Combinatorics of higher order renormalization group equations

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Much delayed joint work with Loïc Foissy and William Dugan.

Why talk on this? Many words intersect with the workshop title!

Connes-Kreimer Hopf algebra of rooted trees

Let $\mathcal{T}$ be the set of rooted trees without an empty tree.

Augment the polynomial algebra $\mathbb{Q}[\mathcal{T}]$ to a Hopf algebra $\mathcal{H}$ with

$$\Delta(t) = \sum_{\substack{S \subseteq V(t) \\ S \text{ antichain}}} \left(\prod_{v \in S} t_v\right) \otimes \left(t \setminus \prod_{v \in S} t_v\right)$$

t_v = subtree rooted at v

1 when empty

and the rest as it needs to be.

Eg: 

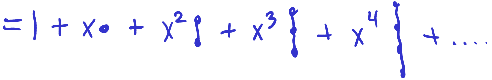
Tree specifications and Dyson-Schwinger equations

Define

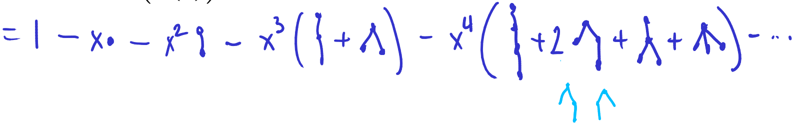
$$B_+(t_1 t_2 \cdots t_k) =$$

Then we can build families of trees. For example:

$$X(x) = 1 + xB_+(X(x))$$



$$X(x) = 1 - xB_+\left(\frac{1}{X(x)}\right)$$



Feynman rules

To take the next step, we need Feynman rules, ϕ, in this context.

Our target space will be $\mathbb{Q}[L]$.

We want $\phi : \mathcal{H} \to \mathbb{Q}[L]$ to be an algebra homomorphism and

$$\phi\big|_{L \leftarrow L_1 + L_2} = \phi\big|_{L \leftarrow L_1} \star \phi\big|_{L \leftarrow L_2}$$

where $\star$ is the convolution product: $f \star g = m(f \otimes g)\Delta$.

The algebraically minded will see that we are essentially just speaking of the exponential map on the associated Lie algebra here.

Tree factorial and general tree Feynman rules

Concretely what can Feynman rules on trees look like? Work it out recursively.

The simplest case and always the leading behaviour is given by the tree factorial Feynman rules.

$$t \mapsto \frac{L^{|t|}}{t!}$$

vertices

where

$$t! = \prod_v |t_v|$$

$|t|!/t!$ counts the number of increasing labellings of a plane tree.
General tree Feynman rules:

$$\phi(f) = \sum_{S \subseteq E(f)} \left(\prod_{t \in (f \setminus S)} \sigma(t) \right) \frac{L^{|f/f \setminus S|}}{(f/(f \setminus S))!}$$

Renormalization group equation

When we apply ϕ to the combinatorial Dyson-Schwinger equations these are the physical Dyson-Schwinger equations.
The Green function is $G(x, L) = \phi(X(x))$.

Then the renormalization group equation is

$$\left(\frac{\partial}{\partial L} + \beta(x)\frac{\partial}{\partial x} - \gamma(x)\right) G(x, L) = 0$$

$$\frac{\partial}{\partial L} G(x,L) = \left(\beta(x)\frac{\partial}{\partial x} - \gamma(x)\right) G(x,L)$$

Higher order renormalization group equations

What about higher x derivatives?

If $G(x, L)$ satisfies

$$\frac{\partial G(x,L)}{\partial L} = \overline{\beta}\left(x, \frac{\partial}{\partial x}\right) G(x,L)$$

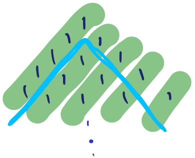
where $\overline{\beta}$ is polynomial in its second argument, then say $G(x, L)$ satisfies a generalized renormalization group equation of *order* the degree of $\overline{\beta}$ in the second argument.

These come from some 2014 conversations with Spencer Bloch and Dirk Kreimer, but all fault is mine.

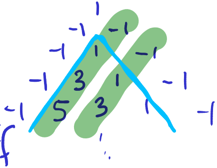
Back to the examples

$X(x) = 1 + xB_+(X(x))$ satisfies the renormalization group equation with $\beta = 0$ — a 0th order renormalization group equation

$$\Delta(l_n) = \sum_{i=0}^{n} l_i \otimes l_{n-i}$$



$X(x) = 1 - xB_+\left(\frac{1}{X(x)}\right)$ satisfies the renormalization group equation (first order)



Meaning at the coefficient level

Generally given a sequence $t_i \in \mathcal{H}$ with t_i homogeneous of degree i, where the t_i generate a Hopf subalgebra, write

$$\Delta(t_n) = t_n \otimes 1 + 1 \otimes t_n + \sum_{i=1}^{n-1} \left(\lambda_{i,n-i}\, t_{n-i} + \text{stuff (products)} \right) \otimes t_i$$

Just consider the $\lambda_{i,j}$ not the outer 1s and make a triangular array as before.

The sequence with ϕ satisfies a kth order renormalization group equation if the leftward diagonals are kth order sequences. This is *strong* kth order.

Strong 0th order

All made of ladders. Precisely, let

$$\sum p_n = \log(1 + \bullet +$$

Then normalizing $\lambda_{1,1} = 1$ every 0th order sequence sums to

- $B_+(\exp(\sum_{k=1}^{n-1} p_k + bp_n))$ or
- $B_+(\exp(\sum_{k\geq 1} p_k))$, that is ladders.

Strong 1st order

The possibilities for the array for a strong first order sequence are

$$\textbf{A:}\quad \lambda_{i,j} = \begin{cases} a_1 i + b & \text{if } j = 1 \\ a_2 i + b & \text{if } j = 2 \\ \frac{a_1 a_2 i}{(j-1)a_1 - (j-2)a_2} + b & \text{otherwise} \end{cases}$$

$$\textbf{B:}\quad \lambda_{i,j} = \begin{cases} a_2 i - 2a_2 & \text{if } j = 2 \\ a_1 i - 2a_1 & \text{otherwise} \end{cases}$$

$$\textbf{C:}\quad \lambda_{i,j} = \begin{cases} a_1 i + 2a_1 & \text{if } j = 1 \\ a_2 i + 4a_2 & \text{if } j = 2 \\ \frac{6a_1 a_2 (2j+i)}{(j-1)j(j+1)a_1 - 2(j-2)j(j+2)a_2} & \text{otherwise} \end{cases}$$

$$\textbf{D:}\quad \lambda_{i,j} = \begin{cases} a_1 i + a_1 & \text{if } j = 1 \\ a_2 i + 2a_2 & \text{if } j = 2 \\ \frac{2a_1 a_2 (i+j)}{(j-1)ja_1 - 2(j-2)ja_2} & \text{otherwise} \end{cases}$$

$$\textbf{E:}\quad \lambda_{i,j} = \begin{cases} a_1 i + b & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases}$$

E is corollas.

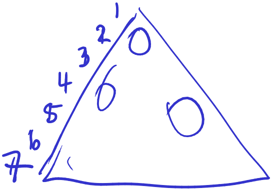
A with $a_1 = a_2$ gives everything from Dyson-Schwinger equations.
A with $a_2 = 0$ is a family of weighted ladders with extra leaves

$$X = B_+\left(\frac{1 + bX}{1 - a_1\bullet}\right)$$

$(a_1+b)\, a_1^{\text{leaf}(t)-1}\, b^{\text{depth}(t)-1}$

Strong higher order

The only strong higher order sequences are scaled corollas. They force 0s in places that would otherwise be trouble.



An interesting failure

So this isn't quite the right notion. A suggestive case is the Connes-Moscovici Hopf algebra.