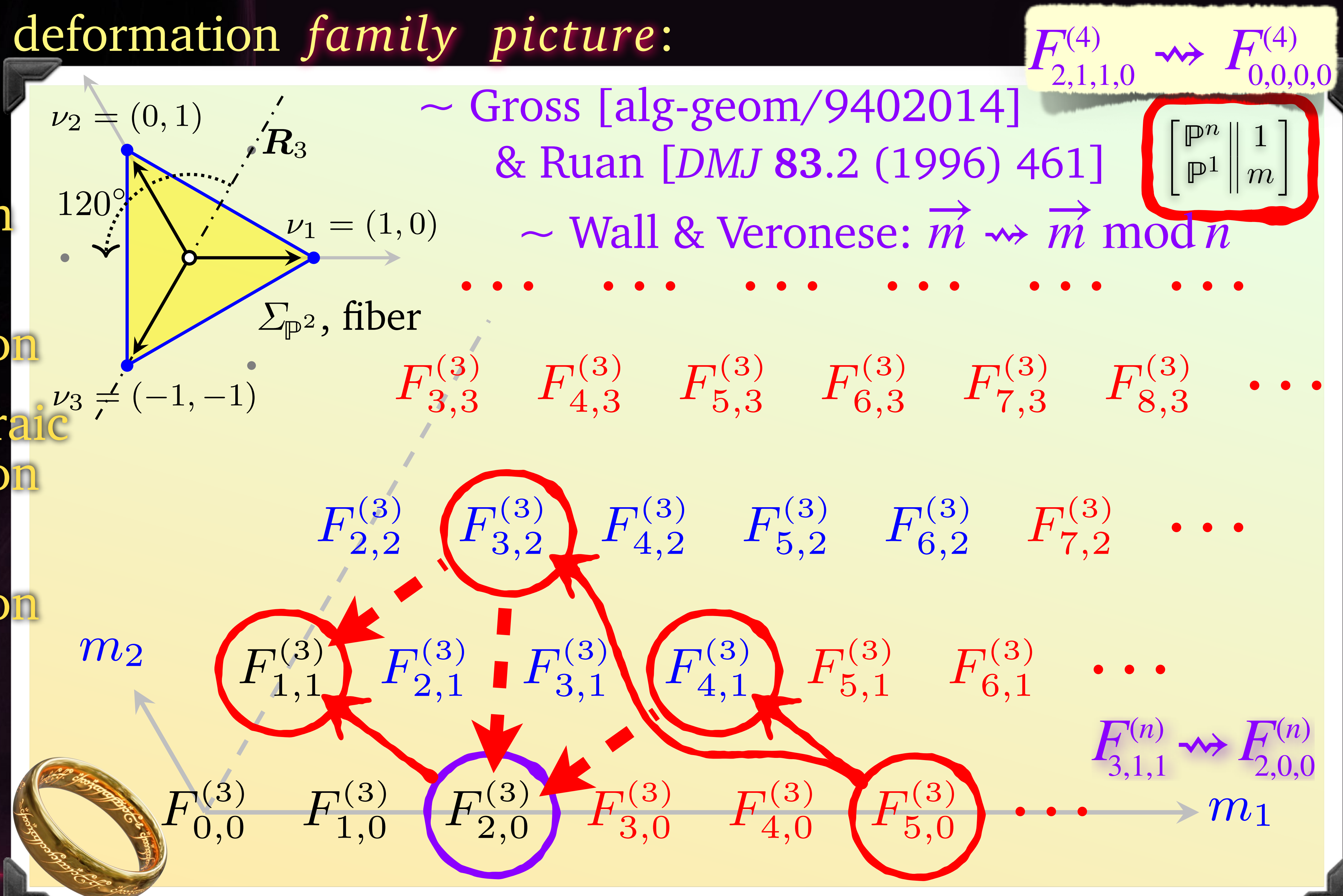
Fractional Fugue

…in well-tempered counterpoint



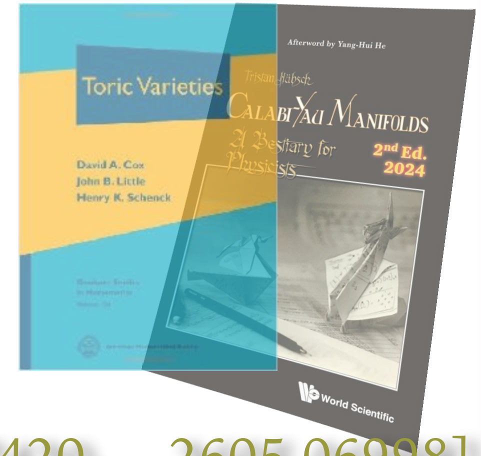
[BH:1606.07420,… 2605.06998]

- A dihedral deformation *family picture*:



- 1951, Hirzebruch
- explicit deformation
- non-algebraic deformation
- additional deformation

Meromorphic March



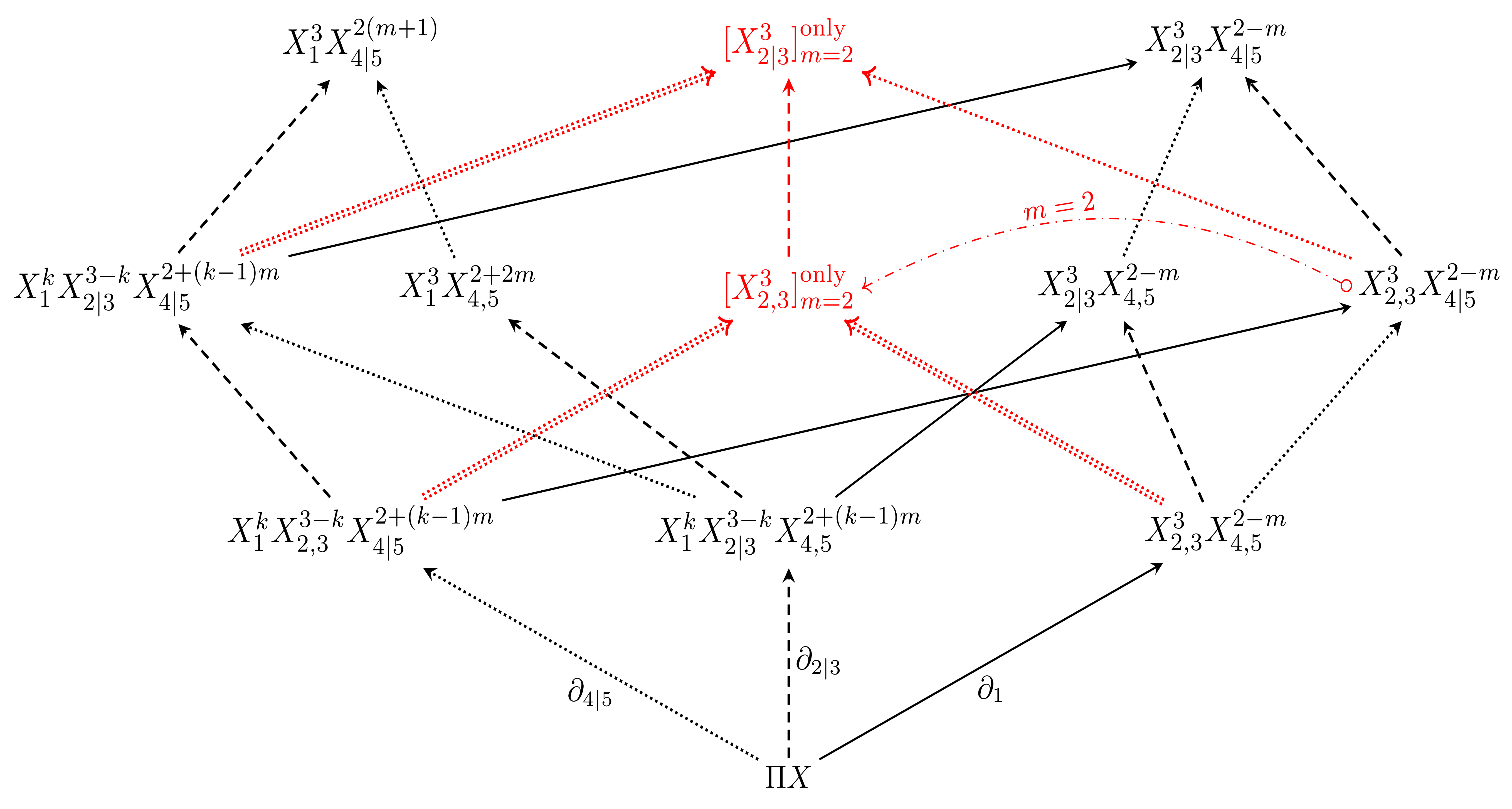
…back to the main motif

[BH:1606.07420,… 2605.06998]

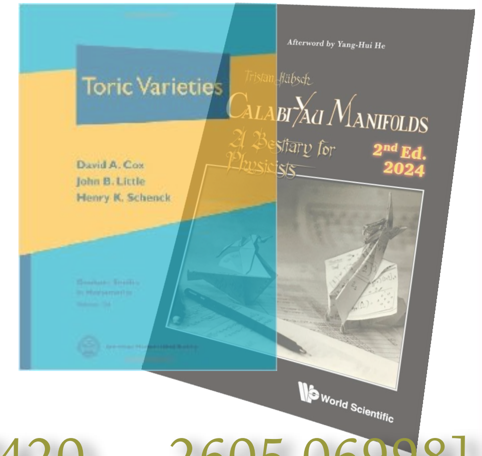
- CY: $F_m^{(n)}[c_1]$, i.e., $F_m^{(n)} \in \left[\begin{matrix}\mathbb{P}^n \\ \mathbb{P}^1\end{matrix}\middle\| \begin{matrix}1 \\ m\end{matrix} \middle| \begin{matrix}n \\ 2-m\end{matrix}\right]$, i.e., need $\deg = \binom{n}{2-m}$
- $f_0(X) = \Pi X := (X_1 = \mathfrak{s}(x,y)) \cdots (X_n = x_n)(X_{n+1} = y_0)(X_{n+2} = y_1)$
  - is not transverse & $\{\Pi X = 0\}$ is singular ("$(n+1)$-horon")
- Deform: $f_0(X) = \Pi X \rightsquigarrow \Pi X + \epsilon^i \delta_i \Pi X$, $\delta_i \Pi X := \mathfrak{m}_i(X)(\partial_i \Pi X)$ where $\deg[\mathfrak{m}_i(X)] = \deg[X_i]$ & $\partial_i \mathfrak{m}_i(X) = 0$
  - in QM, $|n\rangle^{(1)} = \Sigma_{m \neq n}\, c_m |m\rangle^{(0)}$; also, $\langle X_i \rangle = 0 \not\Rightarrow \big\langle \delta_i \Pi X \big\rangle = 0$
  - $(\delta_i \Pi X) \cap (\delta_j \Pi X)$ are X_i- and X_j-independent and delimit $[\delta_i \Pi X]$ & $[\delta_j \Pi X]$ so $\langle X_i \rangle, \langle X_j \rangle = 0 \not\Rightarrow \big\langle [\delta_i \Pi X] \cap [\delta_j \Pi X] \big\rangle = 0$; etc.

they form a poset

Meromorphic March



Meromorphic March



…back to the main motif 

[BH:1606.07420,… 2605.06998]

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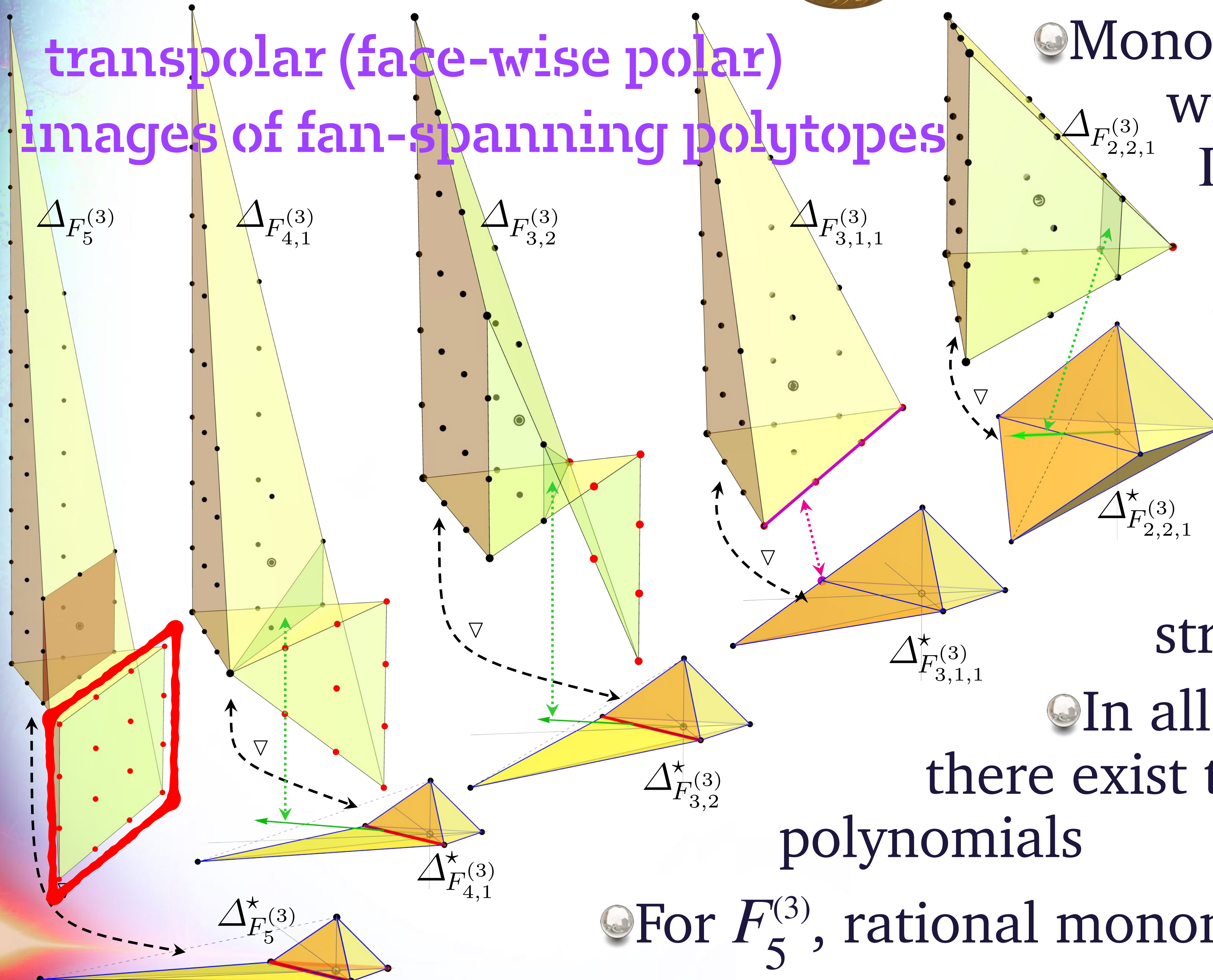
    they form a poset
- Combinatorially, $\Pi X \sqcup \coprod_i [\delta_i \Pi X] \overset{\nabla}{\simeq} \{\Pi X = 0\}$
- Also, $\partial_i \perp \delta_i \Pi X$ & generate a dual poset

Meromorphic March

...back to the main motif 

[BH:1606.07420,... 2605.06998]

transpolar (face-wise polar) images of fan-spanning polytopes



- Monomial collections with $U(1)^2$-charges of ΠX form posets, with "fronts" = facets
- The shape changes in the ϵ_{ik}-space indicating that convexity is not strictly necessary
- In all deformed models, there exist transverse regular polynomials
- For $F_5^{(3)}$, rational monomials are needed

Mirror Minuet

1611.10300 & 2205.12827
2403.07139 & more

& Non-Convex Mirrors $m=3$ —2D *Proof-of-Concept*—

- $X_1{}^2 X_2{}^0 (X_3 \oplus X_4)^{2+1m}$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow \mathbb{P}^4$
$-m$	0	0	0	1	1	$\leftarrow \mathbb{P}^1$

Mirror Minuet

1611.10300 & 2205.12827
2403.07139 & more

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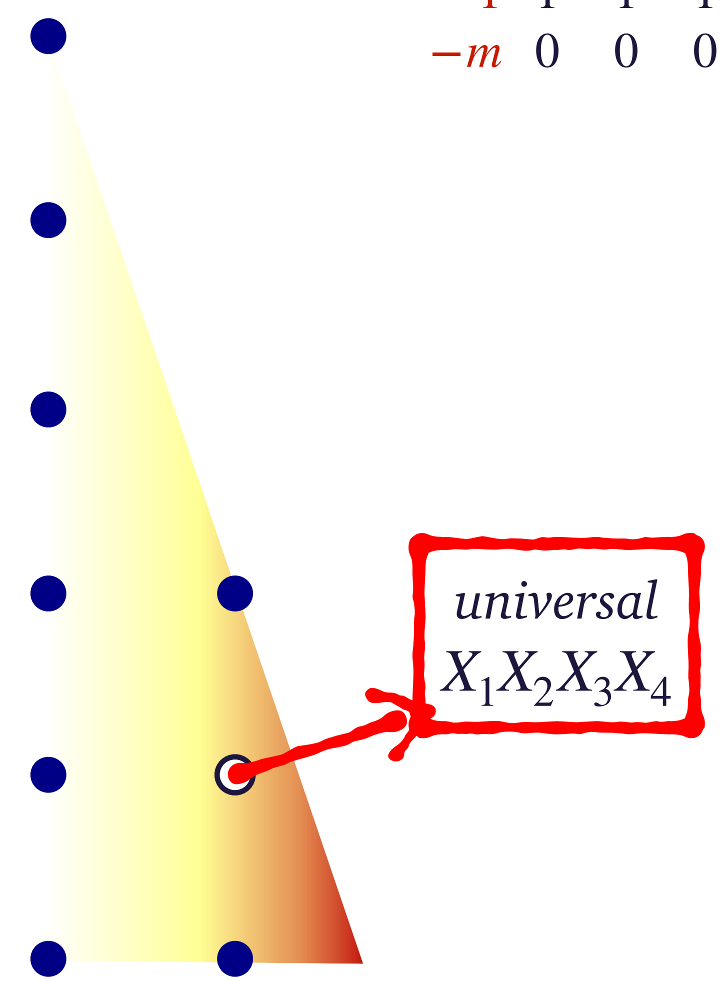
- $X_1{}^2 X_2{}^0 (X_3 \oplus X_4)^{2+1m} \oplus X_1{}^1 X_2{}^1 (X_3 \oplus X_4)^{2+0m}$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow\mathbb{P}^4$
$-m$	0	0	0	1	1	$\leftarrow\mathbb{P}^1$



Mirror Minuet

1611.10300 & 2205.12827
2403.07139 & more

& Non-Convex Mirrors

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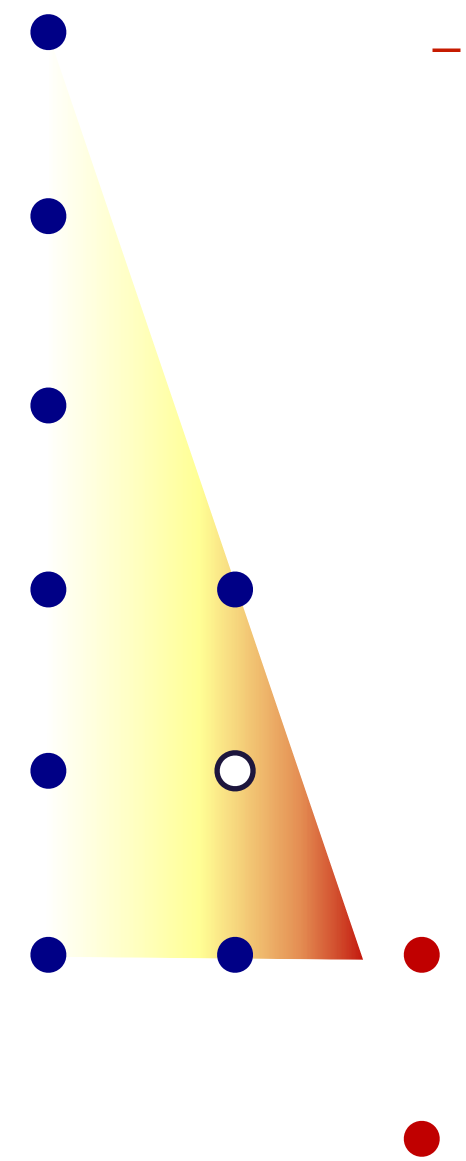

Mirror Minuet

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2403.07139 & more

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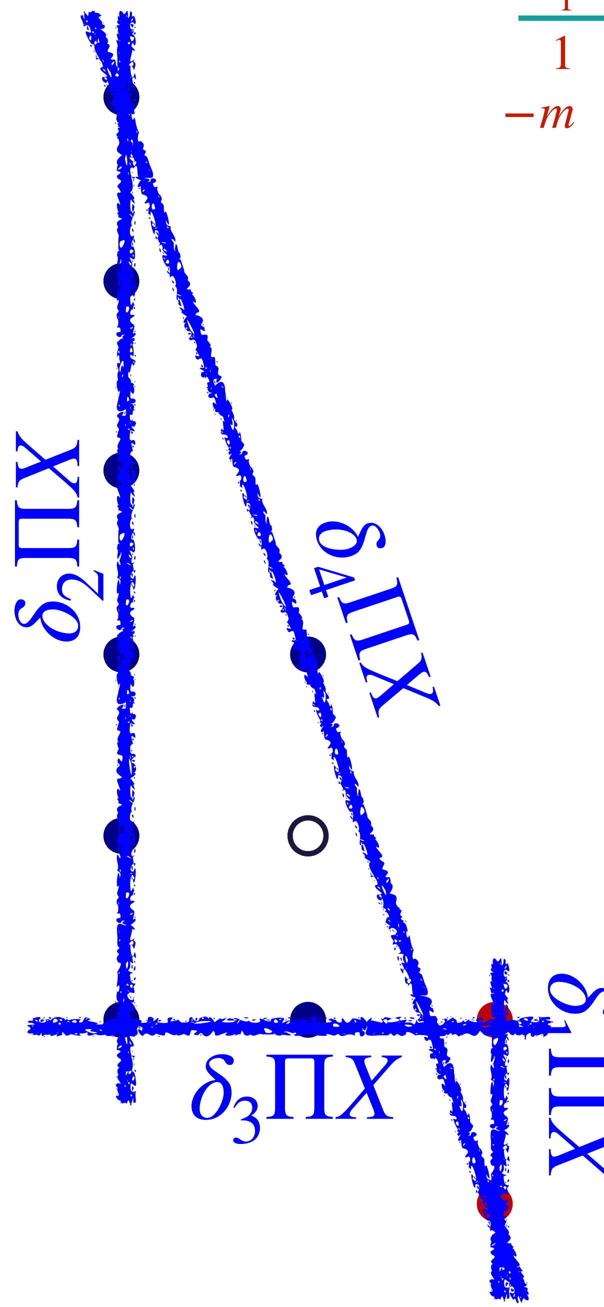

Mirror Minuet

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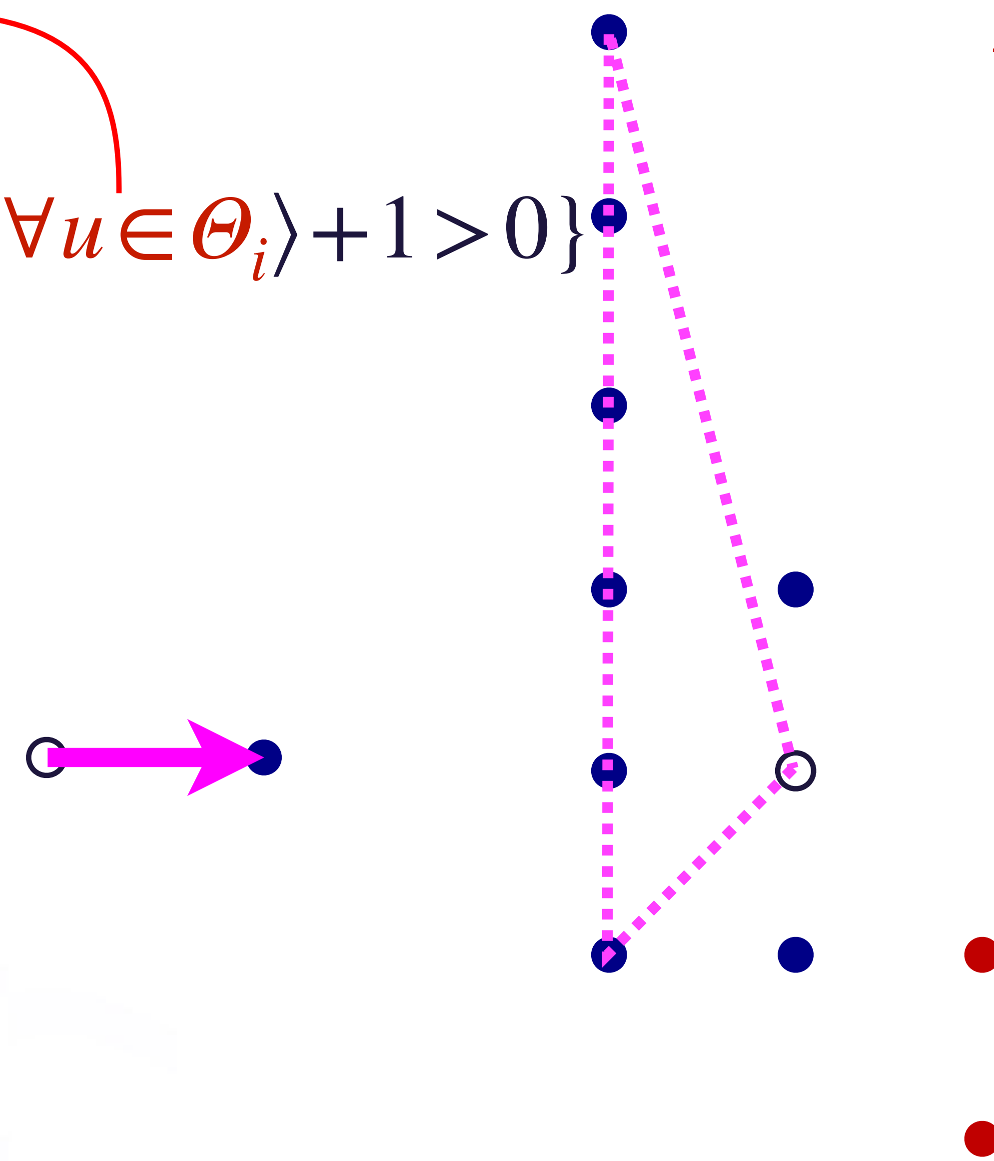
X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow\mathbb{P}^4$
$-m$	0	0	0	1	1	$\leftarrow\mathbb{P}^1$

- *Trans*polar ($\approx$ dual):
  - $\Delta\to\bigcup_i(\text{convex }\Theta_i)$;
  - Compute $\Theta_i\to\Theta_i^\circ:=\{v\colon\langle v|\forall u\in\Theta_i\rangle+1>0\}$



Mirror Minuet

1611.10300 & 2205.12827
2403.07139 & more

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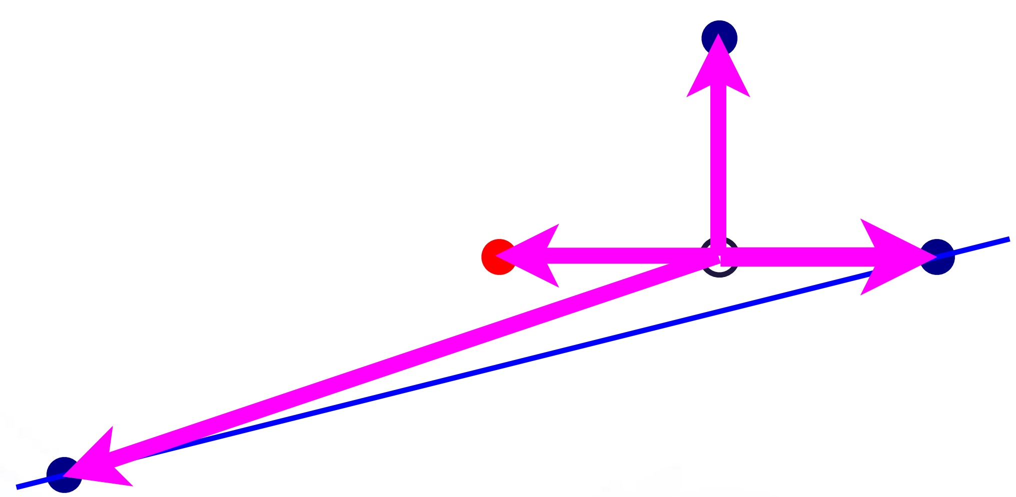

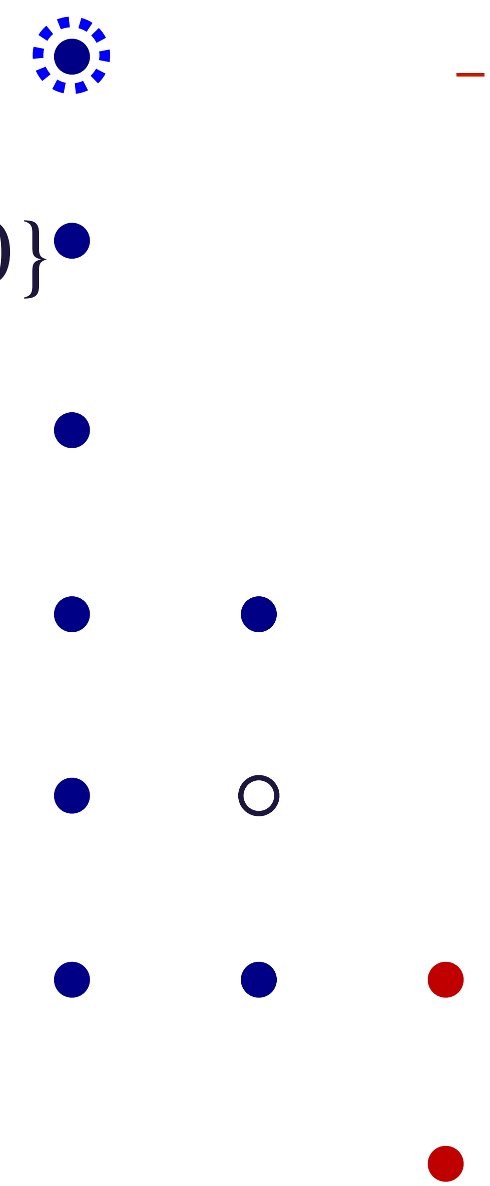

Mirror Minuet

& Non-Convex Mirrors

$m=3$ —2D *Proof-of-Concept*—

1611.10300 & 2205.12827
2403.07139 & more

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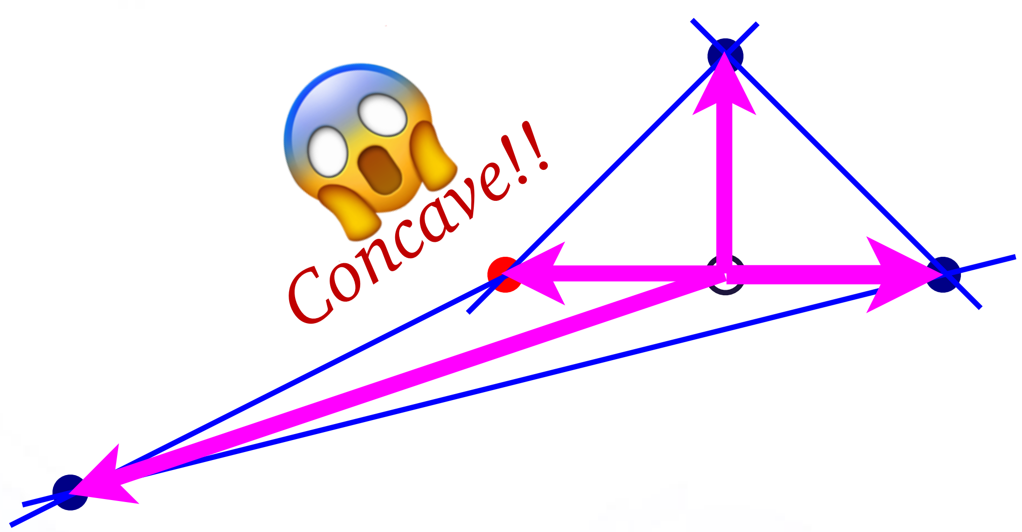

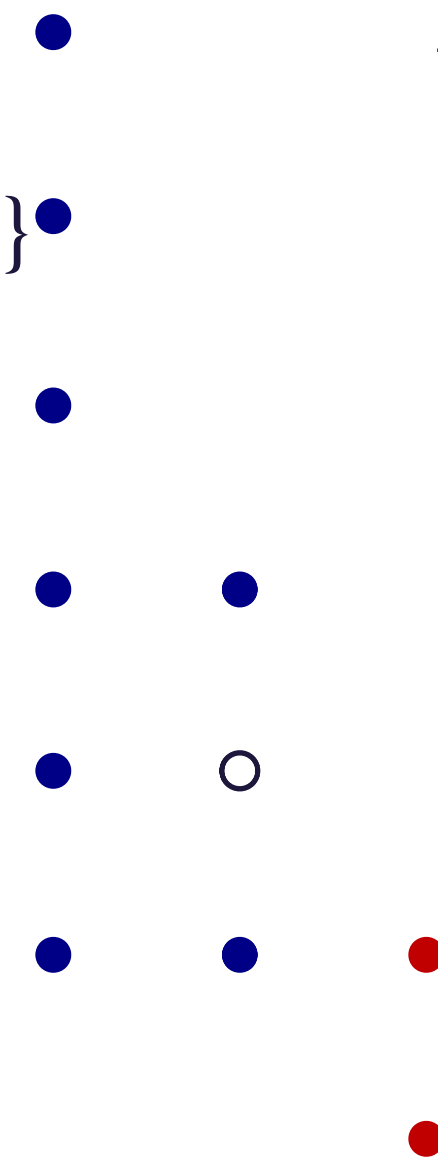

Mirror Minuet

1611.10300 & 2205.12827
2403.07139 & more

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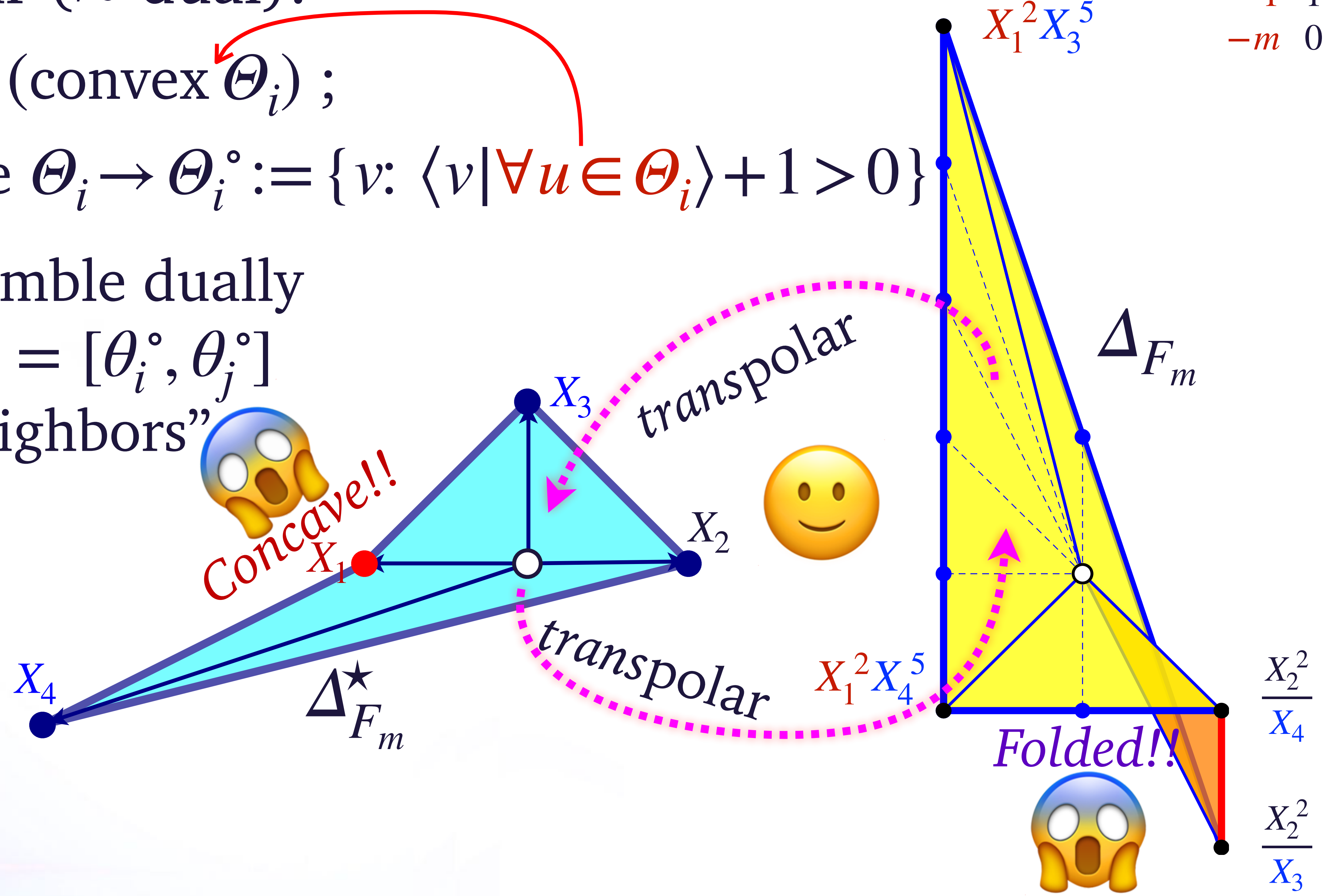
  - (Re)assemble dually $(\theta_i \cap \theta_j)^\circ = [\theta_i^\circ, \theta_j^\circ]$ with “neighbors”



Mirror Minuet

& Non-Convex Mirrors $m=3$ —2D Proof-of-Concept—

1611.10300 & 2205.12827
2403.07139 & more

- $X_1^{\,2}X_2^{\,0}(X_3\oplus X_4)^{2+1m}\oplus X_1^{\,1}X_2^{\,1}(X_3\oplus X_4)^{2+0m}\oplus X_1^{\,0}X_2^{\,2}(X_3\oplus X_4)^{2-1m}$
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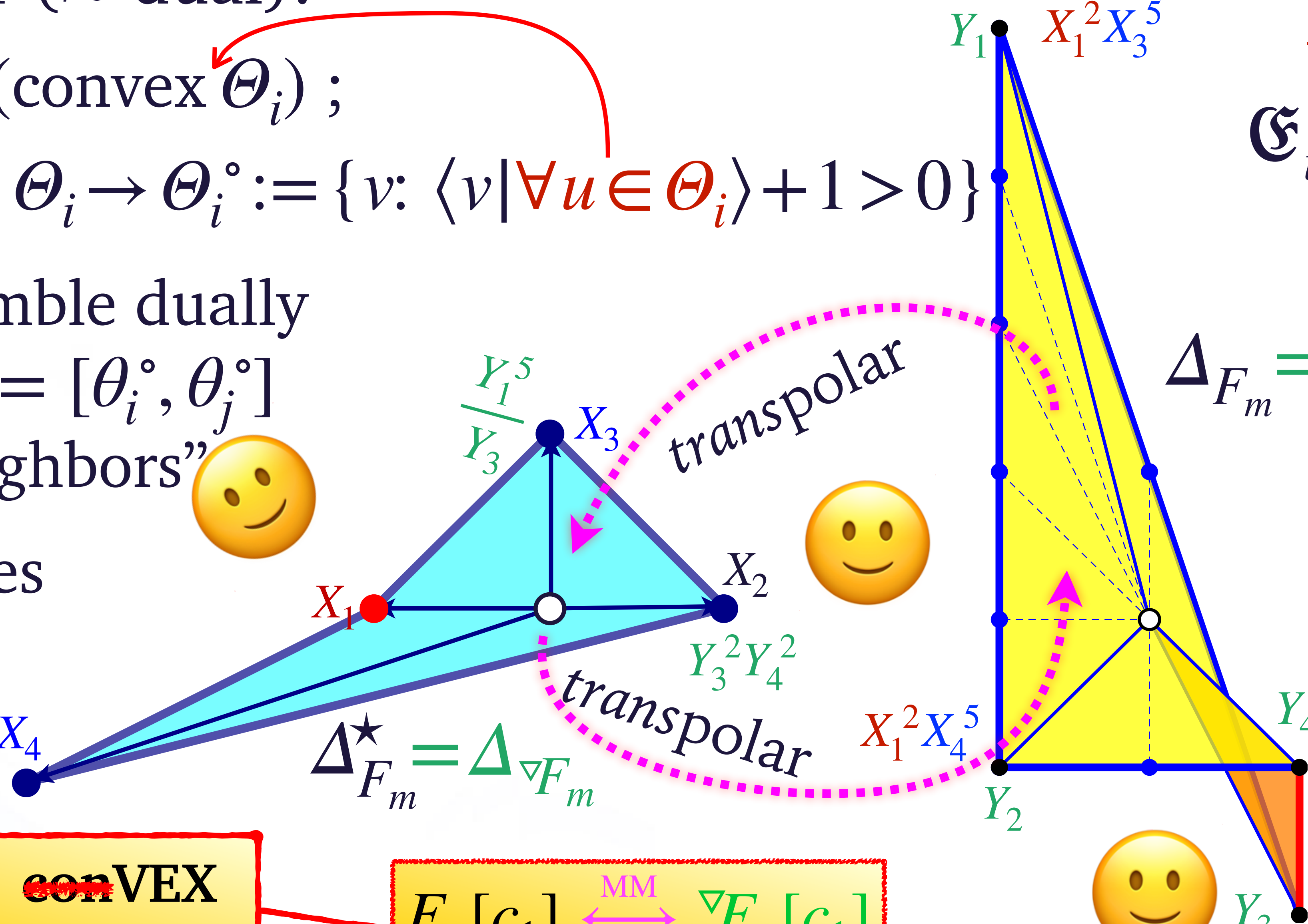
  - (Re)assemble dually $(\theta_i\cap\theta_j)^{\circ}=[\theta_i^{\circ},\theta_j^{\circ}]$ with "neighbors"
  - Swap roles $X_i\leftrightarrow Y_I$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow\mathbb{P}^4$
$-m$	0	0	0	1	1	$\leftarrow\mathbb{P}^1$

$$\mathfrak{C}_{ij}=\langle u_i,v_j\rangle+1$$
$$u_i\in\Delta^\star_X,\ v_j\in\Delta_X$$



'92: Khovanskii +Pukhlikov
'93: Karshon +Tolman
'99: Hattori +Masuda
+lots of

(pre)complex algebraic geometry

~~con~~VEX polytopes

$F_m[c_1]\xleftrightarrow{\text{MM}}{}^{\nabla}F_m[c_1]$

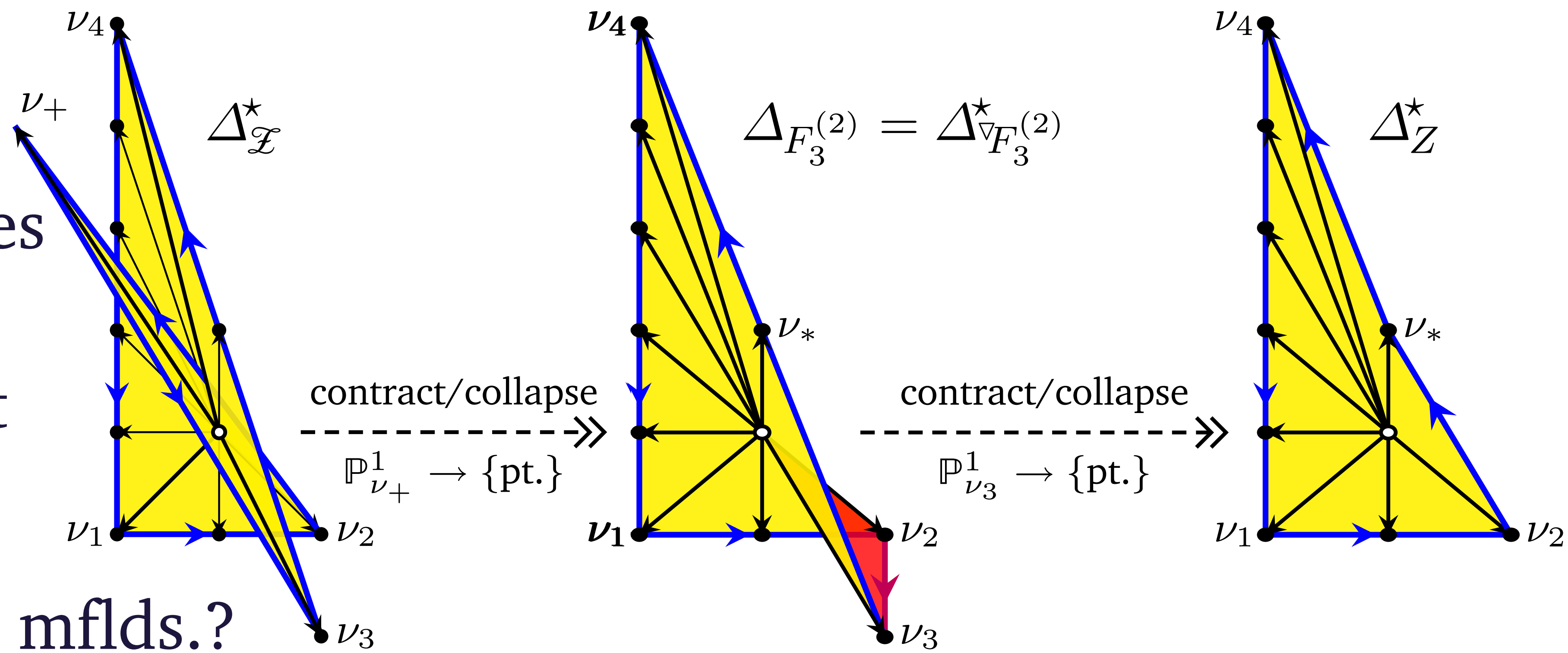
(pre)symplectic geometry

Mirror Minuet

& Non-Convex Mirrors $m=3$ —2D *Proof-of-Concept*—

1611.10300 & 2205.12827
2403.07139 & more

- What of self-crossing multifans & multitopes that span them?
- Good question!
  - “sandwiched” btw. toric varieties & torus mflds.
  - $(S^1)^n$-equivariant
  - $\mathbb{C}^\times \approx S^1 \times \mathbb{R}_+$
  - topological torus mflds.?
    - Top cone orientation ↔ rel. orientation btw. M & cpx$[T_M \otimes \underline{\mathbb{R}}^{2k}]$…
    - They have cohomology rings generated by Cox-like coords & “$\mathscr{K}^*_M$”



- Guess: obverse top-cones obstruct global cpx. structure & break supersymmetry…

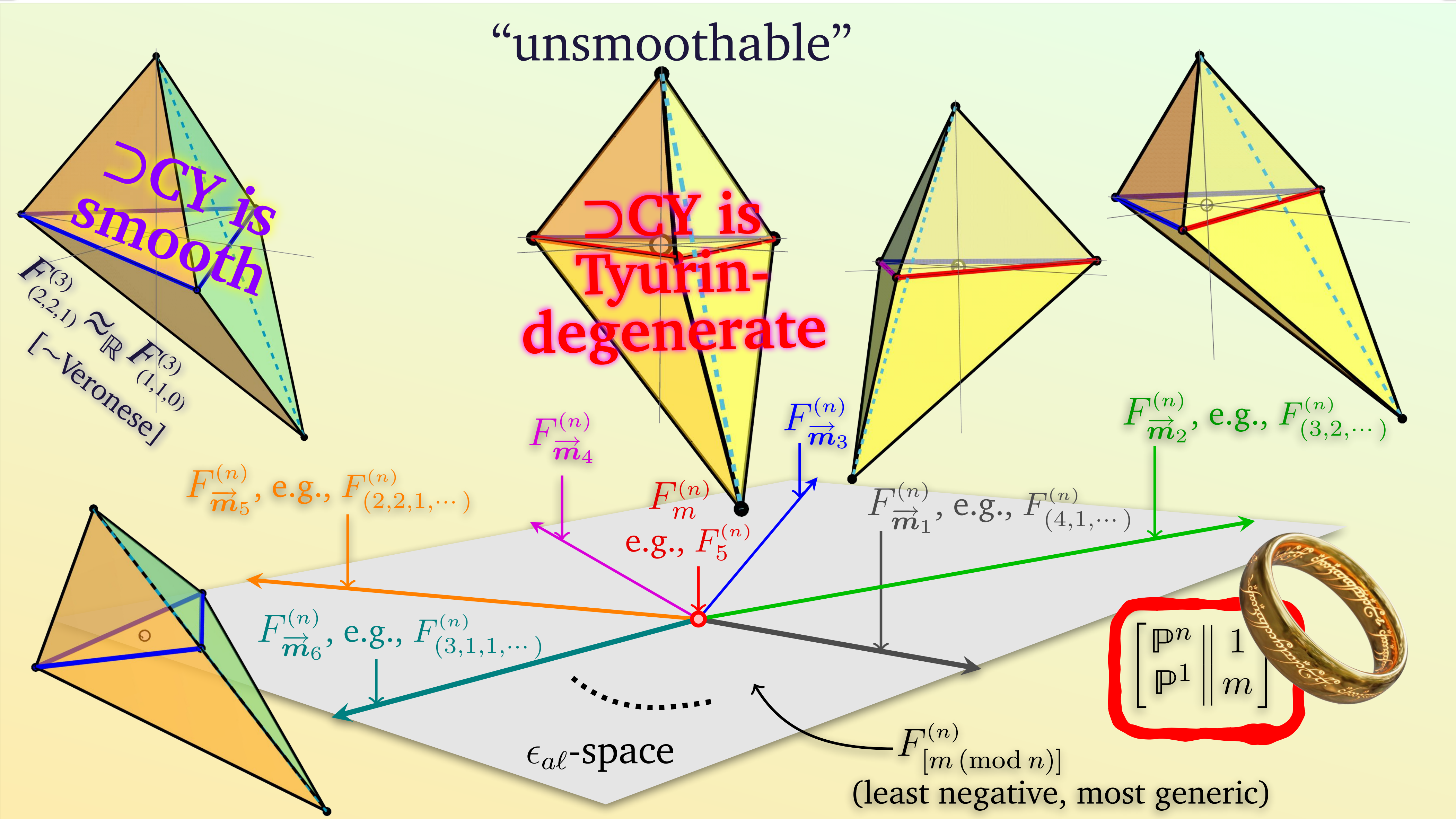
Conjectural Coda

Summary

—Proof-of-Concept—

1611.10300 & 2205.12827
2403.07139 & more

- CY(n−1)-folds in Hirzebruch n-folds, $X^{(n-1)}_m \in F^{(n)}_m[c_1]$



A deformation family picture

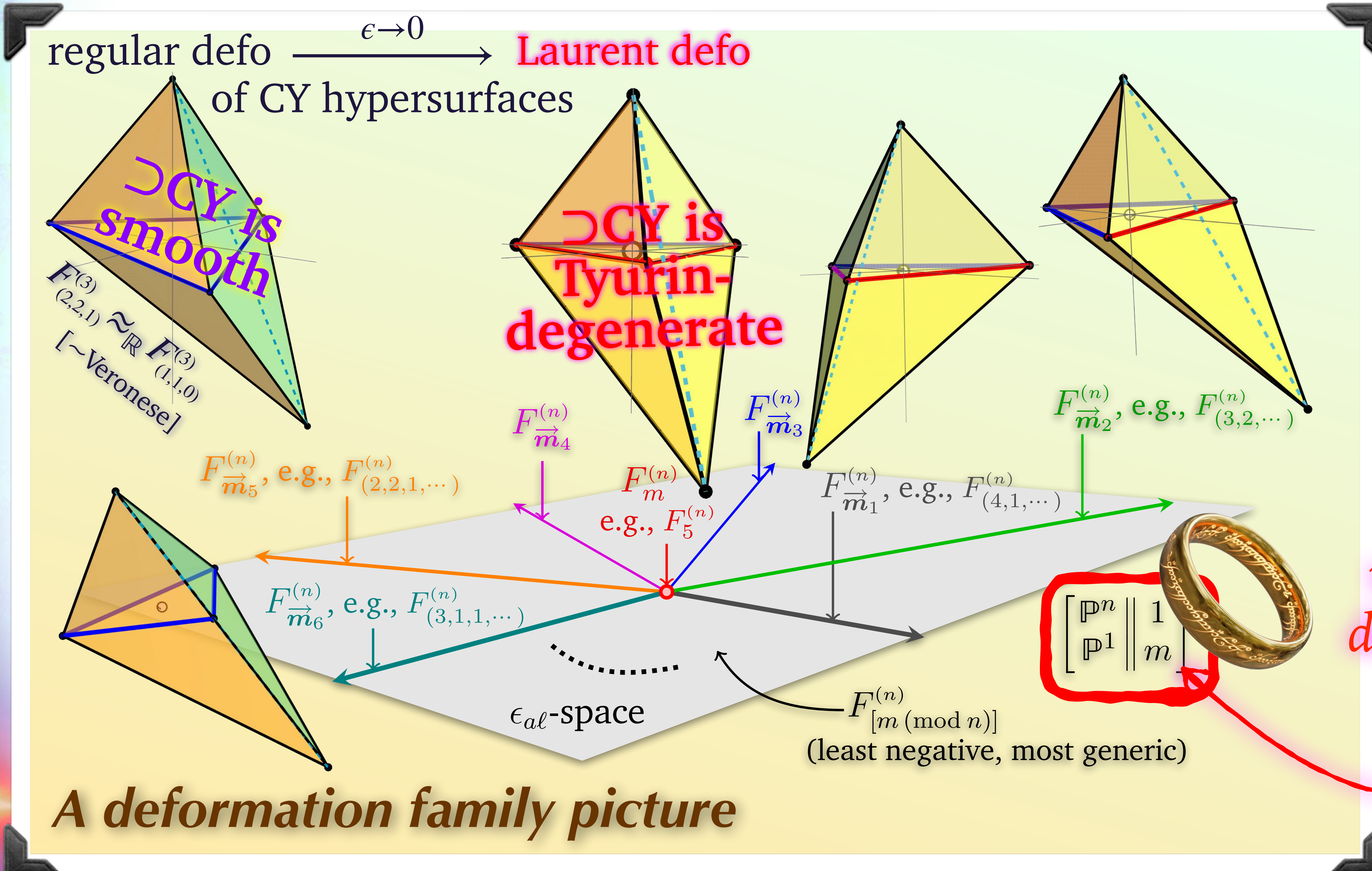
Conjectural Coda

1611.10300 & 2205.12827
2403.07139 & more

Summary

—Proof-of-Concept—

- CY(n−1)-folds in Hirzebruch n-folds, $X_m^{(n-1)} \in F_m^{(n)}[c_1]$



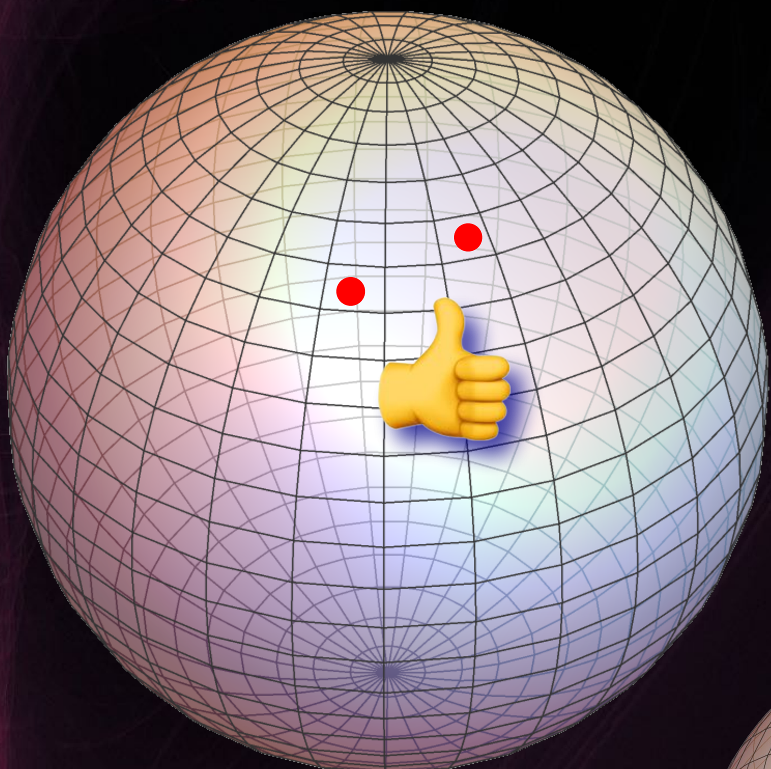
each distinct $F_m^{(n)}$ harbors multiple transp. mirror models

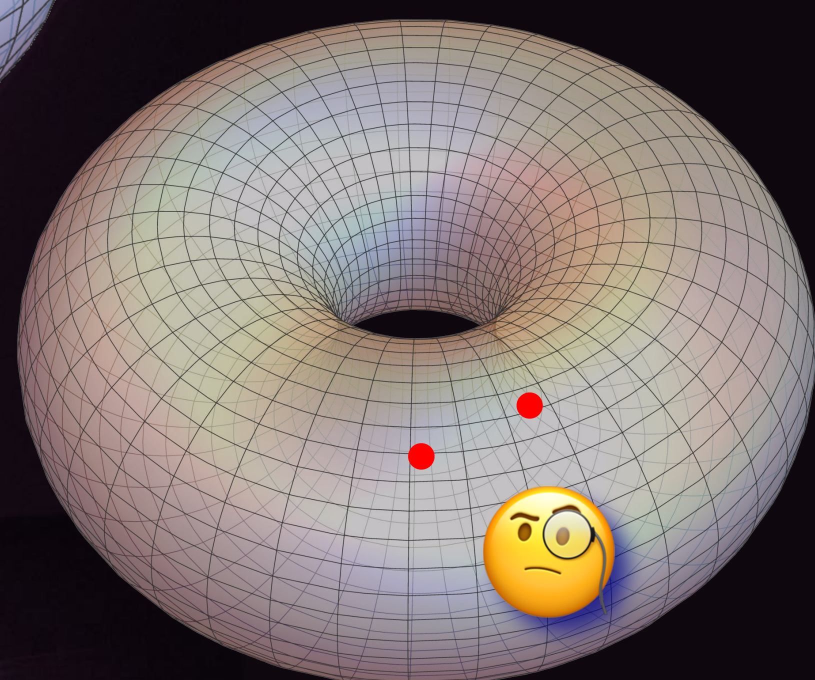
all within the deformation family

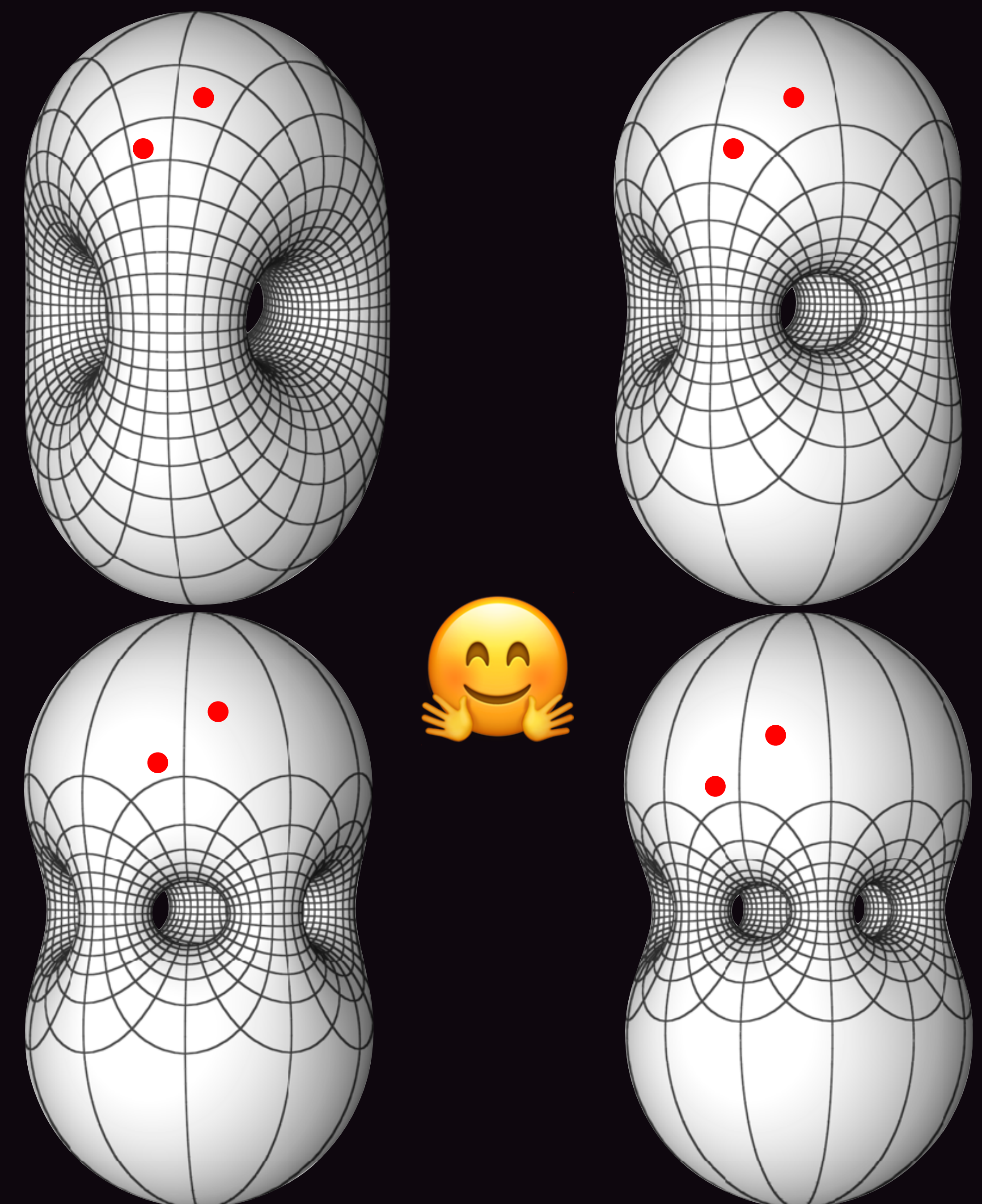
How Hard Can it Be?

Constructing CY ⊂ Some "Nice" Ambient Space

- Reduce to 0 dimensions: $\mathbb{P}^4[5] \to \mathbb{P}^3[4] \to \mathbb{P}^2[3] \to \mathbb{P}^1[2]$





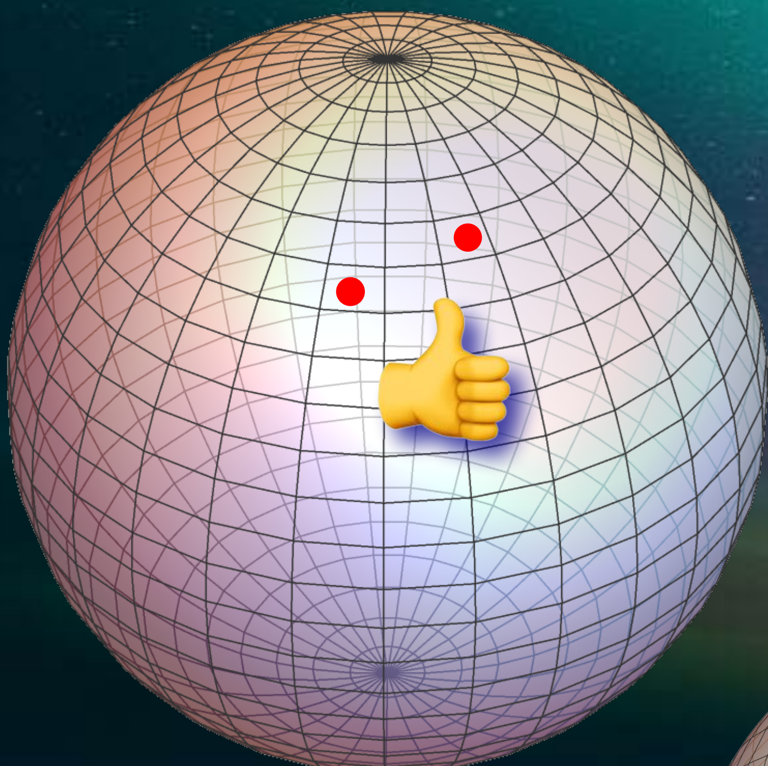


How Hard Can it Be?

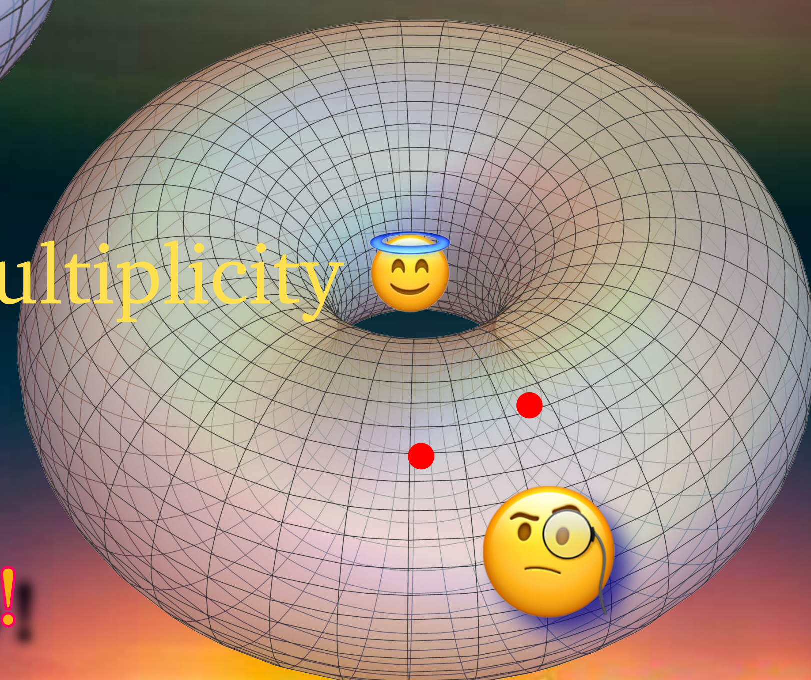
1611.10300
2205.12827
2403.07139
2501.11684
2502.08002
2605.06998
2605.24724
to be cont'd

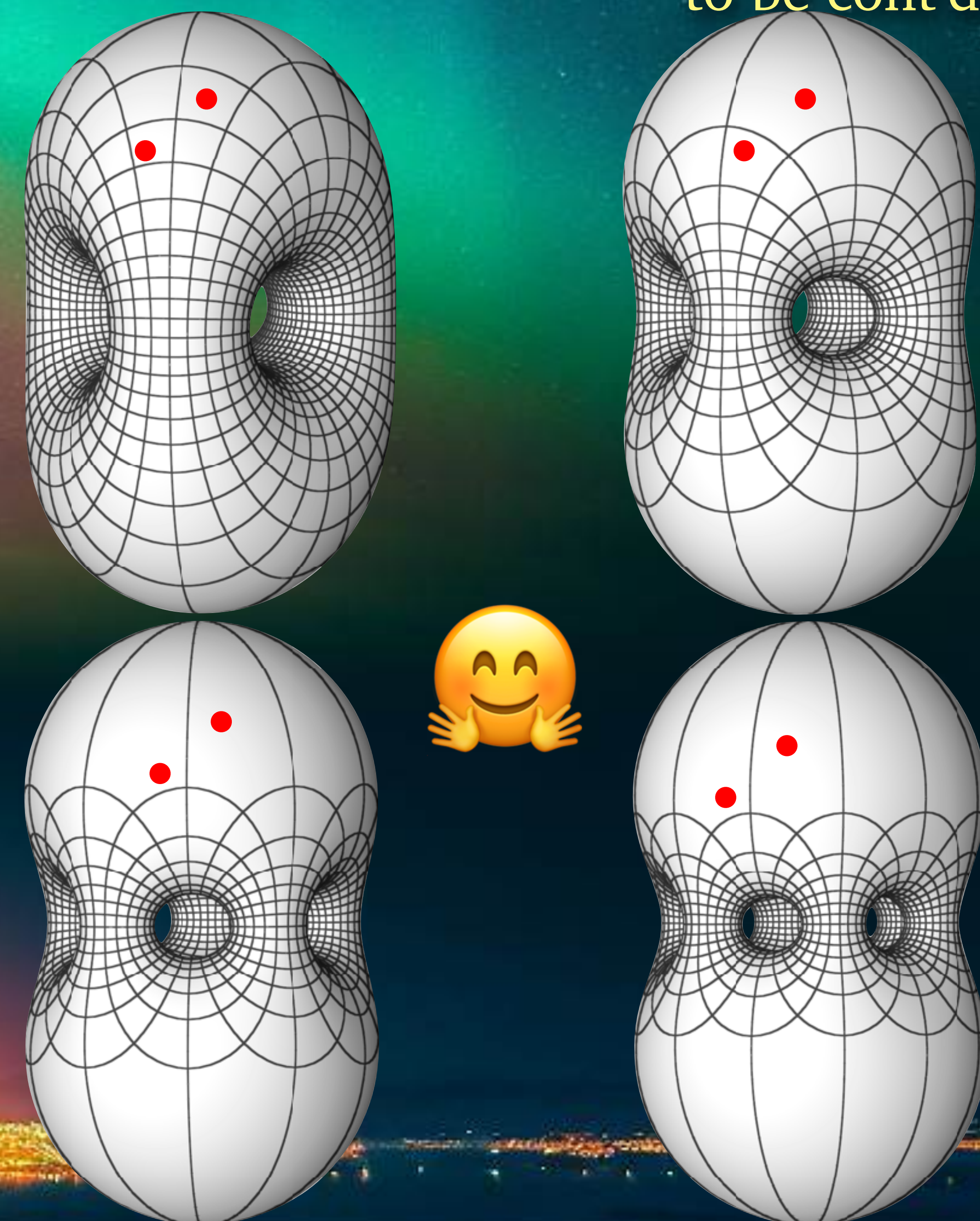
Constructing CY $\subset$ Some "Nice" Ambient Space

- Reduce to 0 dimensions: $\mathbb{P}^4[5] \to \mathbb{P}^3[4] \to \mathbb{P}^2[3] \to \mathbb{P}^1[2]$



Just add more dimensions





Counting with multiplicity

$\mathcal{X} \in A[c_1]$

Eldorado Ho!


1611.10300
2205.12827
2403.07139
2501.11684
2502.08002
2605.06998
2605.24724
to be cont'd
Thank You!
Eldorado Ho!