# STRICT REVERSE MATHEMATICS/1

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A considerable portion of these two talks is based on

[Fr21] The emergence of (strict) reverse mathematics, https://u.osu.edu/friedman.8/foundational-adventures/downloadable-manuscripts/, 116, 110 pages, December 19, 2021.

This first talk focuses on SRM for ω based countable mathematics. The largest part of current RM addresses what I call ω based countable mathematics. The second talk focuses on SRM for Z based finite mathematics and $\mathfrak{R}$ based analysis.

1. Origins of strict reverse mathematics
2. What does strictly mathematical mean?
3. SRM for $Z_2$ fragments
4. SRM for the wilderness
5. The system ETF
6. Research programs for ETF
7. SRM for $RCA_0$
8. SRM for $WKL_0$
9. SRM for $ACA_0$
10. SRM for $\Pi^1_1$-$CA_0$
11. SRM for $ATR_0$

## 1. ORIGINS OF STRICT REVERSE MATHEMATICS

My original conception of RM was really what I now call Strict Reverse Mathematics. RM was my compromise put forward in the mid 1970's to facilitate a clear development of a new area in the foundations of mathematics. Much earlier, in the late 1960's, I wanted to impress the mathematicians at the Stanford University

**mathematics teas by showing how the beloved formal systems of mathematical logic are intrinsic and form permanent fixtures of the mathematical landscape.**

**In order to achieve this, I could not simply start discussions with a formal system with nonlogical axioms. The idea that principal formal systems of mathematical logic are in an appropriate sense "equivalent" to certain mathematical theorems was clearly envisioned and expressed. For this, one cannot really start with any formal system. Of course, in contrast, RM starts with a formal system right from the beginning, normally $RCA_0$.**

**We can see my early attempts to carry out this strict reverse mathematics program in the manuscripts available on my downloadable manuscripts page https://u.osu.edu/friedman.8/foundational-adventures/downloadable-manuscripts/**

**The Analysis Of Mathematical Texts, And Their Calibration In Terms Of Intrinsic Strength I, April 3, 1975, 7 pages.**
**The Analysis Of Mathematical Texts, And Their Calibration In Terms Of Intrinsic Strength II, April 8, 1975, 5 pages.**
**The Analysis Of Mathematical Texts, And Their Calibration In Terms Of Intrinsic Strength III , May 19, 1975, 26 pages.**
**The Analysis Of Mathematical Texts, And Their Calibration In Terms Of Intrinsic Strength IV, August 15, 1975, 32 pages.**
**The Logical Strength Of Mathematical Statements, October 15, 1975, 1 page.**
**The Logical Strength Of Mathematical Statements I, August, 1976, 20 pages.**

**In the first of the above, I wrote "In 1969, I discovered that a certain subsystem of second order arithmetic based on a mathematical statement (that every perfect tree which does not have at most countably many paths, has a perfect subtree), was provably equivalent to a logical principle (the weak $\Pi^1_1$ axiom of choice) modulo a weak base theory (comprehension for arithmetic formulae)"**

**But the emphasis in these six manuscripts was on giving a strictly mathematical base theory for $\mathfrak{R}$ based real analysis and reverse a myriad of strictly mathematical theorems to various standard formal systems in the sense of logical strength. Actually this calibration program can be traced back to my publication about the high strength of Borel Determinacy:**

Friedman, H. [1971], Higher set theory and mathematical practice, Annals of Mathematical Logic 2, 325-357.

In a way, the aim of these six writings were actually more ambitious than present SRM in this sense. I was talking about treating raw mathematical text. Raw mathematical text proceeds only at most semiformally. Considerable amount of logical structure is only "implied". However, it is possible in the future that needed logical structure could be added to raw text using AI, based on an appropriate inventory of examples of "implied" logical structure. This would bring the raw text vision alive.

Already by 1974 I was working with the compromise of using a formal system for the base theory with my two manuscripts founding RM as we know it today.

[Fr75] Some Systems of Second Order Arithmetic and Their Use, Proceedings of the 1974 International Congress of Mathematicians, Vol. 1, (1975), pp. 235-242
[Fr76] Subsystems of Second Order Arithmetic with Restricted Induction I,II, abstracts, J. of Symbolic Logic, Vol. 41, No. 2 (1976), pp. 557-559, RMAbsracts1976

with provable equivalences with formal systems.

However I still had the SRM idea very much in mind in [Fr76]. There you see an axiomatization of $RCA_0$ using functions only, proposed as a base theory, but also a strictly mathematical system I called ETF (elementary theory of functions). I claimed that ETF is equivalent to $RCA_0$ without proof. I regard this as the founding operational moment of SRM.

As $RCA_0$ with its schemes were readily accepted for RM by the interested researchers, I did not bother to back up that claim. At least until I wrote [Fr21] where a full proof appears.

Soon after [Fr75],[Fr76], the RM program took off mostly using the set version of $RCA_0$ in its current form. In RM practice, one generally uses $RCA_0$ with both sets and functions of several variables.

## 2. WHAT DOES STRICTLY MATHEMATICAL MEAN?

We have always maintained from the outset that although a desired precise definition of "strictly mathematical" cannot be given in the foreseeable future, the notion can be made precise enough so that a great deal of relevant f.o.m. and rich mathematical logic can be productively motivated by it. Certainly it is (at this time) much more precise than notions like "important mathematics", "beautiful mathematics", "simple proof", and so forth.

A precise guide to whether a concept is strictly mathematical can be obtained by counting the number of references to that concept found on the internet. Of course, there one must be searching in and around that concept, not some particular inflexible embodiment of that concept. Another guide is the "diversity" of the references under various cultural measures. For example "range of a function" not only has a huge number of references, but also great diversity of the (sources of) references.

At this early point in the development of SRM, it is not important to dwell on what "strictly mathematical" means until a significant dispute arises. But we do need to emphasize that the richer concept of "more mathematical" plays an important role - perhaps a more important role - and there one expects even more disputes. However, this is to be expected, akin to "harder", "deeper", "simpler", "more important", "more beautiful" that is in common use in mathematical culture (and literature). Mathematics develops just fine with this murkiness, and so will SRM.

## 3. SRM FOR $Z_2$ FRAGMENTS

In this section, we will focus on the goals for SRM applied to finite fragments of $Z_2$. We assume $L[Z_2]$ is the official language from

[Si09] S.G. Simpson, Subsystems of second order Arithmetic, Perspectives in Mathematical Logic, 2009.

which is $(\omega,S(\omega),0,1,+,\cdot,<,\in,=)$ with sorts $\omega,S(\omega)$ and = only on sort $\omega$.

SRM/1. Let S be a notable finitely axiomatized fragment of $Z_2$. We look for a logically equivalent formal system T consisting entirely of finitely many strictly mathematical theorems. We continue to look for alternative T which are simpler, shorter, more mathematical, or more interesting.

Unfortunately, none of the commonly studied fragments of $Z_2$ seem to be treatable in this way. This is because $L[Z_2]$ is just too primitive to support this.

HOWEVER, SRM/1 is very much viable if we formulate $Z_2$ with functions of several variables, or with both sets and functions of several variables. We will explore SRM/1 for the famous five formulated augmented with 1,2,3-ary functions.

For Z2 with its usual language, we need to weaken SRM/1 as follows:

SRM/2. Let S be a notable finitely axiomatized fragment of $Z_2$. We look for a finite set T of strictly mathematical statements with a faithful interpretation of T into S that is the identity on L[S]. We continue to look for alternative T in simpler languages, with shorter, more mathematical, or more interesting axioms.

Note that $L[S] \subseteq L[T]$ is required since $\pi$ is the identity on L[S]. Faithful here means that $\varphi$ is provable in T if and only if $\pi(\varphi)$ is provable in S.

We shall see that in the examples of SRM/2 given here for the famous five, the faithful interpretations of T in S are really "definitional". However, there is no clear treatment of definitional extensions in many sorted free logic with function and relation variables - our official underlying logic - in common use, and we are not sure what exact forms the interpretations are going to take as SRM gets fully developed. Perhaps we may even run into interpretations that are not faithful, but in any case we want that T proves S and we have an interpretation of T into S which is the identity on L[S].

It is easier and sometimes trivial to carry out SRM/2 if we allow S to use a sufficiently generous expansion of the language of $Z_2$. For treating fragments of $Z_2$, the challenge is to adhere to L[ETF] and modest extensions thereof.

## 4. SRM FOR THE WILDERNESS

In the realm of $\omega$ based countable mathematics we clearly have a large number of notable finitely axiomatized formal systems within $Z_2$, including the famous five and many more.

**We also have a large number of notable finitely axiomatized systems within PA in the realm of Z based finite mathematics. We discuss SRM for these in the second lecture.**

**However, in the realm of $\Re$ based real analysis, we do not really have notable formal systems. The closest we come to them is through the usual RM systems where reals are treated by coding, with rival systems of coding studied. Mathematicians greatly prefer to treat $\Re$ as primitive, not wedded to one particular definition, knowing that there are multiple interpretations, and then worry about things when it is time to worry about things, which will be never.**

**What we do have is tons of strictly mathematical theorems based on $\Re$. In SRM, we take the strictly mathematical approach of treating $\Re$ as a sort with the ordered field of real numbers as axioms. This is very different from choosing some particular coding of reals as Cauchy sequences or Dedekind cuts. I will talk about this in the third lecture.**

**Another important area of mathematical wilderness is where we look at mathematics that is proved where no clear tangible logical principle is being used. There is just special arguments and special constructions. Here the usual non mathematical elements used in the usual formal systems are just too crude for RM for this realm, and SRM is required, where the relevant mathematical theorems are taken as stated.**

**We formulate the goals of SRM in the wilderness as follows.**

**SRM for wilderness. In a rich mathematical realm where there are no generally accepted ready made formal systems, develop formal systems consisting entirely of strictly mathematical theorems, which provide a transparent logical organization of that rich mathematical realm through logical implications, logical equivalences, interpretations, conservative extensions, and the like.**

## 5. THE SYSTEM ETF

**ETF was introduced in [Fr76]. The language of ETF has four sorts. One ranging over $\omega$, and the others ranging over 1-ary, 2-ary, 3-ary functions from $\omega$ into $\omega$. We have the constant 0 of sort $\omega$, 1-ary function symbol S on $\omega$, and = between terms of sort $\omega$ only, with the usual underlying logic. The axiom names are:**

1. Successor Axioms.
2. Initial Function Axioms.
3. Composition Axioms.
4. Primitive Recursion Axiom.
5. Permutation Axiom.
6. Rudimentary Induction Axiom.

Here is the detailed axiomatization given in section 6 of [Fr21] which is a slight simplification of the original ETF of [Fr76] (in ways documented in [Fr21]). For specificity below, n,m,r are the first three variables over sort ω (as they were in [Fr76]).

1. SUCCESSOR AXIOMS
i. $S(n) \neq 0$
ii. $S(n) = S(m) \rightarrow n = m$
iii. $n \neq 0 \rightarrow (\exists m)(S(m) = n)$
2. INITIAL FUNCTION AXIOMS
i. There exists 1-ary, 2-ary, 3-ary functions that are constantly any given n. Recall n is a variable of sort ω.
ii. The three 3-ary projection functions exist. The two 2-ary projection functions exist. The 1-ary identity function exists.
iii. S(n) defines a 1-ary function.
3. COMPOSITION AXIOMS
i. $(\exists f)(\forall n,m,r)(f(n,m,r) = g(n,m))$
ii. $(\exists f)(\forall n,m,r)(f(n,m,r) = g(n))$
iii. $(\exists f)(\forall n,m)(f(n,m) = g(n,m,m))$
iv. $(\exists f)(\forall n)(f(n) = g(n,n,n))$
v. $(\exists f)(\forall n,m,r)(f(n,m,r) = g(h_1(n,m,r),h_2(n,m,r),h_3(n,m,r)))$
4. PRIMITIVE RECURSION AXIOM
$(\exists f)(f(n,0) = g(n) \wedge (\forall m)(f(n,S(m)) = h(n,m,f(n,m))))$.
5. PERMUTATION AXIOM.
Every 1-ary function that maps ω one-one onto ω has an inverse.
6. RUDIMENTARY INDUCTION AXIOM
$f(0) = g(0) \wedge (\forall n)(f(n) = g(n) \rightarrow f(S(n)) = g(S(n))) \rightarrow f(n) = g(n)$.

**Is ETF strictly mathematical? It is basically obvious that 1-6 above either actually appear in the mathematical literature or are special cases of statements actually appearing in the mathematical literature that are usually stated semiformally. In particular, with 3, there is the usual semiformally given notion of "functions defined by substitution from other functions" used extensively throughout mathematics, and i-v are fairly simple special cases. We can probe more deeply here into exact matches with the actual literature, but this does not seem rewarding at least at this early stage of SRM.**

**[Ba22] Ilnur I. Batyrshin, Countable strict reverse mathematics, ArXiv, August 19, 2022,** https://arxiv.org/pdf/2209.00108**.**

**is a very nice exposition of the basics of SRM and ETF based on [Fr21], and shows, among other things, that ETF is logically equivalent to 1,2,3,4,6 plus "functions defined by minimalization (μ operator)". As an example of a "more mathematical" judgment, the Permutation Axiom is "more mathematical" than minimalization - a judgment like this can probably be "documented" using searches of the mathematical literature. There can be a general protocol for gathering evidence of "more mathematical" using internet searches and AI.**

## 6. RESEARCH PROGRAMS FOR ETF

**Already there are a number of detailed questions that arise if we focus intensely on ETF. First two formally precise programs.**

**PROGRAM 1. Note that there are exactly 14 axioms of ETF. Determine the logical consequence relation among the $2^{14}$ subsets. Is this just reverse inclusion (doubtful)? Also consider logical consequence for the many sublanguages of the full language. I.e., one subset proves all sentences provable in another that lie in a given sublanguage. Obviously this naturally lends itself to natural partial results.**

**PROGRAM 2. Determine the interpretability relation among the $2^{14}$ subsets of the ETF axioms. How much linearity and non linearity is there here? Again, this naturally lends itself to natural partial results.**

**We believe that research into these two programs should require the development of some interesting new techniques.**

**Now for more flexible open ended programs.**

**PROGRAM 3. Are there alternatives to ETF in the same language that are as strictly mathematical or more so than ETF, and are logically equivalent to ETF or mutually interpretable with ETF? What if we use only some of the four sorts?**

**How can we go weaker than ETF for an even richer SRM targeting more mathematics? A well known step along these lines in the RM context is by using $RCA_0$*. This system originated in**

**[Si09] S.G. Simpson, Subsystems of second order Arithmetic, Perspectives in Mathematical Logic, 2009.**

**and is used as a base theory for many reversals including those discussed there at the end of that book. $RCA_0$* is the fragment of $RCA_0$ obtained by replacing the $\Sigma^0_1$ induction scheme by the $\Delta^0_0$ induction scheme, and adding exponentiation to the language with usual defining axioms for exponentiation.**

**Here are some refences that use $RCA_0$* as base theory.**

**Factorization of polynomials and $\Sigma^0_1$ induction, S.G. Simpson, R.L. Smith, Annals of Pure and Applied Logic 31 (1986), 289-306.**

**Categorical characterizations of the natural numbers require primitive recursion, Kołodziejczyk, Leszek Aleksander; Yokoyama, Keita, Ann. Pure Appl. Logic 166 (2015), no. 2, 219–231.**

**Weaker cousins of Ramsey's theorem over a weak base theory, Fiori-Carones, Marta; Kołodziejczyk, Leszek Aleksander; Kowalik, Katarzyna W., Ann. Pure Appl. Logic 172 (2021), no. 10, Paper No. 103028, 22 pp.**

**Reverse Mathematics, problems, reductions and proofs, D.D. Dzhafarov and C. Mummert, Theory and Applications of Computability. Springer Nature, Cham, 2022, xix + 488 pp. in section 6.6**

**This suggests the use of certain fragments of ETF as base theories for SRM. Perhaps the most obvious interesting idea is to simply remove the Permutation Axiom from ETF. In the proof in section 10 of [Fr21] that $RCA_0$ with functions and ETF are logically equivalent, we see 30 lemmas, the first 25 of which are proved in ETF\PERM. Already we see one reversal over**

ETF\PERM to ETF in [Ba22] mentioned above, reversing minimalization (μ-operator).

However, ETF\PERM is quite different from $RCA_0$* because of the Primitive Recursion axiom. Instead of removing PERM, we can keep PERM but remove PRIM. Here we would look for mathematically sensitive weakenings of PRIM and look for implications among each other and derivations of PRIM, all over ETF\PRIM.

Particularly interesting here may be working with various forms of the Exponentiation Axiom (easily formulated strictly mathematically over ETF), and also various mathematical theorems from which we can derive Exponentiation over ETF\PRIM.

Alternatively, we can simply focus on reversing over ETF\PRIM + EXP somewhat akin to how in $RCA_0$* we add back exponentiation.

In the abaove discussion of ETF\PRIM we are keeping PERM. We may want to go further and also drop PERM. All of these ideas need to be carefully explored. There is the overall intriguing question of whether exponentiation can be proved to be essential for a variety of theorems in countable mathematics that do not directly involve any finite counting.

## 7. SRM FOR $RCA_0$

Does ETF actually realize the most ambitious SRM/1 goal for $RCA_0$ because, as claimed in [Fr76], that $RCA_0$ and ETF are logically equivalent? No, because [Fr76] was referring to $RCA_0$ formulated with (ω,1-ary,2-ary,3-ary,0,S) and not with the language of ETF which is (ω,S(ω),0,1,+,•,<,∈).

Below we will continue to use $RCA_0$ as usual with sets only, and ETF as usual with functions only. We write $RCA_0$[f,s] for $RCA_0$ augmented with L[ETF], which needs no explanation as it appears implicitly and sometimes explicitly in the literature and folklore via standard coding. We write ETF[f,s] for ETF augmented with L[$RCA_0$]. This needs explanation since it is required to be strictly mathematical.

The f,s indicates we have both functions and sets. L[ETF[f,s]] = L[ETF] ∪ L[$Z_2$] in the obvious sense. We mark that L[ETF] ∩ L[$Z_2$] consists of just the sort ω with the constant 0. ETF[f,s] extends ETF with the following additional axioms:

1. $1 = S(0)$, $n+0 = n$, $n+(m+1) = n+m$, $n\bullet 0 = 0$, $n\bullet(m+1) = n\bullet m + n$, $n < m \leftrightarrow (\exists r)(m = (n+r)+1)$.
2. There are 2-ary functions which respectively agree with +,• everywhere.
3. There is a set consisting of the zeroes of any given 1-ary function.
4. There is a 1-ary function whose zeroes are the same as the elements of any given set.

Obviously ETF[f,s] is strictly mathematical - in fact just as strictly mathematical as ETF is.

From the logical equivalence of $RCA_0$ with functions only and ETF claimed in [Fr76] and proved in [Fr21], we obtain the following.

A. $RCA_0$[f,s] and ETF[f,s] are logically equivalent.
B. There is a faithful interpretation of ETF into $RCA_0$ which is the identity on $L[Z_2]$. Same with ETF[f,s].

## 8. SRM FOR $WKL_0$

A good way to achieve $WKL_0$ in SRM is to use the following strictly mathematical theorem in L[ETF[f,s]].

I) If $f,g:\omega \to \omega$ have no common value, then some set contains all values of f and no values of g

It is well known that $WKL_0$ is equivalent to the coded version I)' of I) over $RCA_0$. From this we obtain the following.

A. $WKL_0$[f,s] and ETF[f,s] + I) are logically equivalent.
B. There is a faithful interpretation of ETF[f,s] + I) into $WKL_0$ which is the identity on $L[Z_2]$.

## 9. SRM FOR $ACA_0$

A good way to achieve $ACA_0$ in SRM is to use the following strictly mathematical theorem in L[ETF[f,s]].

II) The set of all values of any given $f:\omega \to \omega$ exists

It is well known that $ACA_0$ is equivalent to the coded version II)' of II) over $RCA_0$. From this we obtain the following.

A. $ACA_0$[f,s] and ETF[f,s] + II) are logically equivalent.

**B. There is a faithful interpretation of ETF[f,s] + II) into $ACA_0$ which is the identity on $L[Z_2]$.**

## 10. SRM FOR $\Pi^1_1$-$CA_0$

**A good way to achieve $\Pi^1_1$-$CA_0$ in SRM is to use ETF[f,s] together with the following strictly mathematical theorem in L[ETF[f,s]].**

**III) For every $f:\omega \to \omega$ there is a largest $A \subseteq f[A]$**

**$\Pi^1_1$-$CA_0$ is equivalent to the coded version III)' of III) over $RCA_0$. I attach Appendix A for a proof sketch. From this we obtain the following.**

**A. $\Pi^1_1$-$CA_0$[f,s] and ETF[f,s] + III) are logically equivalent.**
**B. There is a faithful interpretation of ETF[f,s] + III) into $\Pi^1_1$-$CA_0$ which is the identity on $L[Z_2]$.**

## 11. SRM FOR $ATR_0$

**A good way to achieve $ATR_0$ in SRM is to use ETF[f,s,Q,$<_Q$,...] which is ETF[f,s] extended to accommodate Q and $<_Q$ (discussed below), together with the following strictly mathematical theorem in L[ETF[f,s,Q,$<_Q$,...]].**

**IV) Any two sets of rationals have an order continuous embedding from one of them into the other one**

**Here order continuous embedding means one-one and preserves limits (in the order topology). Or simply one-one continuous in the order topology.**

**$ATR_0$ is equivalent to the coded version IV)' of IV) over $RCA_0$, as proved in**

**[Fr05] H. Friedman, Metamathematics of comparability, in: Reverse Mathematics 2001, Lecture Notes in Logic, ed. S.G. Simpson, p. 201.**

**From this we obtain the following.**

**A. $ATR_0$[f,s,Q,$<_Q$,...] and ETF[f,s,Q,$<_Q$,...] + IV) are logically equivalent.**
**B. There is a faithful interpretation of ETF[f,s,Q,$<_Q$,...] + IV) into $ATR_0$ which is the identity on $L[Z_2]$.**

It remains to document the strictly mathematical ETF[f,s,Q,$<_Q$,...] to support this. This illustrates the spadework that needs to be done in the proper development of SRM. Since different researchers may do such spadework differently in the details, there needs to be the

OFFICIAL SRM WEBSITE

which tracks the needed extensions of ETF[f,s] that arise to support SRM. It tracks official strictly mathematical extensions of ETF[f,s] so that all SRM researchers are on the same page. It also contains statements of SRM results with references to their proofs. It needs to be run by committee.

When introducing the rationals into ETF[f,s] it's probably best to introduce the ordered ring of integers first and the ordered ring of rationals second, with Z,Q as separate sorts. Let's first focus on the ordered ring of integers. A lot of sorts need to be added. Adding the ring of integers can be construed as a special case of "adding a structure" where the domain elements are structureless points (like urelementes in set theory). Whenever a structure is added one adds a new sort for its domain. But then we also need to add new sorts for the 1,2,3-ary functions from various of the sorts into various of the sorts. There will be some special functions between sorts like | | from Z into $\omega$.

A decision needs to be made about where equality is allowed. I recommend that equality be allowed only on a given sort and that identity maps be used between sorts as appropriate. Whether equality should be required on every one sort is not clear. The committee needs to grapple with these questions and adjust the SRM website accordingly if there is a change of rules.

The SRM website should become the "honest friendly public formalization of mathematics of record". It is more honest and more friendly than what is normally done for proof assistants. We emphasize that the site only references the literature for proofs.

Let me close with a list of notable fragments of $Z_2$ whose SRM treatment needs further effort.

$\Delta^0_1$-$CA_0$
$RCA_0$*
$WKL_0$*
$WWKL_0$
ACA'

**$ACA_0^+$**
**$\Delta^1_1$-$CA_0$**
**$\sum^1_1$-IND**
**$\sum^1_1$-$AC_0$**
**$\sum^1_1$-$DC_0$**
**$\sum^1_1$-$TI_0$**
**$\Pi^1_2$-$TI_0$**
**$\Delta^1_2$-$CA_0$**
**$\Pi^1_1$-$TR_0$**
**$\sum^1_2$-IND**
**$\sum^1_2$-$CA_0$**
**$ID_1$**
**$ID_2$**

**Note that the last two are not strictly in $L[Z_2]$.**

**APPENDIX A**

**We prove the following.**

**THEOREM. "For every $f:\omega \to \omega$ there is a largest $A \subseteq f[A]$" is provably equivalent to $\Pi^1_1$-$CA_0$ over $RCA_0$.**

**It is obvious from $\Pi^1_1$-$CA_0$ that the union of all such sets is such a set.**

**Now for the reversal to $\Pi^1_1$-$CA_0$.**

**A finite sequence tree is a set T of finite sequences from $\omega$ closed under initial segments (so < > in T). An infinite path through T is an f: $\omega \to$ T where f(0) = < > and each f(n+1) extends f(n) by a single integer. The following is well known.**

**LEMMA. "In every finite sequence tree T, the union of the infinite paths exists" is provably equivalent to $\Pi^1_1$-$CA_0$ over $RCA_0$.**

**Now start with finite sequence tree T. Let f:T $\to$ T, where x $\in$ T\{< >} extends f(x) by one number, and f(< >) = < >. What do the A $\subseteq$ f[A] look like?**

**Let x $\in$ A. Then let f(y) = x, y $\in$ A. Hence y extends x by one number or y = x = < >. Hence**

**1) x ∈ A lies on an infinite path through T or is < >.**

**In particular, every element of A has a single integer extension in A, and any infinite path through T is one of these A's.**

**Let A be the maximum with A ⊆ f[A]. Then A is the union of all infinite paths through T together with < >.**

**I'll stop here. I plan a major revision of some of [Fr21] where this will be fully developed (redeveloped), together with some more general language extension approaches that should be very effective. This should involves treating the wider realm of countable mathematics and not just ω based countable mathematics. What I wrote in [Fr21] about this needs to be revisited.**