Cumulants & Spreadability

Combinatorial proba = moment method

X, distr = moments $m_n(X) = \mathbb{E}X^n$

independence $\longleftrightarrow$ $\mathbb{E}f(X)g(Y) = \mathbb{E}f(X)\,\mathbb{E}g(Y)$

$X \perp\!\!\!\perp Y$

$$m_n(X+Y) = \sum_{k=0}^{n} \binom{n}{k} m_k(X)\, m_{n-k}(Y)$$

"formal group law"

Lie algebra?

$$\mathbb{E}\, e^{t(X+Y)} = \mathbb{E}\, e^{tX}\, \mathbb{E}\, e^{tY}$$

$$\Rightarrow \mathcal{F}_X(t) = \log \mathbb{E}\, e^{tX} = \log \sum_{n=0}^{\infty} \frac{m_n}{n!} t^n$$

$$= \sum_{n=1}^{\infty} \frac{\kappa_n(X)}{n!} t^n$$

$\Rightarrow$ κ_n = cumulants

$$\kappa_n(X+Y) = \kappa_n(X) + \kappa_n(Y)$$

<u>ncps</u> : algebra $\mathcal{A}$ with 1 over $\mathbb{C}$, usually C^*-alg

$\varphi : \mathcal{A} \to \mathbb{C}$ $\quad \varphi(1) = 1$ $\qquad$ φ state

$X \in \mathcal{A}$ called n.c.r.v.

distr = $\quad m_n(X) = \varphi(X^n)$

1) $\mathcal{A} = L^\infty(\Omega)$ $\qquad \varphi(X) = \mathbb{E}X$

2) QM $\quad X$ = observable

$$\varphi(X) = \langle X\xi, \xi \rangle \quad \text{vector state}$$

3) random matrices

$$\mathcal{A} = L^\infty(\Omega; M_n)$$

$$\varphi(X) = \frac{1}{n}\mathbb{E}\,\mathrm{Tr}(X)$$

independence

classical indep = tensor indep

$$\mathcal{U} = \mathcal{A} \otimes \mathcal{A}, \quad \tilde{\varphi} = \varphi \otimes \varphi$$

$$\tilde{\varphi}(X \otimes Y) = \varphi(X) \cdot \varphi(Y)$$

i.e. $X \otimes I \perp\!\!\!\perp I \otimes Y$

Voiculescu's free independence

$\mathcal{U} = \mathcal{A} * \mathcal{A}$ unital free product

$\tilde{\varphi} = \varphi * \varphi$

$\mathcal{A} \overset{1}{\hookrightarrow} \mathcal{A}^{(1)} * \mathcal{A}^{(2)}$ $\qquad$ $\mathcal{A} \overset{2}{\hookrightarrow} \mathcal{A}^{(1)} * \mathcal{A}^{(2)}$

$X_j \in \mathcal{A}^{(1)}_{i_j}$ $\qquad$ $Y_j \in \mathcal{A}^{(2)}_{i_j}$ $\qquad$ $\varphi(X_j) = 0$

$\varphi(Y_j) = 0$

$\Rightarrow \varphi(X_1 Y_1 X_2 Y_2 \ldots) = 0$

Boolean independence

$\mathcal{U} = \mathcal{A} * \mathcal{A}$ non-unital free product

$\mathcal{A} \overset{1}{\hookrightarrow} \mathcal{U} \qquad \mathcal{A} \overset{2}{\hookrightarrow} \mathcal{U}$

$X_j \in \mathcal{A}^{(1)} \qquad Y_j \in \mathcal{A}^{(2)}$

$$\varphi(X_1 Y_1 X_2 Y_2 \ldots) = \varphi(X_1)\varphi(Y_1)\varphi(X_2)\varphi(Y_2)\ldots$$

$\to$ corresponding convolution, cumulants

Common framework

idea (X, Y) independent

$\Leftrightarrow (X, Y) \overset{d}{\sim} (X^{(1)}, Y^{(2)})$

where $(X^{(1)}, Y^{(1)})$ and $(X^{(2)}, Y^{(2)})$ are iid copies of (X, Y)

e.g. class. $X, Y \in L^\infty(\Omega)$ are indep $\Leftrightarrow$

$$(X, Y) \overset{d}{\cong} (X \otimes 1, 1 \otimes Y) \text{ in } L^\infty(\Omega \times \Omega)$$

Def $(\mathcal{A}, \varphi)$ ncps

An exchangeability syst is a ncps $(\mathcal{U}, \tilde{\varphi})$

togeth with $\iota_j : \mathcal{A} \longrightarrow \mathcal{U}$ embeddings, $j \in \mathbb{N}$

$$X \longmapsto X^{(j)}$$

st. 1) $\varphi = \tilde{\varphi} \circ \iota_j \quad \forall j$ (state-preserving)

2) $\tilde{\varphi}$ is exchangeable, i.e. invariant under $\mathfrak{S}_\infty$

$$\tilde{\varphi}\left(X_1^{(i_1)} X_2^{(i_2)} \cdots X_n^{(i_n)}\right) = \tilde{\varphi}\left(X_1^{(\sigma(i_1))} X_2^{(\sigma(i_2))} \cdots X_n^{(\sigma(i_n))}\right)$$

$$\forall \sigma \in \mathfrak{S}_\infty$$

e.g. $\tilde{\varphi}\left(X_1^{(1)} X_2^{(3)} X_3^{(1)} X_4^{(2)} X_5^{(3)}\right)$

$$\sigma : \begin{array}{l} 1 \longmapsto 5 \\ 2 \longmapsto 4 \\ 3 \longmapsto 6 \\ \vdots \end{array} \qquad = \tilde{\varphi}\left(X_1^{(5)} X_2^{(6)} X_3^{(5)} X_4^{(4)} X_5^{(6)}\right)$$

$$= (*)$$

e.g. $\mathcal{U} = \mathcal{A}^{\otimes\infty}$, $\tilde{\varphi} = \varphi^{\otimes\infty}$

j-th tensor

$$X^{(j)} = I \otimes \dots \otimes I \otimes X \otimes I \otimes \dots \overline{\underline{I}}$$

orbit of the action: $13123 \xrightarrow{\sigma} 56546 \dots$

$\pi = \ker 131\,23 \in \mathcal{P}(5)$

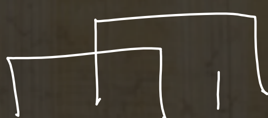
set partition

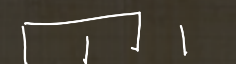
$$(*) = \varphi_\pi(X_1 \dots X_5)$$

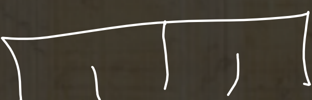
e.g. classical tensor $\quad \varphi_\pi(X_1 \ldots X_5) = \varphi(X_1 X_3) \cdot \varphi(\underbrace{X_2 X_5}_{\perp\!\!\!\perp}) \cdot \varphi(X_4)$

assume $\{X_1, X_3, X_5\} \perp\!\!\!\perp \{X_2, X_4\}$

$$= \varphi(X_1 X_3)\, \varphi(X_2)\, \varphi(X_5) \cdot \varphi(X_4)$$

$\pi =$ 

$=\varphi$  $(X_1 \ldots X_5)$

$\eta =$  (indep)

 $= \pi \wedge \eta$ meet operation in lattice $P(5)$

$$\pi \wedge \eta = \{A \cap B \mid A \in \pi,\ B \in \eta\} \setminus \{\emptyset\}$$

<u>Def</u> $\mathcal{E} = (\mathcal{U}, \tilde{\varphi}, (\iota_j))$ exch sgsh for $(\mathcal{A}, \varphi)$

subalg $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$ are $\mathcal{E}$-indep

if $\varphi_\pi(X_1 \dots X_n) = \varphi_{\underline{\pi \wedge \eta}}(X \dots X_n)$

whenever X_i can be partitioned into $\eta = \{I, J\}$

s.t. $\{X_i\}_{\underline{I}} \subseteq \mathcal{A}_1 \quad \{X_j\}_J \subseteq \mathcal{A}_2$

<u>Cumulants</u> Rota's ~~dot~~ dot operator

$$N \cdot X = X^{(1)} + X^{(2)} + \dots + X^{(N)}$$

$$\tilde{\varphi}((N \cdot X_1) \dots (N \cdot X_n)) = \text{poly in } N$$

$$= N \cdot \underbrace{K_n(X_1 \ldots X_n)}_{\text{cumulat}} + O(N^2)$$

e.g. $$\tilde{\varphi}(N \cdot X) = N \cdot \underbrace{\varphi(X)}_{K_1(X)}$$

$$\tilde{\varphi}\big((N\cdot X)(N\cdot Y)\big) = \tilde{\varphi}\Big(\sum_{i,j} X^{(i)} Y^{(j)}\Big)$$

$$= \sum_{i=j} \tilde{\varphi}(X^{(i)} Y^{(i)}) + \sum_{i \neq j} \tilde{\varphi}(X^{(i)} Y^{(j)})$$

$$= N \cdot \varphi(XY) + N(N-1)\, \varphi_{1,1}(X,Y)$$

$$= N\big(\varphi(XY) - \varphi_{1,1}(X,Y)\big) + N^2 \varphi_{1,1}(X,Y)$$

class / $\mathbb{E}[\cdot]$ / Bar $= N \cdot (\varphi(XY) - \varphi(X)\varphi(Y))$

$K_2(X, Y)$

Prop $K_n(X_1 \dots X_n) = \sum_{\pi \in P(n)} \varphi_\pi (X_1 \dots X_n) \mu(\pi, \hat{1}_n)$

$$K_\pi = \sum_{\sigma \leq \pi} \varphi_\sigma \mu(\sigma, \pi)$$

Thm $A_1, A_2 \subseteq A$ indep

$\iff K_n(X_1 \dots X_n)$ vanishes

whenever $\exists\ \eta = \{I, J\}$ s.t. $\{X_i\}_I \subseteq A_1$

$\{X_j\}_J \subseteq A_2$

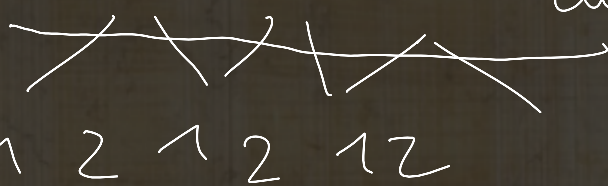
Another notion of indep:

monotone independence of Muraki:

$\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$ $\qquad X_j \in \mathcal{A}_1, \quad Y_j \in \mathcal{A}_2$

$$\varphi(X_1Y_1X_2Y_2\ldots) = \varphi(X_1X_2\ldots X_n)\varphi(Y_1)\varphi(Y_2)\ldots\varphi(Y_n)$$

cut peaks



asymmetric: $X \perp\!\!\!\perp Y \not\Rightarrow Y \perp\!\!\!\perp X$

$\longrightarrow$ order important

monotone convolution $\rhd$

$\Rightarrow$ $\mu \rhd \nu \neq \nu \rhd \mu$ unless $\mu = \nu$

$$(X,Y) \overset{d}{\sim} (X^{(1)}, Y^{(2)}) \sim (X^{(1)}, Y^{(3)})$$

$$\overset{d}{\not\sim} (X^{(2)}, Y^{(1)})$$

quasi-exchangeable or spreadable

<u>Def</u> <u>A spreadability system</u>

as before

$$\tilde{\varphi}(X^{(i_1)} \dots X^{(i_n)}) = \tilde{\varphi}(X^{(h(i_1))} \dots X^{(h(i_n))})$$

$\forall\, h : \mathbb{N} \to \mathbb{N}$ order preserving

Def $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$ are called ς-indep

if $\varphi_{\bar\pi}(X_1 \dots X_n) = \varphi_{\bar\pi \curlywedge \eta}(X_1 \dots X_n)$

ordered set partitions

$\pi \curlywedge \eta = (B_1 \cap C_1, B_1 \cap C_2, \dots B_1 \cap C_\ell, B_2 \cap C_1 \dots) \setminus \{\emptyset\}$

$\bar\pi = (B_1, \dots, B_k) \quad \eta = (C_1, \dots, C_\ell)$

Solomon-Tits alg.

Cumulants as before

$$K_n(X \dots X_n) = \sum_{\pi \in OSP(n)} \frac{(-1)^{|\pi|-1}}{|\pi|} \varphi_{\bar\pi}(X \dots X_n)$$

convolution non commutative

→ Lie algebra n.c.

mixed cumulants do not vanish

$$K_n(X_1 \dots X_n) = \sum_{\tau \le \eta} K_\tau(X_1 \dots X_n)\, g(\tau, \eta)$$

η is partition into independent subsets

$$K_6\left(X_1^{(3)} X_2^{(1)} X_3^{(2)} X_4^{(3)} X_5^{(1)} X_6^{(3)}\right)$$

$$\eta = 3\,1\,2\,3\,1\,3 = (25/3/146)$$

$$\tau = 1\,3\,5\,1\,4\,2 \simeq (14/6/2/5/3)$$

$$3\;3\;1\;1\;2$$

$$g(\tau, \eta) = \text{coeff of } X_3^2 X_1^2 X_2$$

$$\text{in } \log e^{X_1} e^{X_2} e^{X_3}$$

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realize Solomon-Tits alg $\subseteq WQSym^*$

annihat $\longleftrightarrow$ Euler idempotents