

STRICT REVERSE MATHEMATICS/3

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This third talk concerns the development of SRM for $\aleph$ based real analysis. This is a wilderness in the sense that there is no detailed set of well established benchmarking formal systems that have been developed for this purpose like the well known subsystems of Z_2 and the well known subsystems of PA. Generally speaking, formalized real analysis for reverse mathematics purposes has been done via codings back into $L[Z_2]$, and thereby considered as part of traditional RM. This is definitely not the right way to go about any kind of SRM for $\aleph$ based real analysis.

We have seen that SRM anchored in two mathematical realms which already have well studied formal systems can be sufficiently well defined to support active investigations. Most succinctly, given a target non mathematical standard system, and given a strictly mathematical language, find a finite set of strictly mathematical theorems in that language that corresponds closely to that target. Also develop strictly mathematical base theories and obtain reversals over them. In the realm of ω based countable mathematics, we have seen that we can basically import the whole of RM with base theory RCA_0 to SRM with base theory ETF . However, in the realm of Z based finite mathematics, we can't just import finite RM based on fragments of PA, but have to introduce finite sets or finite sequences of integers with carefully designed strictly mathematical base theories (FSTZ, FSQZ), and solve some internalization issues.

There are a few major relationships between theories that we encounter in SRM (and RM).

1. T is interpretable in S .
2. S, T are mutually interpretable.
3. T is a regular extension of S if and only if T proves S and there is an interpretation of T into S which is the identity on $L[S]$.
4. T is a faithful extension of S if and only if there is a faithful interpretation of T into S which is the identity on $L[S]$.

4 is what we strive for most in SRM developments. We saw 4 throughout lecture 1 in the SRM of ω based countable mathematics. We saw both 4 and the weaker 3 in lecture 2 in the SRM of $\mathbb{Z}$ based finite mathematics.

We now come to a vitally important wilderness - the realm of $\mathbb{R}$ based analysis.

MAJOR OVERALL GOAL. Take a book on real analysis like

Real Analysis, H.L. Royden, Macmillan, 1968.

Through SRM formalization, give a detailed logical analysis. Example findings might look like this:

The SRM of (such and such portions of or modifications of) the book forms a finite set of logically presented strictly mathematical theorems from the book which forms a faithful extension of,
regular extension of, interpretable in, or mutually interpretable with one of RCA_0 , WKL_0 , ACA_0 , ATR_0 , $\Pi^1_1\text{-CA}_0$ or other standard formal system

Work roughly along this line includes

G. Takeuti, Two applications of logic in mathematics, Princeton University Press, published 2015, copyright 1978, 148 pp.

"In Part Two, he develops classical analysis including complex analysis in Peano's arithmetic, showing that any arithmetical theorem proved in analytic number theory is a theorem in Peano's arithmetic. In doing so, the author applies Gentzen's cut elimination theorem." - Princeton University Press.

The rest of this talk is to give some illustration as to how such an SRM development might begin to take shape. I have not actually seriously engaged in such a project above, with a real real analysis text, so I am wandering in the wilderness.

We start with the motivating one sorted structure

$(\mathbb{R}, <, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, 0, 1, -, +, \cdot)$, unary $-$. Thus we are viewing $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, where all three are taken as primitive. We use the standard axioms for this structure, which includes that every element of $\mathbb{Z}$ is isolated.

I think there is an interesting completeness theorem here. First the wrong completeness "theorem".

WRONG. The set of all true universal sentences in $(\mathbb{R}, <, N, Z, Q, 0, 1, -, +, \cdot)$ is decidable with a nice axiomatization.

Of course we are looking for a nice axiomatization, but this isn't even decidable by H10.

CORRECT? The set of all simple true universal sentences in $(\mathbb{R}, <, N, Z, Q, 0, 1, -, +, \cdot)$ is very decidable with a nice axiomatization.

Universally quantified propositional combinations of atomic formulas where there is at most one atomic formula without N, Z, Q , and that atomic formula is linear or quadratic? E.g., Carl Ludwig Siegel proved that quadratic Diophantine equations over Z are decidable. But nice axiomatization?? The decidability was also done over Q . In any case here "simple" will make nice axiomatizations very likely.

We can push this idea of "simple true universal" and even more aggressively "simple true (not necessarily universal)" further as we go along to give principled versions of complex axiomatizations.

Moving on, we need finite sequences. We do this by moving to the 2-sorted structure with sorts $\mathbb{R}$ and $\mathbb{R}^*$. We have to add additional primitives. The most obvious ones are

1. Constant $< >$ in $\mathbb{R}^*$.
2. lth from $\mathbb{R}^*$ into N . All values fall under predicate N .
3. Injection from $\mathbb{R}$ into $\mathbb{R}^*$.
4. Concatenation from $\mathbb{R}^*, \mathbb{R}^*$ into $\mathbb{R}^*$.
5. Application from $\mathbb{R}^*, \mathbb{R}$ into $\mathbb{R}$. This is coordinate extraction, and undefined unless the second argument lies in N^+ .

These symbols come with obvious axioms which I won't spell out here. Most interesting is $lth(cat(x_{\mathbb{R}^*}, y_{\mathbb{R}^*})) = lth(x_{\mathbb{R}^*}) + lth(y_{\mathbb{R}^*})$. Simple true treatment again?

So now the intended interpretation is

$$(\mathbb{R}, \mathbb{R}^*, <_{\mathbb{R}}, N_{\mathbb{R}}, Z_{\mathbb{R}}, Q_{\mathbb{R}}, 0_{\mathbb{R}}, 1_{\mathbb{R}}, -_{\mathbb{R} \rightarrow \mathbb{R}}, +_{\mathbb{R} \rightarrow \mathbb{R}}, \cdot_{\mathbb{R} \rightarrow \mathbb{R}}, \\ <>_{\mathbb{R}^*}, lth_{\mathbb{R}^*}, \mathbb{R} \rightarrow \mathbb{R}, inj_{\mathbb{R} \rightarrow \mathbb{R}^*}, cat_{\mathbb{R} \rightarrow \mathbb{R}}, app_{\mathbb{R} \rightarrow \mathbb{R}})$$

But we need a sort for partial functions from $\mathbb{R}^*$ into $\mathbb{R}^*$, something allowed by our rich underlying logic (free many sorted logic with partial function variables and relativized quantifiers). Actually this will ruin the plan to have a faithful extension of RCA_0 , as this would require us making false mathematical statements that limit the partial functions from $\mathbb{R}^*$ into $\mathbb{R}^*$. (But we can later relativize to certain ones and still have faithful extensions). Instead we use a sort for partial functions from $\mathbb{Q}^*$ into $\mathbb{R}^*$. For this to be allowed in our underlying logic, we need $\mathbb{Q}^*$ to either be a sort or a (unary) relation symbol. We choose the latter. Then it is natural to add new unary predicates N^*, Z^*, Q^* on sort $\mathbb{R}^*$, arriving at the intended interpretation

$$(\mathbb{R}, \mathbb{R}^*, \text{PF}(\mathbb{Q}^*_{\mathbb{R}^*}, \mathbb{R}^*), <_{\mathbb{R}}, N_{\mathbb{R}}, Z_{\mathbb{R}}, Q_{\mathbb{R}}, N^*_{\mathbb{R}^*}, Z^*_{\mathbb{R}^*}, Q^*_{\mathbb{R}^*}, 0_{\mathbb{R}}, 1_{\mathbb{R}}, \\ -_{\mathbb{R} \rightarrow \mathbb{R}}, +_{\mathbb{R} \rightarrow \mathbb{R}}, \cdot_{\mathbb{R} \rightarrow \mathbb{R}}, <>_{\mathbb{R}^*}, \text{lth}_{\mathbb{R}^* \rightarrow \mathbb{R}}, \text{inj}_{\mathbb{R} \rightarrow \mathbb{R}^*}, \text{cat}_{\mathbb{R}, \mathbb{R} \rightarrow \mathbb{R}}, \\ \text{app}_{\mathbb{R}, \mathbb{R} \rightarrow \mathbb{R}}, \text{val}_{\text{PF}(\mathbb{Q}^*_{\mathbb{R}^*}, \mathbb{R}^*), \mathbb{Q}^*_{\mathbb{R}^*} \rightarrow \mathbb{R}^*})$$

with the axioms that these three new unary predicates carve out the elements of $\mathbb{R}^*$ all of whose coordinates fall under N, Z, Q , respectively, using application (app). Also see the $\text{val}_{\text{PF}(\mathbb{Q}^*_{\mathbb{R}^*}, \mathbb{R}^*), \mathbb{Q}^*_{\mathbb{R}^*} \rightarrow \mathbb{R}^*}$ which maps $\text{PF}(\mathbb{Q}^*_{\mathbb{R}^*}, \mathbb{R}^*), \mathbb{Q}^*_{\mathbb{R}^*}$ into $\mathbb{R}^*$, for the action of the elements of the new sort $\text{PF}(\mathbb{Q}^*_{\mathbb{R}^*}, \mathbb{R}^*)$. This is the environment for what we call sequential analysis.

We are finally in a position to lay down the axioms that come to grips with the real analysis. First of all, we need to import ETF.

1. We need initial axioms and composition here much like we had in ETF. Successor axioms are not needed because we have the ordered ring axioms with discrete $\mathbb{Z}$. Sort ω in ETF is identified with $N_{\mathbb{R}}$ here. 1,2,3-ary functions in ETF are identified with elements of $\text{PF}(\mathbb{Q}^*_{\mathbb{R}^*}, \mathbb{R}^*)$ that map $N_{\mathbb{R}}, N_{\mathbb{R}^2}, N_{\mathbb{R}^3}$ into $\mathbb{R}^*$, respectively. Also the f in $\text{PF}(\mathbb{Q}^*_{\mathbb{R}^*}, \mathbb{R}^*)$ correspond exactly to certain g in $\text{PF}(\mathbb{Q}^*_{\mathbb{R}^*}, \mathbb{R})$ via val and inj .

A Cauchy sequence is an $f: \mathbb{N} \rightarrow \mathbb{R}$ such that there exists $g: \mathbb{N} \rightarrow \mathbb{N} \setminus \{0\}$ with for all $m, n > r$, $|f(m) - f(n)| < 1/g(r)$.

2. Every Cauchy sequence converges.

3. Every element of $\mathbb{R}$ is the limit of a Cauchy sequence $f: \mathbb{N} \rightarrow \mathbb{Q}$.

We need a form of 2,3 with uniformity.

A g -Cauchy sequence is an $f:N \rightarrow Q$ such that $g:N \rightarrow N \setminus \{0\}$ and for all $m, n > r$, $|f(m) - f(n)| < 1/g(r)$.

4. Let $f:N \rightarrow \mathbb{R}$. There exists $g:N^2 \rightarrow Q$ and $h:N^2 \rightarrow N \setminus [0]$ such that for all n , $(g)_n$ is an $(h)_n$ -Cauchy sequence which limit $f(n)$.

5. Let $g:N^2 \rightarrow \mathbb{R}$ and $h:N^2 \rightarrow N$ be such that for all n , $(g)_n$ is an $(h)_n$ -Cauchy sequence. There exists $f:N \rightarrow \mathbb{R}$ such that for all n , $f(n)$ is the limit of $(g)_n$.

This system now pins down $\mathbb{R}$ and $PF(Q^*, \mathbb{R}^*)$ in the right way so that we now have a faithful extension of RCA_0 which does some basic real analysis.

But what about having $PF(\mathbb{R}^*, \mathbb{R}^*)$ or even just $PF(\mathbb{R}, \mathbb{R})$? For venturing into such third order contexts, we actually should begin with $S(\mathbb{R}^*)$.

Here to have a faithful extension of RCA_0 or even $\Pi^1_1\text{-}CA_0$, we need to pin down $S(\mathbb{R}^*)$. This involves restricting $S(\mathbb{R}^*)$.

The natural restriction in this context is, on the intended interpretation, the Borel subsets of $\mathbb{R}^*$. But this is too big a leap at this point in the SRM development. Far too much SRM machinery is needed to take the plunge into arbitrary Borel subsets of $\mathbb{R}^*$ at this point. So we adopt the incremental approach, treating increasingly large families of subsets of $\mathbb{R}$ incrementally, stopping to look at strictly mathematical statements that arise, their logical relationships, and reversals.

A very important target is the semi algebraic subsets of the $\mathbb{R}^n$. Expect all standard theorems in this context to be provable in our present base theory.

The idea now is to climb slowly and carefully up the low levels of the Borel hierarchy of subsets of $\mathbb{R}^*$, starting with countable sets and open sets. Investigate the status of all of the relevant mathematical theorems from descriptive set theory, establishing implications and also proofs that certain combinations are faithful extensions of the standard benchmarks from RM. Limit points, closures, interiors, boundaries,

exteriors. Also condensation points (of closures) without introducing third order notions.

Then treat the n -th level of the Borel hierarchy of subsets of $\mathbb{R}^*$, for arbitrary n . Most naturally, this is done using trees where the leaves are assigned open intervals in $\mathbb{R}$ with rational endpoints, given by those endpoints, and $\cup$ splitting indicates union and 1 -splitting (no splitting) indicates complement, or variants thereof. Then we introduce "real x lies at node u " as a primitive with associated axioms. This avoids introducing third order notions.

Then move to the full transfinite Borel hierarchy via well founded trees. Move cautiously into third order by comparing the transfinite Borel hierarchy with the characterization of Borel sets as the least sigma algebra containing the intervals.

Another major challenge is to work out the SRM of

Classical descriptive set theory, A. Kechris, Springer-Verlag, 1991.