Skew-product systems over infinite interval exchange transformations

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Problem:

- Study the dynamics and ergodic properties of skew-product extensions of infinite interval exchange transformations.

Motivation:

- First return maps of flows on translation surfaces with wild singularities.

Methods:

- Symbolic dynamics.
- Theory of essential values.

Results:

- Non-ergodicity criteria.
- Estimates for discrepancy and diffusion coefficients.

The talk is based on:

H. Bruin, O. Lukina, *Skew-product systems over infinite interval exchange transformations*, preprint.

H. Bruin, O. Lukina, *Rotated odometers*, J. Lond. Math. Soc., 107 (2023), 1983–2024.

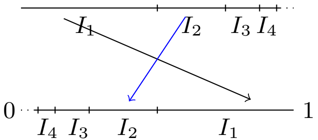
H. Bruin and O. Lukina, *Rotated odometers and actions on rooted trees*, Fundam. Math. **260** (2023), 233-249.

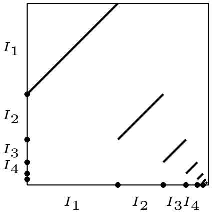
Von Neumann-Kakutani map (the dyadic odometer)

Let $I = [0,1)$. The **von Neumann-Kakutani map** $A : I \to I$

$$A(x) = x - (1 - 3 \cdot 2^{1-n}) \qquad \text{if } x \in [1 - 2^{1-n}, 1 - 2^{-n}),\ n \geq 1,$$

is an infinite IET.





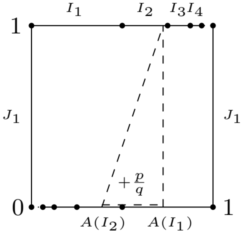
Graph of A

The map A has a countable number of discontinuities at the points

$$\{1 - 2^{-n} \mid n \geq 1\}.$$

Flows on Loch Ness monsters

A surface of infinite genus and a wild singularity **out of a square**:



Identify the edges using A

$\Longrightarrow$

The surface

- Has infinite genus
- Has one end
- Has finite area
- Has a **wild singularity** in the metric completion

The first return map of a flow of rational slope is described by an exchange of infinite number of subintervals in $I = [0, 1)$, called a **rotated odometer**.

Rotated odometers

Let $q \geq 2$, and consider a rotation: $x \mapsto x + p/q \mod 1$.

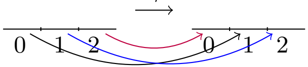
This induces: - a permutation π,
- a finite IET $R_\pi : I \to I$.

Given **any** permutation π of q symbols, the **rotated odometer** is an IET

$$F_\pi = A \circ R_\pi : I \to I.$$

Example:

$x \mapsto x + 1/3 \mod 1$



Permutation $\pi = (012)$

Theorem (Bruin, Lukina 2023)
Any rotated odometer can be realized as the first return map of a flow of rational slope on a translation surface with a wild singularity and possibly a finite number of cone angle singularities.

Dynamics of rotated odometers

Let π be a permutation of q symbols, and consider a rotated odometer

$$F_\pi = A \circ R_\pi : I \to I.$$

Some results (Bruin and Lukina 2023)

- A rotated odometer (I, F_π) may have intervals of periodic points (up to a countable number of periods).

- There is a unique aperiodic subsystem (I_{np}, F_π) with at most q invariant ergodic measures.

- (I_{np}, F_π) has the unique minimal set (I_{min}, F_π) which is uniquely ergodic.

- (I_{min}, F_π) and (I_{np}, F_π) may or may not have the dyadic odometer as a maximal equicontinuous factor.

Methods: Renormalization

Let $N = \min\{n \mid 2^{-n} < q^{-1}\}$,

and consider sections $L_k = [0, 2^{-kN})$, $L_0 = I$.

Theorem (Bruin, Lukina 2023)

Let $F_{\pi,k} : L_k \to L_k$, $k \geq 1$ be the first return maps. For $k \geq 1$:

1. There exist permutations π_k of q symbols,
2. and finite IET $R_{\pi,k} : L_k \to L_k$ such that

$$F_{\pi,k} = A_k \circ R_{\pi,k},$$

where A_k is a scaled copy of the von Neumann-Kakutani map A,

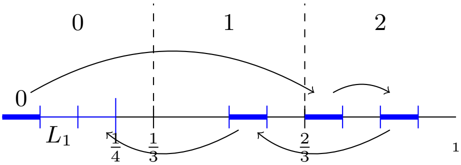
$$A_k(x) = \frac{1}{2^{kN}} A\left(2^{kN} x\right).$$

3. Moreover, the sequence $(\pi_k)_{k \geq 1}$ is pre-periodic.

Methods: Coding of orbits

Partition L_k into q intervals of equal length, call this partition $\mathcal{P}_{q,k}$.

Number the sets in $\mathcal{P}_{3,0}$ and in $\mathcal{P}_{3,1}$ from left to right.



Follow the orbit of a set of $\mathcal{P}_{3,1}$ in L_1 and record the number of the set in $\mathcal{P}_{3,0}$ visited by the orbit. This gives *substitution words*, for instance (see the picture)

$$0 \mapsto 0221$$

Methods: S-adic systems

For each $k \geq 1$, recording the set of $\mathcal{P}_{q,k-1}$ in L_{k-1} visited by the orbit of an interval in $\mathcal{P}_{q,k}$ under F_{k-1}, we obtain substitutions

$$\chi_k(i),\, 0 \leq i \leq q-1 \,\text{with alphabet}\, \mathcal{A} = \{0, 1, \ldots, q-1\}.$$

Lemma

The sequence $\{\chi_k\}_{k\geq 1}$ is eventually periodic.

Thus we can pass to an eventually constant sequence of substitutions, and restrict to the study of *stationary* sequences, i.e.

$$\chi_k = \chi_1 = \chi \quad \text{for all} \quad k \geq 1.$$

Substitution

A **substitution** $\chi : \mathcal{A} \to \mathcal{A}^*$ assigns to every $a \in \mathcal{A}$ are word $\chi(a) \in \mathcal{A}^*$.

This extends to $\mathcal{A}^*$ and Σ by concatenation:

$$\chi(b_1 \cdots b_r) = \chi(b_1)\chi(b_2) \cdots \chi(b_r).$$

Example: $\mathcal{A} = \{0, 1, 2\}$,

$$\chi(0) = 0221, \quad \chi(1) = 0221, \quad \chi(2) = 0011.$$

The associated matrix of χ is

$$M = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

Remark: For many rotated odometers, M is not primitive, and so there is no fixed point for χ. However, we can still use substitution words to code the dynamics of arbitrary long pieces of orbits.

Skew-products over rotated odometers

Let $F_\pi : I \to I$ be a rotated odometer, and let μ be an ergodic invariant measure.

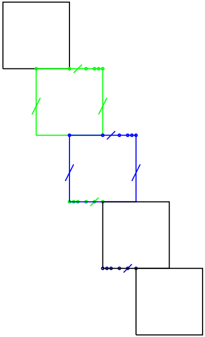
Define a *skew-function* by

$$\psi(x) = \begin{cases} 1, & 0 \leq x < \frac{1}{2}, \\ -1, & \frac{1}{2} \leq x < 1. \end{cases}$$

The skew-product of F_π and ψ is given by

$$T_\pi : I \times \mathbb{Z} \to I \times \mathbb{Z}, \qquad (x, n) \mapsto (F_\pi(x), n + \psi(x)),$$

with invariant measure $\mu \otimes \nu$, where ν is the counting measure on $\mathbb{Z}$.



Then $(I \times \mathbb{Z}, T_\pi, \mu \otimes \nu)$ is the first return map for a lifted flow (or rational slope) on a staircase, which is a $\mathbb{Z}$-to-one cover of the infinite genus surface of finite area that we considered earlier.

Results (Bruin and Lukina, 2024):

1. Two criteria for non-ergodicity of the skew-product $(I \times \mathbb{Z}, T_\pi, \mathrm{Leb} \otimes \nu)$.
2. An estimate on the discrepancy of the orbit of a typical point of (I, F_π, Leb).
3. An estimate on the diffusion coefficient of $(I \times \mathbb{Z}, T_\pi, \mathrm{Leb} \otimes \nu)$.

To answer 1, we use theory of essential values, see:

K. Schmidt, *Cocycles on ergodic transformation groups.* Macmillan Lectures in Mathematics, **Vol. 1**. Macmillan Co. of India, Ltd., Delhi, 1977.

Cocycles and essential values

Given a rotated odometer (I, F_π, μ), and a skew-function $\psi : X \to \mathbb{Z}$, define a function $\Psi : X \times \mathbb{Z} \to \mathbb{Z}$ by

$$\Psi(x,n) = \begin{cases} \psi(F_\pi^{n-1}x) + \cdots + \psi(F_\pi(x)) + \psi(x), & n \geq 1 \\ 0, & n = 0, \\ -\psi(F_\pi^{n}(x)) - \cdots - \psi(F_\pi^{-1}(x)), & n \leq -1. \end{cases}$$

Then Ψ is a *cocycle*, i.e. it satisfies:

- For every $n_1, n_2 \in \mathbb{Z}$ and every $x \in X$:

$$\Psi(x, n_1 + n_2) = \Psi(F_\pi^{n_2}(x), n_1) + \Psi(n_2, x). \tag{1}$$

- $\mu\left(\bigcup_{n\in\mathbb{Z}} (\{x : F_\pi^n(x) = x\} \cap \{\Psi(x,n) \neq 0\})\right) = 0.$

Denote the partial unweighted ergodic sums by

$$S_n\psi(x) = \sum_{i=0}^{n-1} \psi(F_\pi^i(x)).$$

Definition (Essential values)

The element $e \in \mathbb{Z}$ is an *essential value* of the cocycle Ψ if and only if for every positive measure Borel set A there exists an $n \in \mathbb{Z}$ such that

$$\mu\left(A \cap F_\pi^{-n}(A) \cap \{x \in X : S_n\psi(x) = e\}\right) > 0. \tag{2}$$

Remark: 0 is always an essential value, with $n = 0$.

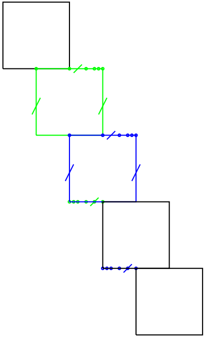
0 may also be a non-trivial essential value if the above definition holds for all positive measure sets with $n \neq 0$, n depending on a set.

Recurrence

We are interested in the dynamics of the flow on the staircase:

- Do all flow lines go to infinity, or do they stay within a bounded subsurface on the staircase?
- If the lines go to infinity, do they ever return to where they started, and how often?

So we may ask if the system $(I \times \mathbb{Z}, T_\pi, \mu \otimes \nu)$ is *recurrent* to the section $I \times \{0\}$.



Since the measure $\mu \otimes \nu$ is infinite, the Poincaré Recurrence Theorem does not apply; instead, one asks if 0 is a **non-trivial** essential value of (I, F_π, μ).

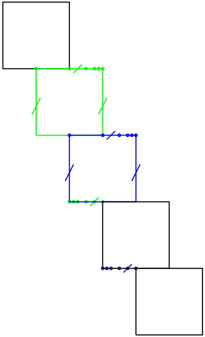
Transience

To formally describe trajectories escaping to infinity, consider the one-point compactification

$$\mathbb{Z}^* = \mathbb{Z} \cup \{\infty\}.$$

Then ∞ is an essential value for the cocycle Ψ (with associated skew-function ψ) if for every $N \in \mathbb{N}$, and every positive Borel measure set A there exists an $n \in \mathbb{Z}$ such that

$$\mu\left(A \cap F^{-n}(A) \cap \{x \in X : |S_n\psi(x)| \geq N\}\right) > 0.$$



Ergodicity of skew-products

Denote by $E^*(\Psi) \subseteq \mathbb{Z}^*$ the set of essential values for cocycle Ψ.

We will use the following theorem which is well-known:

Theorem

Let $F : X \to X$ be an ergodic transformation with probability measure μ, and let Ψ be a cocycle for the skew-product T. Then:

1. $E^*(\Psi)$ is a closed non-empty subset of $\mathbb{Z}^*$,
2. $E(\Psi) = E^*(\Psi) \cap \mathbb{Z}$ is a closed subgroup of $\mathbb{Z}$,
3. T is ergodic if and only if $E(\Psi) = \mathbb{Z}$.

Recurrence of rotated odometers

We will concentrate on the case when a rotated odometer (I, F_π) has no periodic points, and so Lebesgue measure Leb is ergodic.

In this case recurrence follows from the result of Atkinson 1976:

Theorem

Let (X, F) be an ergodic transformation with ergodic probability measure μ. Then the skew-function $\psi : X \to \mathbb{Z}$ has integral $\int_X \psi \, d\mu = 0$ if and only if the corresponding skew-product is recurrent.

Corollary: Let (I, F_π, Leb) be a rotated odometer with ergodic Lebesgue measure. Then 0 is a non-trivial essential value of the skew-product $(I \times \mathbb{Z}, T_\pi, \mathrm{Leb} \otimes \nu)$ with skew-function ψ as above.

Criteria for non-ergodicity

Theorem (Bruin, Lukina 2024)

Let (I, F_π, Leb) be a rotated odometer with ergodic Lebesgue measure, and consider the skew-product $(I \times \mathbb{Z}, T_\pi, \mathrm{Leb} \otimes \nu)$ with skew-function ψ as before.

Let χ be the associated substitution with alphabet $\mathcal{A} = \{0, \ldots, q\}$, and

$$\mathbf{d} := \gcd\{\psi(\chi(a)) : a \in \mathcal{A}\}.$$

Then the subgroup $E(\Psi)$ of essential values is contained in $\mathbf{d}\mathbb{Z}$. In particular, if $\mathbf{d} > 1$, then 1 is not an essential value.

Remark: Criteria for non-ergodicity of $\mathbb{Z}^d$-extensions of finite IETs appear, for instance, in Fraczek and Hubert 2018, Fraczek and Ulcigrai 2014.

Example:

$\pi = (1,7,4)(2,5)(3,6)$

$q = 9$

$\#\{\text{ergodic measures}\} = 1$

$E^*(\Psi) \subset 2\mathbb{Z} \cup \{\infty\}$

$$M = \begin{pmatrix} 4 & 3 & 1 & 2 & 0 & 1 & 3 & 4 & 1 & 3 \\ 3 & 6 & 3 & 3 & 1 & 1 & 3 & 3 & 2 & 3 \\ 1 & 2 & 3 & 3 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 3 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 3 & 3 & 2 & 1 & 1 & 0 & 4 & 3 & 1 & 4 \end{pmatrix}$$

$a \in \mathcal{A}$	0	1	2	3	4^+	4^-	5	6	7	8
Weight $\psi(\chi(a))$	-2	4	4	4	-2	-2	-2	-2	-2	-2

Theorem (Bruin, Lukina 2024)

Let (I, F_π, Leb) be a rotated odometer with ergodic Lebesgue measure, and consider the skew-product $(I \times \mathbb{Z}, T_\pi, \mathrm{Leb} \otimes \nu)$ with skew-function ψ as before.

Assume that all eigenvalues λ_j of the associated matrix M of χ with norm $|\lambda_j| \geq 1$ have weights $\psi(\ell_j) = 0$, with the exception of one, say λ_c, which is Pisot.

Suppose the algebraic and the geometric multiplicities of λ_c are equal.

If ψ is not a coboundary, then the essential values $E^*(\Psi) = \{0, \infty\}$, with 0 non-trivial essential value.

Example:

$\pi = (1,7,4)(2,5)(3,6)$

$q = 9$

$\#\{\text{ergodic measures}\} = 1$

$E^*(\Psi) = \{0, \infty\}$

$$M = \begin{pmatrix} 4 & 3 & 1 & 2 & 0 & 1 & 3 & 4 & 1 & 3 \\ 3 & 6 & 3 & 3 & 1 & 1 & 3 & 3 & 2 & 3 \\ 1 & 2 & 3 & 3 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 3 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 3 & 3 & 2 & 1 & 1 & 0 & 4 & 3 & 1 & 4 \end{pmatrix}$$

$a \in \mathcal{A}$	0	1	2	3	4^+	4^-	5	6	7	8
Weight $\psi(\chi(a))$	-2	4	4	4	-2	-2	-2	-2	-2	-2

Characteristic polynomial: $(x-16)(x-4)^2(x-1)^2x^5$

Eigenvalues	16	$4(\times 2)$	$1(\times 2)$	$0(\times 5)$
Weights of left eigenvectors	0	$-\frac{1}{2}, 1$	$0, 0$	$**$

Discrepancy

Discrepancy of sequences arising as fixed points of primitive substitutions, and fixed points in S-adic sequences, were studied in Adamczewski 2004, Berthé and Delecroix 2014.

For rotated odometers, the substitution matrices are often not primitive.

Theorem (Bruin and Lukina 2024)

Suppose that Leb is ergodic for a stationary rotated odometer (I, F_π).

Suppose the matrix associated to the substitution χ is diagonalizable, and the largest eigenvalue λ_0 has multiplicity 1.

Then for Lebesgue-a.e. x, there is $C = C_x$ such that the F_π-orbit of x has discrepancy

$$\mathfrak{D}_R \leq C_x \cdot R^{\gamma_0 - 1},$$

where $\gamma_0 := \max\left\{\frac{\log|\lambda_1|}{\log \lambda_0}, 0\right\}$.

Open problems

1. Find examples of weakly mixing rotated odometers, or prove that they do not exist.
2. Develop further criteria for (non)-ergodicity of skew-products over rotated odometers.
3. Consider skew-products of rotated odometers with different skew-functions.

References

- B. Adamczewski, *Symbolic discrepancy and self-similar dynamics.* Ann. de l'Institut Fourier, **54** (2004), 2201–2234.
- V. Berthé, V. Delecroix, *Beyond substitutive dynamical systems: S-adic expansions.* RIMS Lecture note "Kokyuroku Bessatu" B46 (2014), 81–123.
- H. Bruin and O. Lukina, *Rotated odometers*. J. Lond. Math. Soc., 107 (2023), 1983-2024.
- H. Bruin and O. Lukina, *Rotated odometers and actions on rooted trees*. Fundam. Math. **260** (2023), 233-249.
- H. Bruin and O. Lukina, *Skew-product systems over infinite interval exchange transformations*. arXiv:2408.14079.
- K. Fraczek, P. Hubert,*Recurrence and non-ergodicity in generalized wind-tree models.* Math. Nachr. **291** (2018), 1686–1711.
- K. Fraczek, C. Ulcigrai, *Non-ergodic $\mathbb{Z}$-periodic billiards and infinite translation surfaces.* Invent. Math. **197** (2014), 241–298.
- K. Schmidt, *Cocycles on ergodic transformation groups.* Macmillan Lectures in Mathematics, **Vol. 1**. Macmillan Co. of India, Ltd., Delhi, 1977.

Thank you for your attention!