Stokes phenomenon and Knizhnik–Zamolodchikov (KZ) equations

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Higher Structures and Field Theory, ESI, Vienna
17-21 Aug 2020

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- Monodromy of KZ, cKZ, qKZ resembles respectively the Yang-Baxter, reflection, and Yang-Baxter equations with spectral parameter.
- A new viewpoint: to study the Stokes phenomenon of these equations. (Advantage: isomonodromy deformation).

Outline

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- Stokes matrices of KZ, cKZ, qKZ equations.
- Possible relation with BV quantization of 2d CohFT, and generalization to KZ 2-connections (in progress with Sheng and Zhu).

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- Borel resummation:

Suppose $\hat{f}(z) = \sum_{k\geq 0} f_k z^k$ with $|f_k| \leq C^k k!$. Formally

$$\hat{f} = \sum_{k=0}^{\infty} f_k z^k = \sum_{k=0}^{\infty} f_k \left(\int_0^{\infty(d)} e^{-t} t^k dt \right) \frac{z^k}{k!} = \int_0^{\infty(d)} e^{-t} \sum_{k=0}^{\infty} f_k \frac{(tz)^k}{k!} dt.$$

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- $\mathbb{BS}_d(\hat{f}) = \frac{1}{z} \int_0^{\infty(d)} e^{-\frac{t}{z}} (\frac{f_k}{k!} t^k) dt.$

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Example

Suppose $f(z) = \sum_{k\geq 0} k! z^k$:

- $\sum_{k=0}^{\infty} t^k = \frac{1}{1-t}$ *(analytically continued to* $t \leq 0$*).*
- *the resummation is* $\int_0^{-\infty} \frac{e^{-t}}{1-tz} dt = \frac{-1}{z} \cdot e^{\frac{-1}{z}} \cdot \Gamma\left(0, \frac{-1}{z}\right).$

ODEs with second order poles

Consider the linear system on z-plane

$$\frac{dF}{dz} = \left(\frac{u}{z^2} + \frac{V}{z}\right) F,$$

where $F(z) \in \mathrm{gl}_n$, $u = \mathrm{diag}(u^1, ..., u^n)$ and $V \in \mathrm{gl}_n(\mathbb{C})$.

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Unique formal fundamental solution:

$$\hat{F}(z) = \hat{H}(z) e^{-\frac{u}{z}} z^{[V]},$$

where $\hat{H}(z) = \mathrm{Id}_n + H_1 z + \cdots$ is a formal sum of matrices.

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Problem: $|H_k| \sim k!$.

Canonical solutions

- Borel resummation (along a direction d):

$$\mathbb{BS}_d(\hat{H}) = \frac{1}{z}\int_0^{\infty(d)} e^{-\frac{t}{z}}\Big(\sum_{k\geq 0}\frac{H_k}{k!}t^k\Big)dt.$$

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Theorem

In each Stokes sector R,

$$F_R(z) := \mathbb{BS}_R(\hat{H})(z)e^{-\frac{u}{z}}z^{[V]}$$

is the unique (therefore canonical) holomorphic solution with the asymptotics $F_R(z) \sim \hat{F}(z)$ at $z = 0$ within R.

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Definition

The **Stokes matrices** $S_\pm$ *of* $\frac{dF}{dz} = \left(\frac{u}{z^2} + \frac{V}{z}\right) F$ *are given by*

$$F_-(z) = F_+(z) \cdot S_+ \text{ in } R_-, \quad F_+(z) = F_-(z) \cdot S_- \text{ in } R_+.$$

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Example

Consider $\frac{dF}{dz} = \frac{1}{z^2}\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} + \frac{1}{z}\begin{pmatrix} t_1 & b_2 \\ b_1 & t_2 \end{pmatrix} F.$

Then $S_+ = \begin{pmatrix} 1 & \frac{2\pi i b_2 (u_2 - u_1)^{t_1 - t_2}}{\Gamma(1-\lambda_1+t_1)\Gamma(1-\lambda_2+t_1)} \\ 0 & 1 \end{pmatrix}.$

Part II

Stokes phenomenon of KZ equations, Yang-Baxter and reflection equations

Summary

- Stokes matrix of $\kappa\frac{dF}{dz} = (u^{(1)} + \frac{\Omega}{z})F$
  $\approx$ R-matrix of quantum groups;
- Stokes matrix of $\frac{dF}{dz} = \left(u^{(1)} + h\frac{2\Omega_{\mathfrak{k}} + C_{\mathfrak{k}}^{(1)}}{z}\right) \cdot F$
  $\approx$ K-matrix of quantum symmetric pairs;
- Stokes matrix of $F(z+p) = \left(\kappa^{-u^{(1)}} + \frac{\kappa^{-u^{(1)}}\Omega}{z}\right)F(z)$
  $\approx$ R-matrix of affine quantum groups.

KZ equations

Set $\mathfrak{g} = \mathrm{gl}_n$, $\Omega = \sum_{1 \le i,j \le n} E_{ij} \otimes E_{ji} \in U(\mathrm{gl}_n)^{\otimes 2}$, and take $u = \mathrm{diag}(u_1, ..., u_n) \in \mathfrak{g}$ and $V \in \mathrm{Rep}(\mathrm{gl}_n)$.

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- Braid group B_n, $\pi_1^{S_n}(\mathbb{C}^n \setminus \{z_i \neq z_j\})$, has generators $b_1, ..., b_{n-1}$ and relations

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1},$$
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- Formal solution $\hat{F}$ at $z = \infty$, whose resummation are different F_σ in different zones $D_\sigma = (\mathrm{Re}(z_{\sigma(1)}) \ll ... \ll \mathrm{Re}(z_{\sigma(n)}))$.

Stokes matrices and Yang-Baxter equations

Figure: Monodromy along $b_i = F_1 \cdot F_{\sigma_{i,i+1}}^{-1} \in \mathrm{End}(V^{\otimes n})$:

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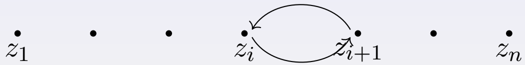

Factorization: the computation reduces to

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for $z = z_{i+1} - z_i$.

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Theorem

For any regular u, the Stokes matrix $S_+ \in \mathrm{End}(V^{\otimes 2})$ satisfies Yang-Baxter equation $S_+^{12} S_+^{13} S_+^{23} = S_+^{23} S_+^{13} S_+^{12}$.

Example: simplest case

Let us take gl_2 and the natural representation V, thus

$$\frac{dF}{dz} = (u + \frac{h\Omega}{z})F,$$

where $u = \mathrm{diag}(u_1, u_1, u_2, u_2)$, and $h\Omega = \left(\begin{array}{cccc} h & 0 & 0 & 0 \\ 0 & 0 & h & 0 \\ 0 & h & 0 & 0 \\ 0 & 0 & 0 & h \end{array}\right)$.

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We get

$$S_+ = \begin{pmatrix} e^h & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2i\sin(\pi h) & 1 & 0 \\ 0 & 0 & 0 & e^h \end{pmatrix}.$$

It is the evaluation of the universal R-matrix of quantum gl_2 on $V \otimes V$.

Cyclotomic KZ equations

- Set $\Omega_{\mathfrak{k}} = \frac{1}{2}\sum_{1\leq i,j\leq n}(E_{ij} - E_{ji}) \otimes (E_{ji} - E_{ij}) \in U(\mathrm{so}_n)^{\otimes 2}$,
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Definition

The cKZ *equation, for a function* $F(z_1, ..., z_n) \in W \otimes V^{\otimes n}$, *is*

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- The braid group on $\mathbb{C}^\times$, $\pi_1^{S_n}((\mathbb{C}^\times)^n \setminus \{z_i \neq z_j\})$, has generators $\tau, b_1, ..., b_{n-1}$ and relations

$$\tau b_1 \tau b_1 = b_1 \tau b_1 \tau, \qquad b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1},$$
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Stokes matrices and reflection equations

Factorization: consider the equation for a $W \otimes V$-valued function $F(z)$

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Theorem

For any u, $K_+ \in \mathrm{End}(W \otimes V)$ satisfies reflection equation $K_+^{12} S_+^{32} K_+^{13} S_+^{32} = S_+^{32} K_+^{13} S_+^{23} K_+^{12} \in \mathrm{End}(W \otimes V \otimes V)$.

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- Monodromy of cKZ by Enriquez, Brochier, De Commer-Neshveyev-Tuset-Yamashita,...

Quantum KZ equations

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Definition (Frenkel-Reshetikhin)

The qKZ equation for a function $F(z_1, ..., z_n) \in V^{\otimes n}$ is

$$\begin{aligned} & F(z_1, ..., z_m + p, ..., z_n) \\ = \; & R^{m,m-1}(z_m - z_{m-1} + p) \cdots R^{m,1}(z_m - z_1 + p)\kappa^{-u^{(m)}} \\ & \times R^{m,n}(z_m - z_n) \cdots R^{m,m+1}(z_m - z_{m+1}) F(z_1, ..., z_n), \end{aligned}$$

where p and κ are parameters, $u \in \mathrm{gl}_n$ is diagonal.

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$$\tilde{F}(y_1, ..., y_m + hp, ..., y_n) = \Big(1 + h\eta u^{(m)} + h \sum_{k \neq m} \frac{\Omega^{k,m}}{y_m - y_k} + o(h)\Big)\tilde{F}.$$

As $h \to 0$, it turns to the KZ equation.

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Proposition (Birkhoff)

(1) There are canonical solutions $F_{\pm}(z)$ asymptotically equal to $\hat{F}(z)$ as $z \to \pm\infty$;
(2) the connection matrix $S(z) = F_{+}(z)^{-1} F_{-}(z)$ is of the form

$$S(z) = S_0 - \frac{S_1}{e^{\frac{2\pi i}{p} z} - 1}.$$

Connection matrices and Yang-Baxter equations

Theorem

The connection matrix $S(z) = S_0 - \frac{S_1}{e^{\frac{2\pi i}{p} z} - 1} \in \mathrm{End}(V^{\otimes 2})$ *of qKZ equation satisfies Yang-Baxter equation with spectral parameter*

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$$p \frac{dF}{dz} = (\eta u^{(1)} + \frac{\Omega}{z}) F. \tag{1}$$

Furthermore, the S_0 and $S_\infty = S_0 + S_1$ in $S(z)$ converge to the Stokes matrices $S_\pm$ of the equation (1) as $h \to 0$.

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Corollary

In particular, it implies that S_+ *satisfies the YB equation.*

Summary

- S_+ of $\kappa\frac{dF}{dz} = (u^{(1)} + \frac{\Omega}{z})F$
  $\approx$ (Algebraic) R-matrix of quantum groups;
- K_+ of $\frac{dF}{dz} = \left(u^{(1)} + h\frac{2\Omega_{\mathfrak{k}} + C_{\mathfrak{k}}^{(1)}}{z}\right) \cdot F$
  $\approx$ (Algebraic) K-matrix of quantum symmetric pairs;
- $S(z)$ of $F(z+p) = \left(\kappa^{-u^{(1)}} + \frac{\kappa^{-u^{(1)}}\Omega}{z}\right)F(z)$
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• In general, to find a Stokes phenomenon interpretation of many objects in the theory of quantum algebras.

Some words on higher structures

- Quantum 2d CohFT via BV formalism: Dotsenko, Sharon, Vaintrob and Vallette arxiv:2006.01649.

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• It rises the question:

$$\begin{array}{ccc} d - \left(u^{(2)} + h\frac{2\Omega_{\mathfrak{k}} + C^{(2)}_{\mathfrak{k}}}{z}\right)dz & \xrightarrow{?} & Quantum\ 2d\ CohFT \\ {\scriptstyle s.c.l}\downarrow & & {\scriptstyle s.c.l}\downarrow \\ d - \left(u + \frac{V}{z}\right)dz & \xrightarrow{Dubrovin} & CohFT\ (Frobenius\ mfld) \end{array}$$

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- There is a notion of classical Yang-Baxter equation for a Lie 2-algebra $\mathfrak{g} = (d : \mathfrak{g}_{-1} \to \mathfrak{g}_0)$. A solution is (r, p), where $r \in \mathfrak{g}_0 \otimes \mathfrak{g}_0$ and $p \in \mathfrak{g}_0 \otimes \mathfrak{g}_0 \otimes \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \otimes \mathfrak{g}_{-1} \otimes \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \otimes \mathfrak{g}_0 \otimes \mathfrak{g}_0$.

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Then given a representation V, one has a flat 2-connection (A, B) over $\mathbb{C}^n \times V^{\otimes n}$

$$A = \sum_i u^{(i)} dz_i + \sum_{i<j} r^{ij}(z)\omega_{ij},$$

$$B = \sum_{i<j<k} \Big(p_{jik}\omega_{ij} \wedge \omega_{ik} + p_{ijk}\omega_{ij} \wedge \omega_{jk} \Big)$$

where $\omega_{ij} := d\log(z_i - z_j)$.
- Problem: singularities.

Thank you very much!