

\mathbb{Q} -Homotopy Algebras & HPL

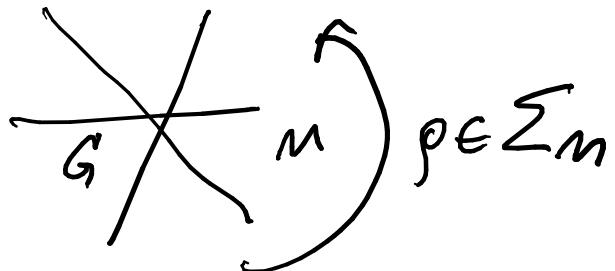
B. ŠURČO, Charles Univ.
Prague

joint work with
M. DOUBEK, K. MÜNSTER, L. PEKSOVÁ, S. PULMANN

ESI 10/09/2020

Cor

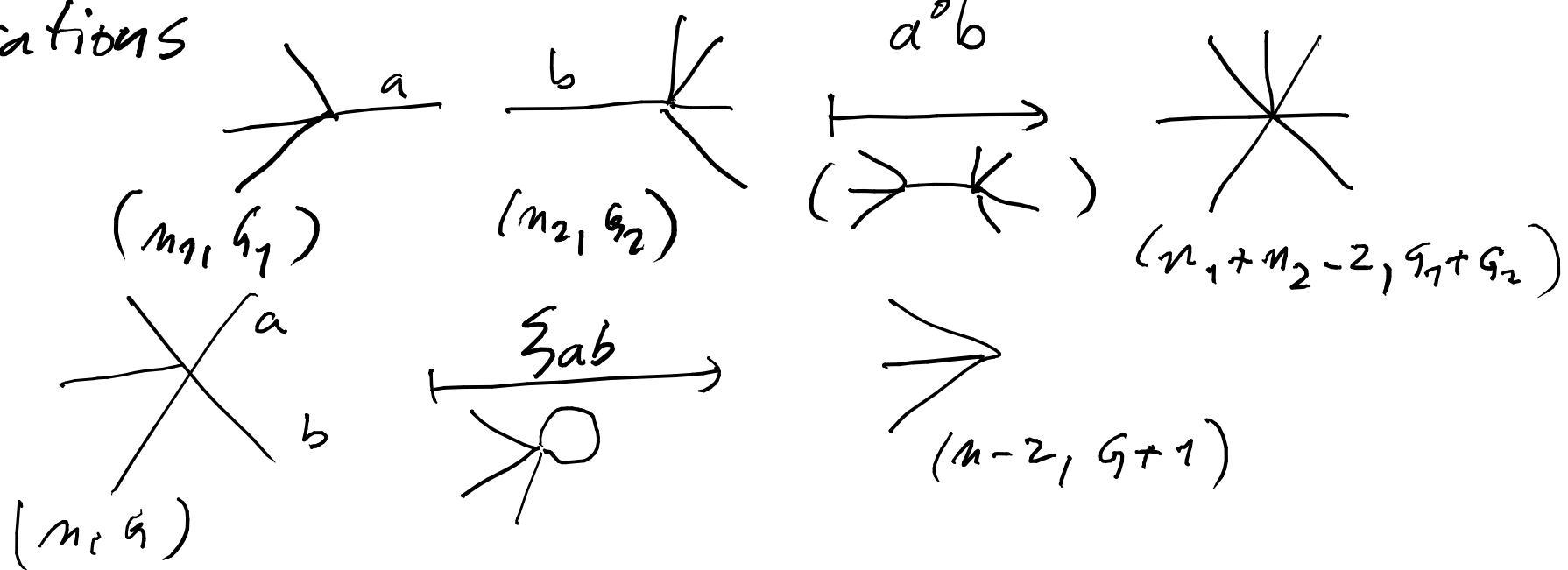
$$2(G-1) + n > 2$$



skeletal revision
(non-skeletal)

$$\{1, \dots, n\} \rightarrow C \text{ sets} \quad |C| = n$$

operations



Modular operad

$P(\cancel{\times}_g^n)$ - collection of A.g.V.s

+ morphisms $\deg = 0$

$$P(\cancel{\times}) \xrightarrow{P(g)} P(g(\cancel{\times})) \quad \Sigma\text{-action}$$

$$P(\cancel{\times}_a) \otimes P(\cancel{\times}_b) \xrightarrow{a^0 b} P(\cancel{\times}_{a+b})$$

$$P(\cancel{\times}^{a,b}) \xrightarrow{\xi_{ab}} P(\cancel{\times})$$

such that:

1. a^ob, ξ_{ab} are Σ -equivariant

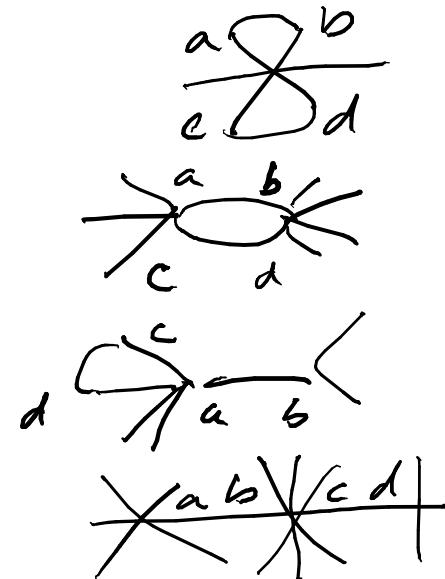
2. $a^ob(x \otimes y) = (-1)^{(x||y)} b^o a(y \otimes x)$

3. $\xi_{ab} \xi_{cd} = \xi_{cd} \xi_{ab}$

4. $\xi_{ab} c^o d = \xi_{cd} a^o b$

5. $a^o b (\xi_{cd} \otimes 1) = \xi_{cd} a^o b$

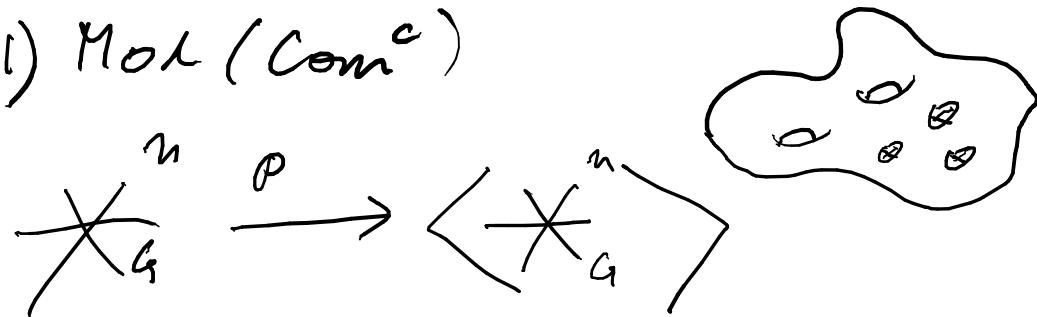
6. $a^o_b (1 \otimes c^o d) = c^o d (a^o b \otimes 1)$



$G = O +$ forgetting ξ_{ab}
cyclic operad

adjoint functor modular envelope

Ex. 1) Mot (Com^c)



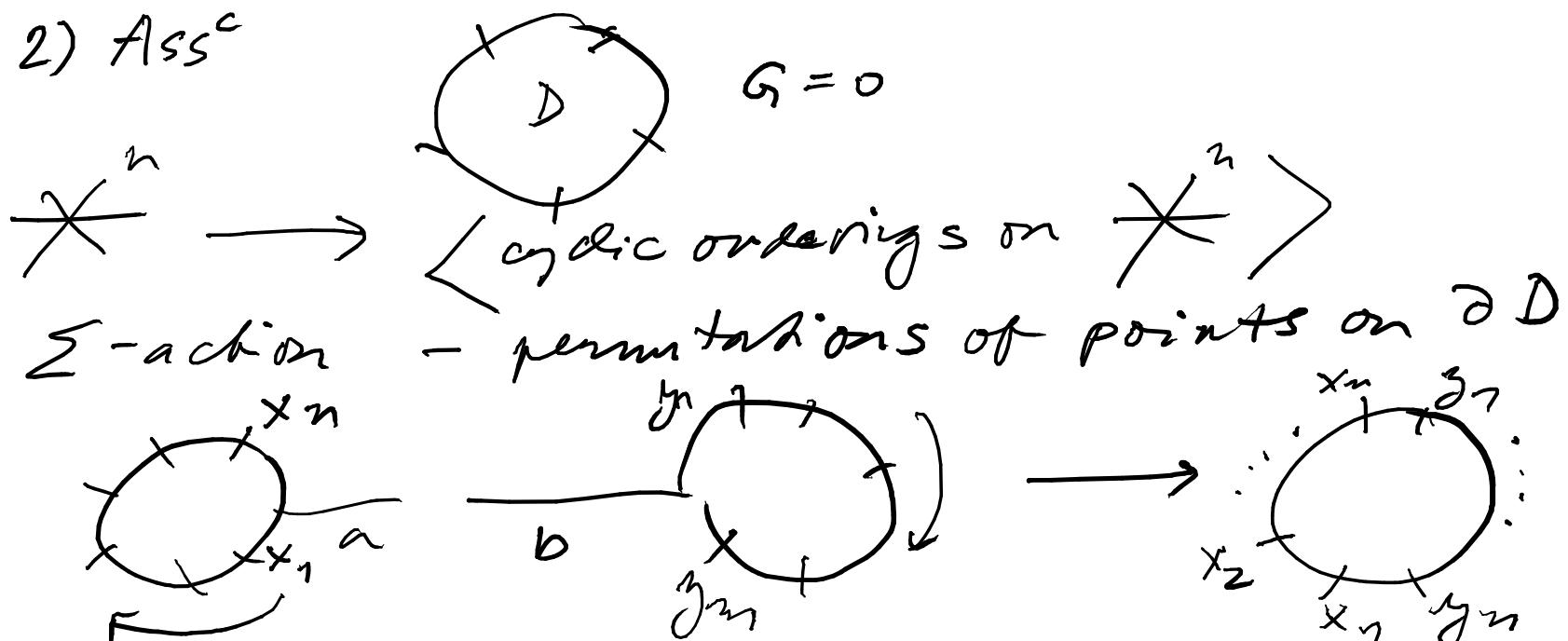
$$G = g \quad (G = 2g + \frac{n}{2} - 1)$$

1-dim v.s. in degree 0
differentiable $d=0$

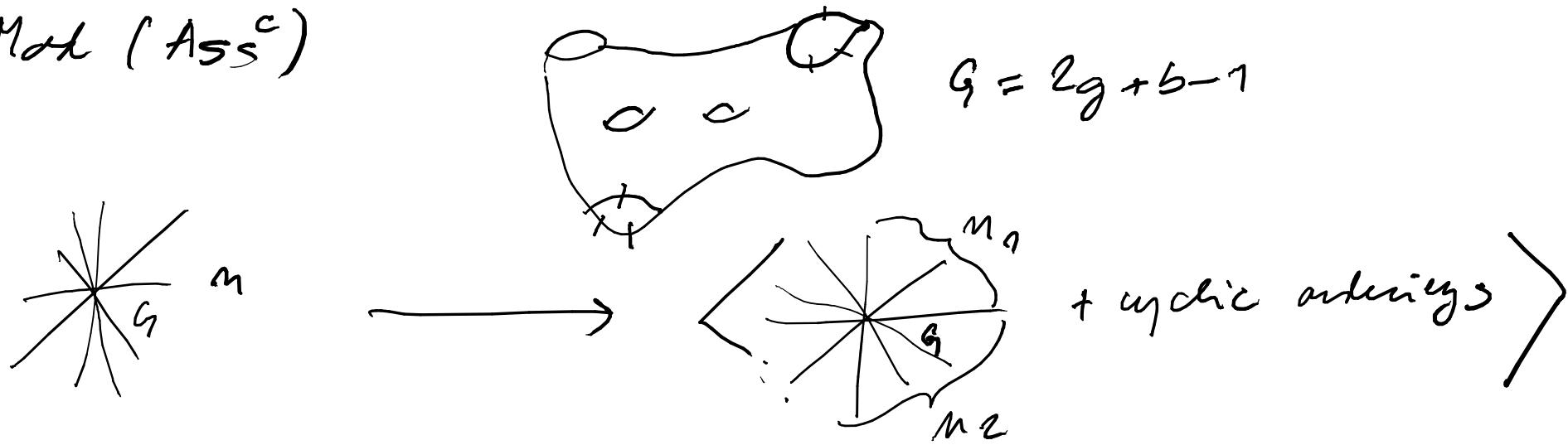
Σ -action trivial

$a^0 b$, ξ_{ab} — obvious

2) Ass^c



3. Mod (Ass^c)



a^b Seizing two surfaces

ξ_{AS} (2 links) - self-seizing of surface within the same boundary component
 - self-seizing using points on different boundary components

4. Quantum open-clock

2-colored, combination of 1. and 3.

$$G = 2g + b + \frac{n}{2} - 1$$

5. super versions

e.g. Mod ($\text{com}^c \ n=1$) type D

4-colored NS-NS, NS-R, R-NS, R-R

Odd modular operads

$a^o b \quad \xi_{ab}$ are at degree $\frac{1}{2}$
 multiplication of axioms

$$\xi_{ab} \xi_{cd} = -\xi_{cd} \xi_{ab}$$

$$\xi_{ab} c^o d = -\xi_{cd} a^o b$$

:

Ex. 1) End V V -d.g. (super) v.s.

w - deg-1 symplectic form, compatible with d

$$\mathcal{P}(n, G) = \text{Hom}(V^{\otimes n}, \mathbb{C}) =: \mathcal{E}_V(n, G) \quad \begin{matrix} \sum \text{-permutation} \\ \text{of inputs} \end{matrix}$$

$a^o b, \xi_{ab}$ contracting inputs with w

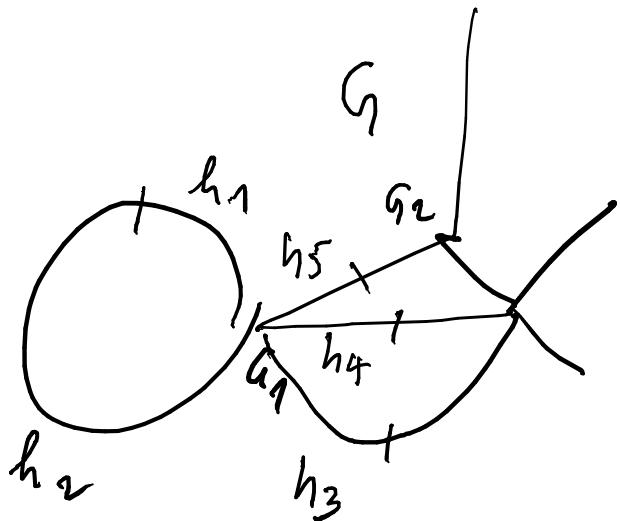
Ex. 2 Feynman transform \mathcal{FP} of a modular operad \mathcal{P}

$$G = \sum g_i \cdot + \# \text{ loops}$$

decorated by

$$(P_1 \otimes P_2 \otimes P_3) \otimes (1^{l_1} \wedge \dots \wedge 1^{l_5})$$

$$P_1 \in \mathcal{P} \left(\begin{array}{c} h_2 \\ h_1 \end{array} \right)^{\#}$$



$$(a \circ b)_{\mathcal{FP}}$$

$$(\xi_{ab})_{\mathcal{FP}}$$

grafting of graphs
and attaching edges, resp.

∂_{FP} - adding edge + modifying decorations

using $(a^0 b)^\#$ and $\xi_{ab}^\#$

$$\partial_{FP} \cancel{\text{---}} = \sum_{\substack{m_1+m_2=n \\ G_1+G_2=G}} \text{---} + \cancel{\text{---}}$$

$G=0, \xi_{ab}$ - trivial - cyclic cobar contr.

Algebra over Feynman transform

Morphism of odd modular operads

$$\mathcal{F}\mathcal{P} \longrightarrow \text{End } V$$

Branikov:

\Leftrightarrow following data

$$\{ m(c, g) \in (\mathcal{P}(c, g) \otimes \mathcal{E}_V(c, g))^\Sigma \}$$

$$(d_{\mathcal{E}_V} - d_{\mathcal{P}}) m(c, g) = \underbrace{(\xi_{ab})_{\mathcal{P}} \otimes (\xi_{ab})_{\mathcal{E}_V}}_{d} m(c \sqcup \{a, b\}, g^{-1}) +$$

$$\frac{1}{2} \sum_{\substack{c_1 \sqcup c_2 = c \\ g_1 + g_2 = g}} \left((a^{\circ b})_{\mathcal{P}} \overset{\Delta}{\otimes} (a^{\circ b})_{\mathcal{E}_V} \right) m(c_1 \sqcup \{a\}, g_1) \otimes m(c_2 \sqcup \{b\}, g_2)$$

(NC) BV-algebra on V

$$((P \otimes \mathcal{E}_V)^\Sigma, d, \Delta, \{\cdot, \cdot\}) \quad \text{d.g. Lie algebra}$$

$d + \Delta$ - differential

$\{\cdot, \cdot\}$ - lie bracket

Action (Interaction part) $\deg S = 0$

$$S_{\text{int}} := \sum_{n, g} t^n m(n, g) \quad \begin{matrix} \text{solution to the BV} \\ \text{QME} \end{matrix}$$

$$dS_{\text{int}} + t \Delta S_{\text{int}} + \frac{1}{2} \{S_{\text{int}}, S_{\text{int}}\} = 0$$

Theorem (Barannikov)

Algebras over $\mathcal{F}\mathcal{P} \Leftrightarrow$ solutions to QME

- Generalization $\mathcal{E}_V \rightarrow$ any odd modular operad
- Action in physics terms

a_i - basis of V

$$a_I = a_{i_1} \otimes \dots \otimes a_{i_n}$$

ϕ^i - dual basis

$$\phi_I = \phi^{i_1} \otimes \dots \otimes \phi^{i_n}$$

$$(\mathcal{P}(n, g) \otimes \sum_r (\eta, g)) \Big|^{\sum_m} \cong \mathcal{P}(n, g) \otimes_{\sum_m} V^\# \otimes^n$$

S_{int} - BV action ; $\Delta, \{\cdot, \cdot\}$ - BV-operation , $\phi = a_i \otimes \phi^i$
string field

$$\text{Ex. 1 } \text{Mod}(Com^c) \quad g = g \quad (\text{or } g = 2g + \frac{n}{2} - 1)$$

$$\mathcal{P}(n, g) \otimes_{\sum_m} V^\# \cong S^n(V^\#) \quad S_{int} \in S(V^\#)[[t_k]]$$

$$S_{int} = \sum_{m, g} \frac{t^g}{g!} f_m^g(a_I) \phi^I$$

graded sym.

$$g=0 L_\infty^c$$

solutions QME

\Rightarrow
long homotopy algebras

Zwiebach
Marke

Ex. 2 Ass^c

$$S_{\text{int}} = \sum_m f_m(q_I) \phi^\pm$$

cyclic sym.

A_∞^c

Stasheff
Zwickbach

Ex. 3 $Hd(Ass^c)$

$$g = 2g + b - 1$$

$$S_{\text{int}} = \sum$$

$$\frac{\hbar^g}{b! m_1! \dots m_b!}$$

Dornbeck, Münster
B.J.

$$\int^{g,b}_{a_{I_1} \dots a_{I_b}}$$

cyclic

$$\phi^{I_1} \dots \phi^{I_b}$$

Quantum A_∞^c

Ex. 4 Quantum open-closed

1+3 combination

Zwickbach

$$g = 2g + b + \frac{n_c}{2} - 1$$

$g=0$ classical
open-closed

Stasheff -
Kajiwara

Ex. 5 type II

Münster,
B.J.,

quantum (super) Lie homotopy algebra

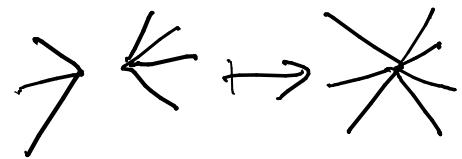
Product?

Dorbek, Pekson
Palmer, B.J.

Modular operads with connected sum #

deg 0

$$\# \quad P(c, g) \otimes Q(c', g') \rightarrow P(c \cup c'; g + g' + 1)$$



$$\# \quad P(g, g) \rightarrow P(g, g+2)$$

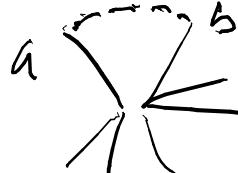


- Σ -equivariant, associative

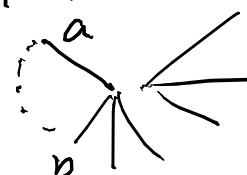
+ obvious compatibility conditions with a^o_b and ξ_{ab}

e.g.

$$\xi_{ab} \# = \# a \circ b$$



$$\# \xi_{ab} = \xi_{ab} \#$$



similar for odd modular operads

Ex. 1 $\text{Mod}(\text{Com}^c)$ ($G = 2g + \frac{u}{2} - 1$)

$(n, g) \# (n', g') \rightarrow (n+n', g+g')$ i.e. connected sum of 2 surfaces

$\#(n, g) \rightarrow (n, g+1)$ i.e. adding handle

Ex. 2 $\text{Mod}(\text{Ass}^c)$ - the same interpretation as above

Ex. 3 End V $E_V(n, g) = \text{Hom}(V^{\otimes n}, \mathbb{C})$

$f \# g$ is induced by \otimes

$\# f = f$ $f \in \text{Hom}(V^{\otimes n}, \mathbb{C})$ - doesn't depend on G

Algebras over \mathcal{FP} gets a product-induced
by $\#_P$ and $\#_{\Sigma^V}$

Prop $(P \otimes_{\Sigma} \Sigma^V, \Delta, \cdot)$ is a BV-algebra

Algebras over $\mathcal{FP} \Leftrightarrow \Delta e^S = 0$

Remark $\Delta(f \cdot g) = \Delta f \cdot g + f \Delta g \stackrel{\text{def}}{=} \{f, g\}$ $\{ \cdot, \cdot \}$ the same
as before

or more precisely $\#$

$$K((x)) \otimes P \otimes_{\Sigma} \Sigma^V / \# f \sim x f$$

laurv

$$(d + o) \cdot l \xrightarrow{\text{Sint}} xl = 0$$

Minimal model (S-matrix)

• direct application at Monological Part. Lense

$$V = H \oplus \text{Im } Q \oplus C_0$$

↑
physical trivial non-physical

decomposition compatible with $\omega \Rightarrow \omega_H = \omega / H$
 $D_H = D / H$

$${}^h\mathcal{C}(V, Q) \xrightleftharpoons[i]{p} (H, 0)$$

$$hQ + Qh = ip - 1 + \dots$$

h, Q, i, p extend to fields $\mathcal{F}(V)$ e.g. $Q = \{\text{Stress}\}$

HPL

$$\overset{\text{def}}{=} \mathcal{C}(\mathcal{F}(V), Q + \Delta) \xrightleftharpoons[\text{?}]{} (\mathcal{F}(H), \Delta_H)$$

$$e^W := p(1 - \Delta h)^{-1} e^{S_{\text{int}}}$$

$$\Delta_H e^W = 0$$

i.e. W defines a top homotopy algebra on H

Theorem $e^W = \int e^{S_{\text{int}}} (= \text{pe}^{\text{propagator}} e^{S_{\text{int}}})$

$$L_C \in \text{Im } Q + C \quad \leftarrow \text{gauge fixing + trivial fields} = 0$$

$$\text{propagator} = [Q, h_0] \quad h_0 = \frac{1}{\#(N, P + \Delta)} h$$

i.e. Minimal model \iff effective action
 $\Delta e^W = 0$ Ward id.

C. Auller
P. Henne
A. Cattaneo
D. Orfodoxian
;

- Works for any algebra over Feynman transform
of a general operad with connected sum.

Dotsenko
Pekcan
Palmer
B. J.

- HPL works also without connected sum (product)

perturbation $\mathcal{Q} \rightarrow \mathcal{Q} + \Delta + \{\mathcal{S}_{\text{int}}, \cdot\}$
gives on \mathcal{H} $\Delta_{\mathcal{H}} + \{\mathcal{W}, \cdot\}_{\mathcal{H}}$

- recursion relation for S-matrix (generalization
of B.-G.)

Marcelli
Sa'man
Koef
B. J.

IBL_∞ (IBA_∞, \dots) world

$$(\Delta + \{S, \cdot\})^2 = 0 \iff BV \text{ QME}$$

More generally D -degree one differential operator
on $\mathcal{F}(V)$ \uparrow
homological

$$D^2 = 0$$

II

Algebras over color construction over preprads

- IBL_∞ point of view

- nice interpretation of open-closed quantum SFT

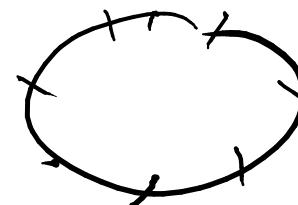
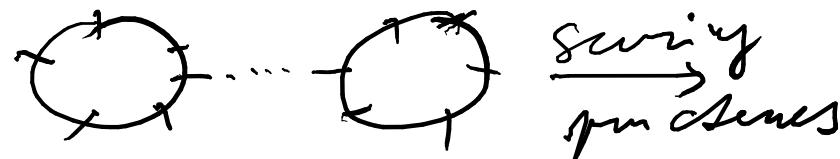
closed SFT $IBL_\infty - \mathcal{L}_c$

$(V_c, \Delta_c + \{S_c, \cdot\})$

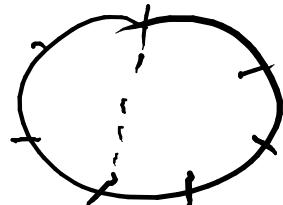
open SFT $IBL - \mathcal{L}_o$

(acylic cochains on $V_o, []_o, \delta_o$)

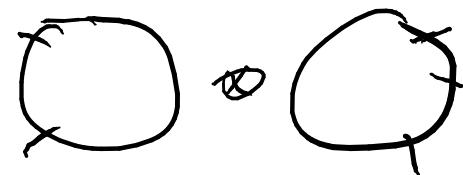
bracket



cobracket



self
swing



- open-closed SFT $\mathcal{L}_c + \mathcal{L}_{oc}$ - all bond closed vertices
 \mathcal{L}_{oc} is an IBL_∞ morphism from \mathcal{L}_c to \mathcal{L}_o

Münster
Sachs

What I didn't discuss

- HPL
 - Explicit homotopy between W and S
 - Morphism between corresponding quantum homotopy algebras
 - Odd symplectic category ...
- HPL works IBL_∞ and generalisations
 (algebras over cobase of propagators)

Zwiebach's construction
of SFT - operadic interpretation

