Non-Crossing Linked Partitions and Infinitesimal Free Multiplicative Convolution

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Def.

- $(\mathcal{A}, \varphi)$: non-commutative Probability space (ncps) if $\mathcal{A}$ is a unital algebra over $\mathbb{C}$ & $\varphi: \mathcal{A} \to \mathbb{C}$ is linear s.t $\varphi(1) = 1$.

- Unital subalgebras $(\mathcal{A}_i)_{i \in I}$ of $\mathcal{A}$ is free if for all $n \geq 1$, $i_1, \dots, i_n \in I$. $i_1 \neq i_2 \neq \cdots \neq i_{n-1} \neq i_n$, $a_j \in \mathcal{A}_{i_j}$, $\varphi(a_j) = 0$. then $\varphi(a_1 \cdots a_n) = 0$.

- $(k_n)_{n \geq 1}$: free Cumulants if $k_n : \mathcal{A}^n \to \mathbb{C}$ is the multilinear functionals, which is defined by the following relation

$$\varphi(a_1 a_2 \cdots a_n) = \sum_{\pi \in NC(n)} k_\pi [a_1, \dots, a_n]$$

where $$k_\pi [a_1, \dots, a_n] = \prod_{\substack{W \in \pi \\ W = \{i_1 < i_2 < \cdots < i_s\}}} k_{|W|}(a_{i_1}, \dots, a_{i_s}).$$

Remarks

- Probability distribution $\xleftrightarrow[\text{condition}]{\text{Carleman}}$ Moments $\xleftrightarrow[\text{formula}]{\text{moment cumulant}}$ free Cumulants

- $$\underbrace{G_a(z)}_{\text{Cauchy transform}} = \sum_{n=0}^{\infty} \frac{\varphi(a^n)}{z^{n+1}} \Longrightarrow \underbrace{R_a(z)}_{\text{R-transform}} = \underbrace{G_a^{\langle -1 \rangle}(z)}_{G_a \circ G_a^{\langle -1 \rangle} = Id} - \frac{1}{z} = \sum_{n=0}^{\infty} \kappa_{n+1}(a, \cdots, a) z^n$$

Theorem (Speicher 1994)

$(\mathcal{A}_i)_{i \in I}$ are free $\Longleftrightarrow$ Vanishing of mixed free Cumulants

Remark

Given a ncps $(\mathcal{A}, \varphi)$, $a, b \in \mathcal{A}$ are free

$$\kappa_n(a+b, a+b, \cdots, a+b) = \kappa_n(a, a, \cdots, a) + \kappa_n(b, b, \cdots, b) + \text{mixed free Cumulants} \to 0$$

$$= \kappa_n(a, a, \cdots, a) + \kappa_n(b, b, \cdots, b)$$

$$R_{a+b}(z) = R_a(z) + R_b(z)$$

Def.

Given $a \in \mathcal{A}$ with $\varphi(a) \neq 0$ then S-transform of a is defined by

$$S_a(z) = \frac{1+z}{z} M_a^{\langle -1 \rangle}(z) \quad \text{where} \quad M_a(z) = \sum_{n=1}^{\infty} \varphi(a^n) z^n$$

Theorem (Voiculescu 1987)

Given a ncps $(\mathcal{A}, \varphi)$, $a, b \in \mathcal{A}$ are free with $\varphi(a) \neq 0$ & $\varphi(b) \neq 0$

then $S_{ab}(z) = S_a(z) \cdot S_b(z)$.

Question

$a, b \in \mathcal{A}$ are free.

$$R_{a+b} = R_a + R_b \quad \longleftrightarrow \quad R_a(z) = \sum_{n=0}^{\infty} k_{n+1}(a, \dots, a) z^n \quad . \quad \varphi(a^n) = \sum_{\pi \in NC(n)} k_\pi[a, \dots, a]$$

$$S_{ab} = S_a \cdot S_b \quad \longleftrightarrow \quad ???$$

Ans: Dykema 2007 (Popa 2010, Mastnak & Nica 2009, Nica 2010)

$$\underset{S_{ab}^{-1}}{\overset{\|}{T_{ab}}} = \underset{S_a^{-1}}{\overset{\|}{T_a}} \cdot \underset{S_b^{-1}}{\overset{\|}{T_b}} \quad \longleftrightarrow \quad T_a(z) = \sum_{n=0}^{\infty} t_n(a, \dots, a) z^n \qquad \varphi(a^n) = \sum_{\pi \in NCL(n)} t_\pi[a, \dots, a]$$

Note

$$f(z) = \sum_{n=1}^{\infty} t_n z^n \quad \& \quad g(z) = \sum_{n=1}^{\infty} s_n z^n$$

$$s_n = \sum_{\substack{\pi \in NC(n) \\ \pi = \{V_1, \cdots, V_\ell\}}} t_{|V_1|} \cdots t_{|V_\ell|} \iff g(z) = f\left(z\,[1+g(z)] \right).$$

Observation [Lemma 6.2 in Mastnak & Nica 2009]

Given a ncps $(\mathcal{A}, \varphi)$, $a \in \mathcal{A}$ with $\varphi(a) = 1$

Then $R_a(z) := \sum_{n=1}^{\infty} k_n(a, \cdots, a) z^n = z + \sum_{n=2}^{\infty} k_n(a, \cdots, a) z^n$

$$M_a = R_a \circ \left(z\,[1 + M_a(z)] \right) \implies S_a(z) = \frac{1}{z} R_a^{\langle -1 \rangle}(z) \implies \frac{R_a(z)}{z} = \frac{1}{S_a(R_a(z))}$$

$f(z) := \frac{1}{S_a(z)} - 1 = \sum_{n=1}^{\infty} t_n z^n$ & $g(z) := \frac{1}{z} R_a(z) - 1 = \sum_{n=1}^{\infty} k_{n+1}(a, \cdots, a) z^n$ satisfies

$$g(z) = f\left(z\,[1 + g(z)] \right) \implies k_n(a, \cdots, a) = \sum_{\substack{\pi \in NC(n-1) \\ \pi = \{V_1, \cdots, V_\ell\}}} t_{|V_1|} \cdots t_{|V_\ell|}$$

Def.

- Let $n \in \mathbb{N}$. E & $F \subseteq [n] := \{1, 2, \dots, n\}$

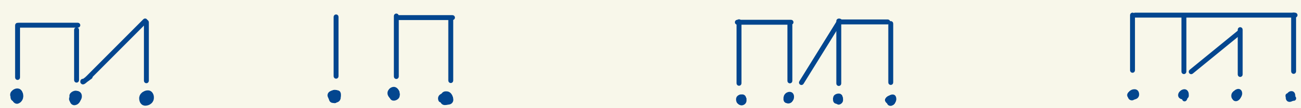
  E & F are **nearly disjoint** if for all $i \in E \cap F$, one of the following holds:

  (i) $i = \min(E)$, $|E| > 1$ & $i \neq \min(F)$

  (ii) $i \neq \min(E)$, $i = \min(F)$ & $|F| > 1$.

Examples of nearly disjoint.

$n=3$: $(1,2)$ & $(2,3)$; (1) & $(2,3)$ $\quad$ $n=4$: $(1,2)$ & $(2,3,4)$; $(1,2,4)$ & $(2,3)$



- π : set of non-empty subsets of $[n]$

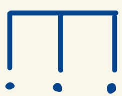
  π is said to be a **non-crossing linked partition** of $[n]$ if the union of π is $[n]$ and for all E & $F \in \pi$, E & F are non-crossing & nearly disjoint.

  $NCL(n)$ = the set of all non-crossing linked partitions of $[n]$.

Examples

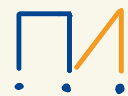
1°. NCL(3)

(1, 2, 3)

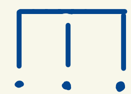


(1, 2) (3)

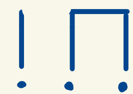
(12) (23)



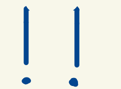
(13) (2)



(1) (2,3)



(1) (2) (3)



2°. NCL(4)

(1) (2,3) (3,4)

(1, 2,3) (3, 4)

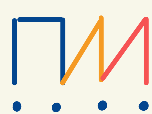
(1,2) (2,3) (4)

(1,2) (2,4) (3)

(1,3) (2) (3,4)

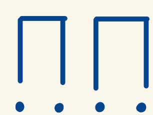
(1,2) (2,3,4)

(1,2) (2,3) (3,4)



(1,2,4) (2,3)

(1,2) (3,4)



...... etc

Notation

- Given $\pi \in NCL(n)$,
  $S(\pi) := \{ k \in [n] \mid k \text{ is not a minimal element in any blocks of } \pi \}$
- Given a ncps $(\mathcal{A}, \varphi)$, $\hat{\mathcal{A}} := \{ a \in \mathcal{A} \mid \varphi(a) \neq 0 \}$

Def.

The t-coefficients $\{ t_n : \hat{\mathcal{A}}^{n+1} \longrightarrow \mathbb{C} \}_{n \geq 0}$ is a sequence of maps that is defined inductively by

$$\varphi(a_1 a_2 \cdots a_n) = \sum_{\pi \in NCL(n)} t_\pi(a_1, \cdots, a_n)$$

where

$$t_\pi(a_1, \cdots, a_n) = \prod_{\substack{V \in \pi \\ V = \{i_1 < \cdots < i_l\}}} t_{l-1}(a_{i_1}, \cdots, a_{i_l}) \prod_{k \in S(\pi)} t_0(a_k)$$

Example

$\pi : (1,2,5)(2,4)(3)$, $S(\pi) = \{4, 5\}$

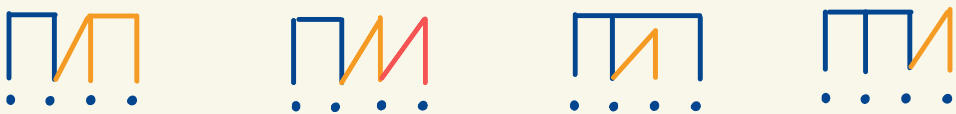
$$t_\pi(a_1, \cdots, a_5) = t_2(a_1, a_2, a_5)\, t_1(a_2, a_4)\, t_0(a_4)\, t_0(a_5)$$

Notation

- Given $\sigma \in NC\mathcal{L}(n)$,
  $i \sim_\sigma j$ if $\exists$ blocks $V_1, \dots, V_s \in \sigma$ s.t. $i \in V_1$ & $j \in V_s$ with $V_k \cap V_{k+1} \neq \emptyset$, $\forall k \in [s-1]$.
- Given $\pi \in NC(n)$, $[\pi] := \{ \sigma \in NC\mathcal{L}(n) \mid i \sim_\pi j \iff i \sim_\sigma j \text{ for all } i, j \in [n] \}$

Example

$(1,2)(2,3,4)$, $(1,2)(2,3)(3,4)$, $(1,2,4)(2,3)$, $(1,2,3)(3,4)$ $\in [1_4]$



property

For any n, $a_1, \dots, a_n \in \hat{\mathcal{A}}$. We have $\kappa_n(a_1, \dots, a_n) = \sum_{\pi \in [1_n]} t_\pi(a_1, \dots, a_n)$.

property.

Unital subalgs $(\mathcal{A}_i)_{i \in I}$ are free $\iff$ vanishing of mixed t-coefficients.

property (single-variable case)

$$\kappa_n(a) = \sum_{\pi \in NC(n-1)} \Big(\prod_{V \in \pi} t_{|V|}(a) \Big) t_0(a)^{n - \#(\pi)}$$

where $\kappa_n(a) := \kappa_n(a, \dots, a)$
$t_n(a) := t_n(a, \dots, a)$.

Def.

- $(\mathcal{A}, \varphi, \varphi')$: infinitesimal non-commutative probability space (incps) if $(\mathcal{A}, \varphi)$ is a ncps & $\varphi' : \mathcal{A} \to \mathbb{C}$ is linear with $\varphi'(1) = 0$.

- Unital subalgs $(\mathcal{A}_i)_{i \in I}$ are infinitesimally free if for all $n \in \mathbb{N}$. $i_1, \cdots, i_n \in I$. $i_1 \neq i_2 \neq \cdots \neq i_n$, $a_j \in \mathcal{A}_{i_j}$ & $\varphi(a_j) = 0$ then

$$\begin{cases} \varphi(a_1 a_2 \cdots a_n) = 0 \\ \varphi'(a_1 a_2 \cdots a_n) = \sum_{j=1}^{n} \varphi'(a_j) \varphi(a_1 a_2 \cdots a_{j-1} a_{j+1} \cdots a_n). \end{cases}$$

- $(\kappa_n')_{n \geq 1}$: infinitesimal free cumulants if $\kappa_n' : \mathcal{A}^n \to \mathbb{C}$ is the multilinear functional that is defined by the equation

$$\varphi'(a_1 a_2 \cdots a_n) = \sum_{\pi \in NC(n)} \partial \kappa_\pi [a_1, a_2, \cdots, a_n]$$

where

$$\partial \kappa_\pi [a_1, \cdots, a_n] = \sum_{\substack{V \in \pi \\ V = \{i_1 < \cdots < i_s\}}} \kappa'_{|V|}(a_{i_1}, \cdots, a_{i_s}) \prod_{\substack{W \neq V \\ W = \{j_1 < \cdots < j_\ell\} \in \pi}} \kappa_{|W|}(a_{j_1}, \cdots, a_{j_\ell}).$$

Theorem (Février & Nica 2010)

Infinitesimal freeness $\iff$ Vanishing of mixed free & inf. free Cumulants.

Def.

Given an incps $(\mathcal{A}, \varphi, \varphi')$ & $x \in \mathcal{A}$, the infinitesimal S-transform of x is defined by

$$\partial S_x(z) = -(M_x^{\langle -1\rangle})'(z) \cdot \partial M_x(M_x^{\langle -1\rangle}(z))$$

where ∂M_x is the inf. moment generating function: $\partial M_x(z) = \sum_{n=1}^{\infty} \varphi'(x^n) z^n$.

In addition,

the infinitesimal T-transform of x is given by $\partial T_x(z) = \dfrac{-\partial S_x(z)}{S_x(z)^2}$.

Theorem (Tseng 2023)

x & $y \in \mathcal{A}$ are inf. free

$\varphi(x) \neq 0$, $\varphi(y) \neq 0$.

$$\Longrightarrow \begin{cases} \partial S_{xy}(z) = \partial S_x(z) \cdot S_y(z) + S_x(z) \cdot \partial S_y(z) \\ \partial T_{xy}(z) = \partial T_x(z) \cdot T_y(z) + T_x(z) \cdot \partial T_y(z). \end{cases}$$

Let $(\mathcal{A}, \varphi, \varphi')$ be an incps & $\pi \in NC(n)$

$\hat{\mathcal{A}} := \{ a \in \mathcal{A} \mid \varphi(a) \neq 0 \}$ & $S(\pi) := \{ k \in [n] \mid k$ is not a minimal elt in any blocks of $\pi \}$

Def.

The infinitesimal t-coefficients $\{ t_n' : \hat{\mathcal{A}}^{n+1} \longrightarrow \mathbb{C} \}_{n \geq 0}$ is a sequence of maps given via the following

$$\varphi'(a_1 a_2 \cdots a_n) = \sum_{\pi \in NC(n)} \partial t_\pi (a_1, \cdots, a_n)$$

where

$$\partial t_\pi (a_1, \cdots, a_n) = \sum_{\substack{V \in \pi \\ V = \{i_1 < \cdots < i_s\}}} t'_{s-1}(a_{i_1}, \cdots, a_{i_s}) \prod_{\substack{W \neq V \\ W = \{j_1 < \cdots < j_r\} \in \pi}} t_{r-1}(a_{j_1}, \cdots, a_{j_r}) \prod_{k \in S(\pi)} t_0(a_k)$$

$$+ \prod_{\substack{V \in \pi \\ V = \{i_1 < \cdots < i_s\}}} t_{s-1}(a_{i_1}, \cdots, a_{i_s}) \sum_{k \in S(\pi)} t_0'(a_k) \prod_{\substack{k' \neq k \\ k' \in S(\pi)}} t_0(a_{k'}).$$

properties (Tseng 2024)

- $a_1, \cdots, a_n \in \hat{\mathcal{A}}$,

  $$\kappa_n'(a_1, \cdots, a_n) = \sum_{\pi \in [1_n]} \partial t_\pi (a_1, \cdots, a_n).$$

- Infinitesimal freeness $\Longleftrightarrow$ Vanishing of mixed t-coeff. & inf. t-coeff.

Property (Single variable Case) (Tseng 2024)

Given an incps $(\mathcal{A}, \varphi, \varphi')$, $a \in \hat{\mathcal{A}}$, then

$$k_n'(a) = \sum_{\pi \in NC(n-1)} \left(\sum_{V \in \pi} t_{|V|}'(a) \prod_{\substack{W \in \pi \\ W \neq V}} t_{|W|}(a) \cdot t_0(a)^{n-\#(\pi)} + \prod_{V \in \pi} t_{|V|}(a) \left(n-\#(\pi)\right) t_0'(a)\, t_0(a)^{n-\#(\pi)-1} \right)$$

Application to Random Matrix Theory

Complex Wishart matrix $X_N := \frac{1}{N} G^* G$ where G is a $M \times N$ Gaussian random Matrix with $N(0,1)$ entries.

If assume $\frac{M}{N} \to c$ & $(M - Nc) \to c'$, then

$\exists$ an incps $(\mathcal{A}, \varphi, \varphi')$ & $x \in \mathcal{A}$ such that
$$\begin{cases} \varphi(x^k) = \lim_{N \to \infty} \frac{1}{N} (E \circ Tr)(X_N^k) \\ \varphi'(x^k) = \lim_{N \to \infty} N \cdot \left\{ \frac{1}{N} (E \circ Tr)(X_N^k) - \varphi(x^k) \right\} \end{cases}$$

$\Longrightarrow$ [Mingo 2019]
$$\begin{cases} \text{indep } \mathbb{C}\text{-Wishart Matrices are asymptotically inf. free.} \\ k_n(x) = c \ \& \ k_n'(x) = c' \text{ for all } n \end{cases}$$

Properties (Tseng 2024)

- $\kappa_n(x) = \begin{cases} c & n=0 \\ 1 & n=1 \\ 0 & n\geq 2 \end{cases}$ & $\kappa_n'(x) = \begin{cases} c' & n=0 \\ 0 & n\geq 1 \end{cases}$ $\iff$ $\begin{cases} T_x(z) = c+z \\ \partial T_x(z) = c' \end{cases}$
- x & y are inf. free with $\begin{cases} \kappa_n(x) = \kappa_n(y) = c \\ \kappa_n'(x) = \kappa_n'(y) = c' \end{cases}$ for all n.

$$\Longrightarrow \begin{cases} T_{xy}(z) = c^2 + 2cz + z^2 \\ \partial T_{xy}(z) = 2c'c + 2cz. \end{cases}$$

Question

- $x, y \in \mathcal{A}$ are free with $\varphi(x) \neq 0$, $\varphi(y) = 0$.

Do we have an analogous process to study the law of xy in the case that $\begin{matrix}\varphi(x)\neq 0 \\ \varphi(y)\neq 0\end{matrix}$?

Ans: Multiplication of free random variables & S-transform: the case of vanishing mean (Speicher & Rao 2007).

Thank You for Your Attention ~