PEPS AND BICATEGORIES

Higher structures and field theory

6 August 2020

Plan

Goal :

understand MPO symmetries of string-net PEPS through TFT

Motivation :

- description of string-net PEPS with boundary

  allowing e.g. for calculation of error thresholds for error-correcting codes based on string nets with boundary

- mapping of PEPS to critical system via “strange correlator”

  allowing e.g. for understanding various CFT structures directly on the lattice

- ......

Goal :

 understand MPO symmetries of string-net PEPS through TFT

Plan :

- reminder about PEPS and MPO
- from MPO symmetries to fusion categories and bimodule categories
- invertible case : a 2-object bicategory / 2-Morita context in fusion categories
- explanation in terms of state-sum topological field theory

work in progress with Jutho Haegeman
& Laurens Lootens
& Christoph Schweigert
& Frank Verstraete

Warmup: MPS

- ☞ efficient approximation to ground states of local gapped Hamiltonians for 1-d lattices
- ☞ element of $\mathcal{H}_{\text{phys}}^{\otimes N}$ for system with N sites
- ☞ $$|\psi(A)\rangle \;=\; \sum_{j_1,j_2,\dots,j_N}^{d} \mathrm{Tr}(A^{j_1} A^{j_2} \cdots A^{j_N})\, |j_1\rangle|j_2\rangle \cdots |j_N\rangle$$

assuming periodic boundary conditions and translational invariance

A $D \times D \times d$ - tensor

$(A^j)_{pq} \equiv A_{jpq}$

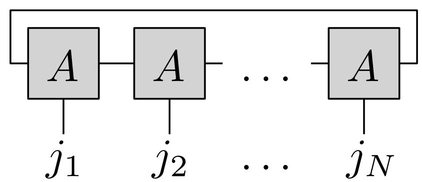
$d = \dim(\mathcal{H}_{\text{phys}})$

$D =$ 'virtual dimension'

Matrix product states

Warmup : MPS

- ☞ efficient approximation to ground states of local gapped Hamiltonians for 1-d lattices
- ☞ element of $\mathcal{H}_{\text{phys}}^{\otimes N}$ for system with N sites
- ☞ $$|\psi(A)\rangle = \sum_{j_1, j_2, \ldots, j_N}^{d} \mathrm{Tr}(A^{j_1} A^{j_2} \cdots A^{j_N}) \, |j_1\rangle |j_2\rangle \cdots |j_N\rangle$$
- ☞ diagrammatically: $|\psi(A)\rangle =$



Matrix product states

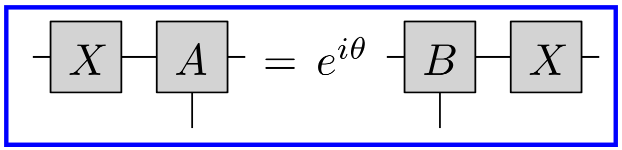
Warmup : MPS

- ☞ efficient approximation to ground states of local gapped Hamiltonians for 1-d lattices
- ☞ element of $\mathcal{H}_{\text{phys}}^{\otimes N}$ for system with N sites
- ☞ $$|\psi(A)\rangle \;=\; \sum_{j_1,j_2,\ldots,j_N}^{d} \mathrm{Tr}(A^{j_1} A^{j_2} \cdots A^{j_N})\, |j_1\rangle |j_2\rangle \cdots |j_N\rangle$$

NB : fundamental theorem of MPS

- ☞ injective MPS : $\{\, A^j \mid j = 1, 2, \ldots, d \,\}$ generate full D^2-dim matrix algebra
- ☞ $|\psi(A)\rangle = |\psi(B)\rangle$ for injective MPS based on tensors A and B

$\Longleftrightarrow$

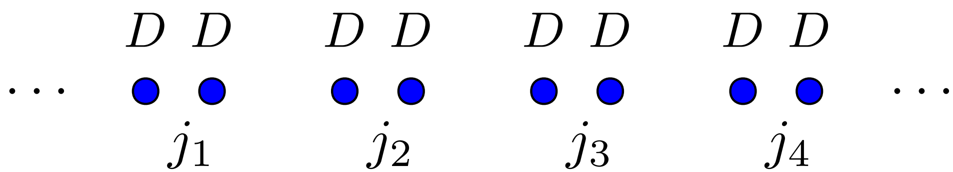


i.e. A and B related up to phase by a virtual gauge transformation

Matrix product states

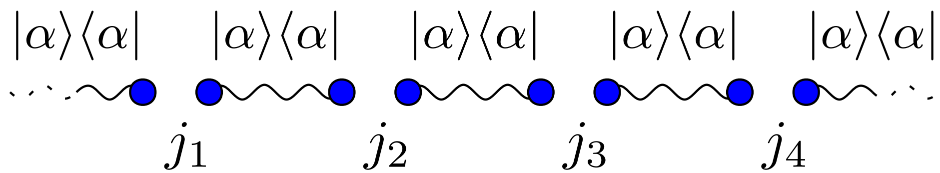
☞ alternative prescription :

- at each site place two D-dim degrees of freedom with on.n. basis $\{|i\rangle\}$:

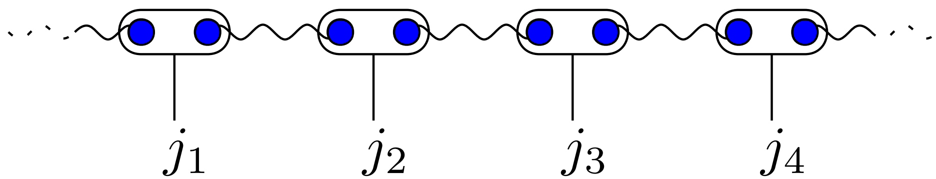


- maximally entangle all pairs of qudits on neighboring sites :

project onto $|\alpha\rangle := \sum_{i=1}^{D} |i\rangle|i\rangle$



- act on the pair of qudits at each site with linear map $f_A : \mathbb{C}^D \otimes \mathbb{C}^D \to \mathbb{C}^d$



$\Longrightarrow$ realize the MPS $|\psi(A)\rangle$ as projected entangled pair state

Projected entangled pair states

☞ in short : an MPS is a projected entangled pair state

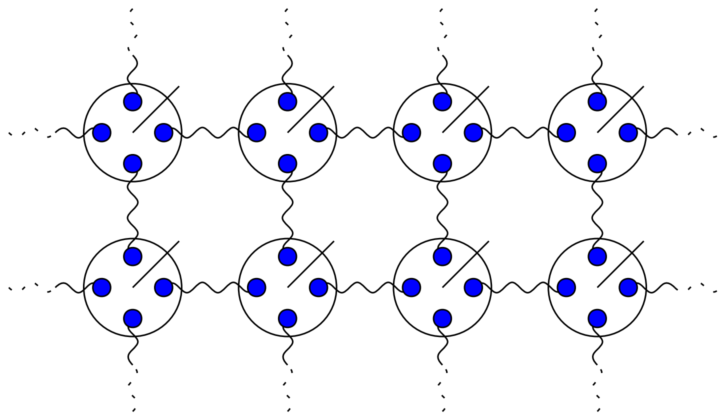
Projected entangled pair states

- ☞ in short : an MPS is a PEPS
- ☞ virtue : generalizes rather directly to any dimension

(term *PEPS* usually reserved for 2-d case)

Projected entangled pair states

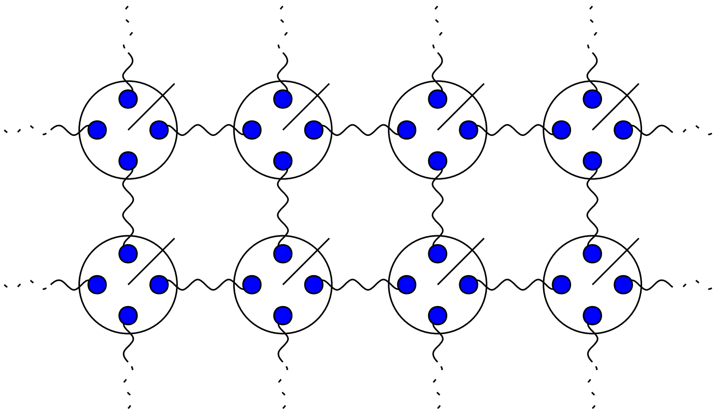
- in short: an MPS is a PEPS
- virtue: generalizes rather directly to any dimension
- concretely in $\mathbf{d = 2}$:



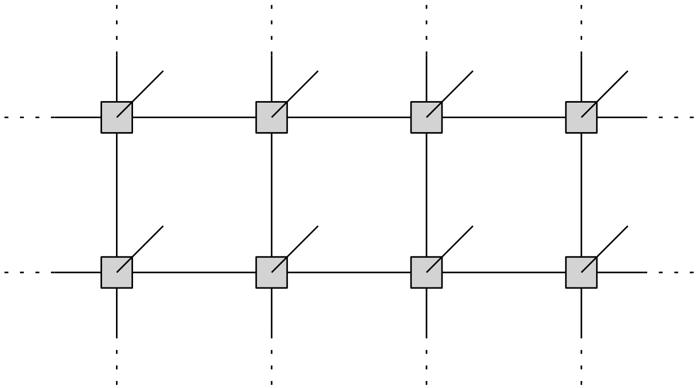
- alternatively via PEPS tensor A with 1 physical leg and n virtual legs
  analogous to standard description of MPS

Projected entangled pair states

- in short: an MPS is a PEPS
- virtue: generalizes rather directly to any dimension
- concretely in $\mathbf{d=2}$:



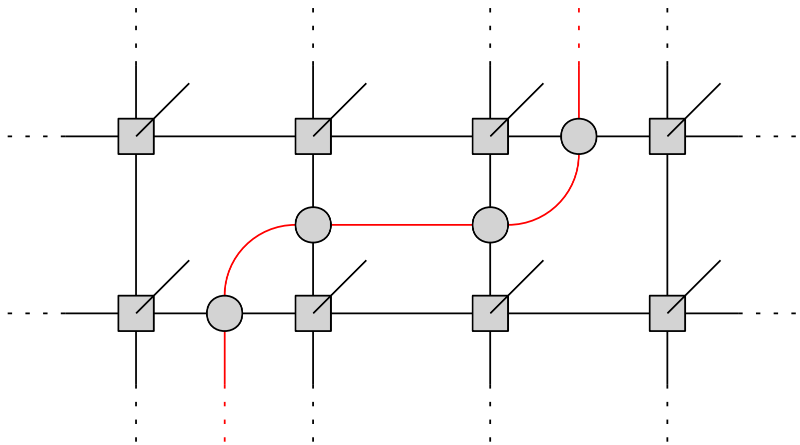
- alternatively via PEPS tensor A with $\mathbf{1}$ physical leg and n virtual legs



  - e.g. for $n=4$:
  - from now on instead: $n=3$ (hexagonal lattice)

Matrix product operators

- properties of PEPS wave function $|\psi(A)\rangle$ for ground states of string-net models $\longleftrightarrow$ symmetries of the PEPS tensor A
- can realize such symmetries through matrix product operators (MPO) $\widehat{B}$ associated with defect lines in the lattice of PEPS tensors:



SCHUCH-CIRAC-PÉREZ-GARCÍA 2010

BUERSCHAPER 2014

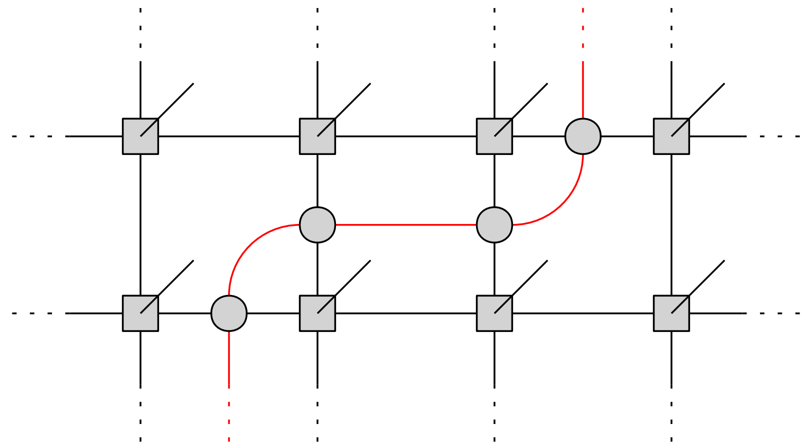
ŞAHINOĞLU-WILLIAMSON-BULTINCK-MARIËN-HAEGEMAN-SCHUCH-VERSTRAETE P2014

BULTINCK-MARIËN-WILLIAMSON-ŞAHINOĞLU-HAEGEMAN-VERSTRAETE 2017

Matrix product operators

- properties of PEPS wave function $|\psi(A)\rangle$
  for ground states of string-net models $\longleftrightarrow$ symmetries of the PEPS tensor A

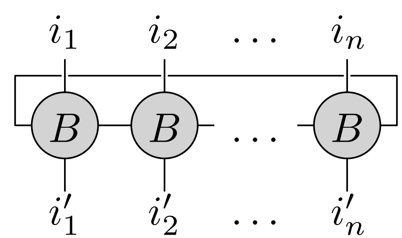
- can realize such symmetries through matrix product operators (MPO) $\widehat{B}$
  associated with defect lines
  in the lattice of PEPS tensors :



- involves MPO tensor B with two external (defect) and two internal legs so that

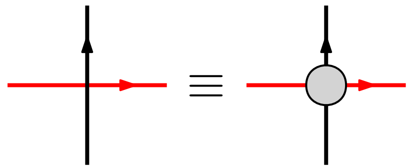
$$\widehat{B} = \sum_{\{i\},\{i'\}=1}^{D} \mathrm{Tr}\left(B^{i_1 i'_1} \cdots B^{i_n i'_n}\right) |i_1 \cdots i_n\rangle\langle i'_1 \cdots i'_n|$$

$=$



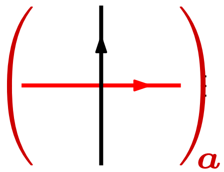
- NB : taking $d_{\mathrm{e}} = d$ instead of $d_{\mathrm{e}} = D$
  allows for alternative use of MPO as operator on an MPS (hence name)

MPO rep's of fusion categories

☞ abbreviated notation : 

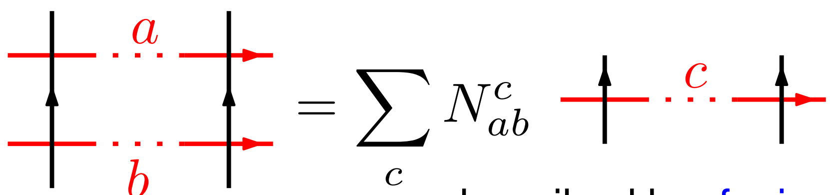
MPO rep's of fusion categories

- abbreviated notation :

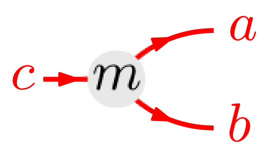


- consider simultaneously whole family of MPO tensors $\boldsymbol{B_a}$ labeled by “‘$\boldsymbol{a}$”
- $\widehat{\boldsymbol{B}}_{\boldsymbol{a}}$ and $\widehat{\boldsymbol{B}}_{\boldsymbol{b}}$ can be fused by concatenation of external legs of $\boldsymbol{B_a}$ and $\boldsymbol{B_b}$
- invoke MPO injectivity / fundamental theorem of MPO

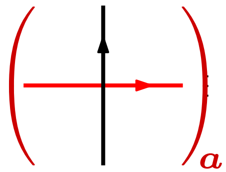
  ⇝ decomposition



described by fusion tensor $\boldsymbol{X_{ab}^{c,m}}$

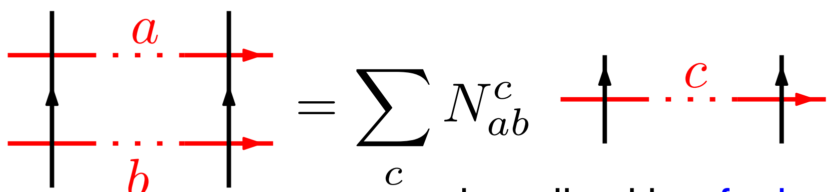


- abbreviated notation :



- consider simultaneously whole family of MPO tensors $\boldsymbol{B_a}$ labeled by "'$\boldsymbol{a}$"
- $\widehat{\boldsymbol{B}}_{\boldsymbol{a}}$ and $\widehat{\boldsymbol{B}}_{\boldsymbol{b}}$ can be fused by concatenation of external legs of $\boldsymbol{B_a}$ and $\boldsymbol{B_b}$
- invoke MPO injectivity / fundamental theorem of MPO

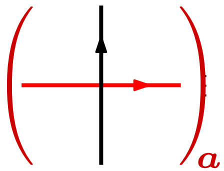
⤳ decomposition



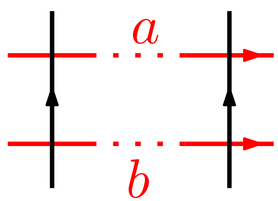
described by fusion tensor $\boldsymbol{X_{ab}^{c,m}}$

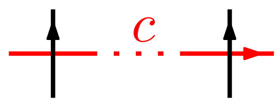
Interpretation : fusion category $\mathcal{C}$ with simple objects labeled by $\boldsymbol{a}$

MPO rep's of fusion categories

- abbreviated notation : 

- consider simultaneously whole family of MPO tensors B_a labeled by "'a"
- $\widehat{B}_a$ and $\widehat{B}_b$ can be fused by concatenation of external legs of B_a and B_b
- invoke MPO injectivity / fundamental theorem of MPO

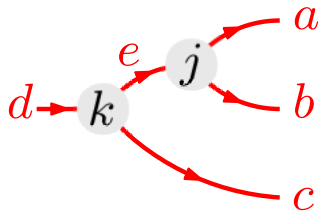
  ⇝ decomposition 

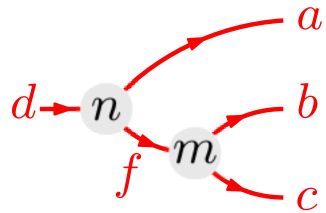
$= \sum_c N_{ab}^c$ 

  described by fusion tensor $X_{ab}^{c,m}$

Interpretation : fusion category $\mathcal{C}$

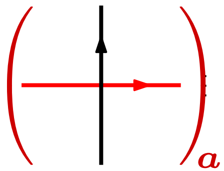
with associativity of fusion encoded in 6j symbols ${}^0\mathbf{F}$ of $\mathcal{C}$:



$= \sum_{f,m,n} \left({}^0\mathbf{F}_d^{abc}\right)_{c,jk}^{f,mn}$


MPO rep's of fusion categories

☞ abbreviated notation :

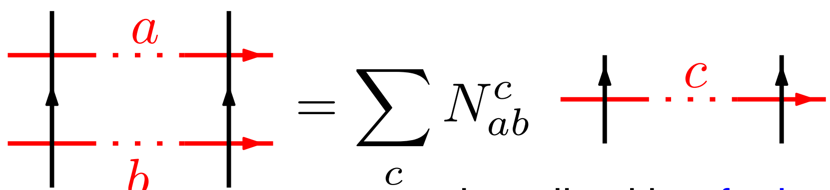


☞ consider simultaneously whole family of MPO tensors B_a labeled by "'a"

☞ $\widehat{B}_a$ and $\widehat{B}_b$ can be fused by concatenation of external legs of B_a and B_b

☞ invoke MPO injectivity / fundamental theorem of MPO

⇝ decomposition



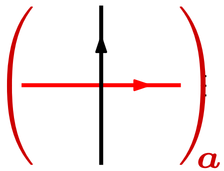
described by fusion tensor $X_{ab}^{c,m}$

More general interpretation : MPO representation of a fusion category $\mathcal{C}$

with associativity of fusion encoded in recoupling identity

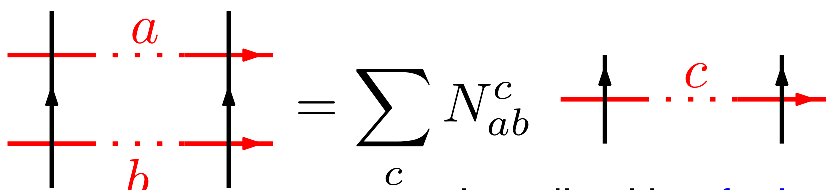
for fusion tensor X

- ☞ abbreviated notation :



- ☞ consider simultaneously whole family of MPO tensors B_a labeled by "'a"
- ☞ $\widehat{B}_a$ and $\widehat{B}_b$ can be fused by concatenation of external legs of B_a and B_b
- ☞ invoke MPO injectivity / fundamental theorem of MPO

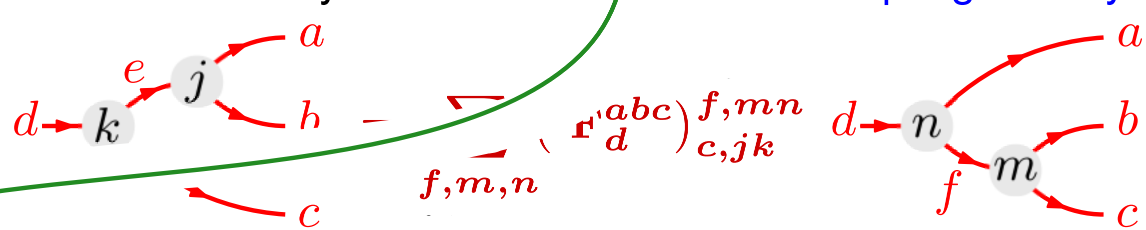
  ⇝ decomposition



described by fusion tensor $X_{ab}^{c,m}$

More general interpretation : MPO representation of a fusion category $\mathcal{C}$

with associativity of fusion enc[oded] in recoupling identity



to be explained

for fusion tensor X

PEPS rep's of fusion categories

☞ consider PEPS description
of ground states of string-net models on honeycomb lattice

BUERSCHAPER-AGUADO-VIDAL 2009

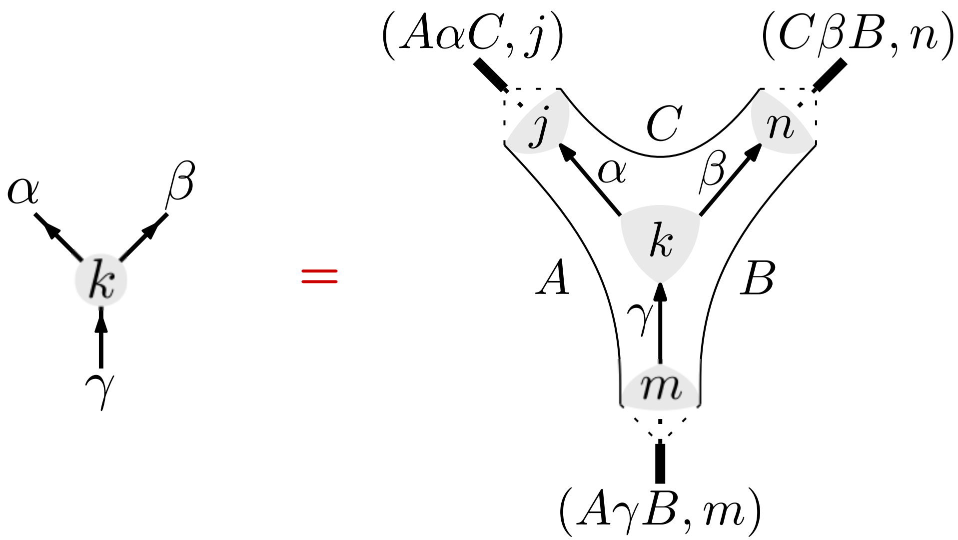
GU-LEVIN-SWINGLE-WEN 2009

and allow for spherical fusion categories with arbitrary fusion multiplicities

PEPS rep's of fusion categories

☞ consider PEPS description
of ground states of string-net models on honeycomb lattice

☞ amounts to realization



of PEPS tensor

with physical leg $(\alpha\beta\gamma, k)$ (sticking out of screen)

and three virtual legs $(A\alpha C, j)$

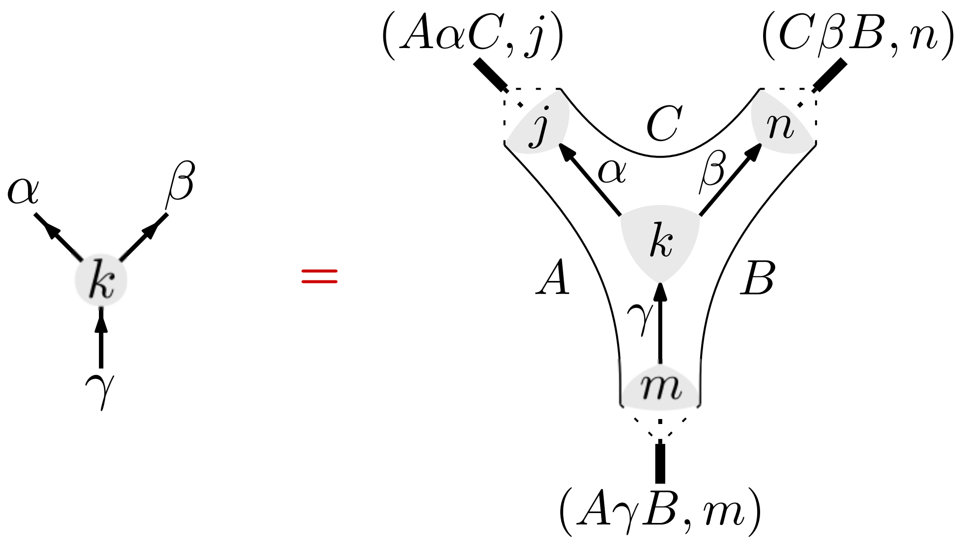
$(C\beta B, n)$

$(A\gamma B, m)$

multiplicity label

PEPS rep's of fusion categories

- consider PEPS description
  of ground states of string-net models on honeycomb lattice
- amounts to realization



Interpretation :

fusion category $\mathcal{D}$ with simple objects labeled by $\alpha, \beta, \dots, A, B, \dots$

and morphisms $j \in \mathrm{Hom}_{\mathcal{D}}(A \otimes \alpha, C)$

$n \in \mathrm{Hom}_{\mathcal{D}}(C \otimes \beta, B)$

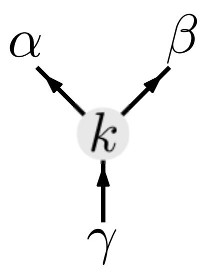
$m \in \mathrm{Hom}_{\mathcal{D}}(A \otimes \gamma, B)$

with PEPS tensor as 6j-symbol ${}^4\mathrm{F}$ of $\mathcal{D}$ $\qquad k \in \mathrm{Hom}_{\mathcal{D}}(\alpha \otimes \beta, \gamma)$

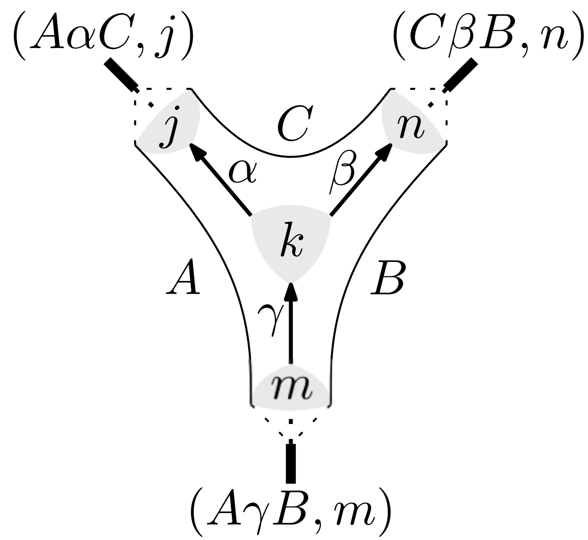
PEPS rep's of fusion categories

☞ consider PEPS description
of ground states of string-net models on honeycomb lattice

☞ amounts to realization



$=$



Interpretation :

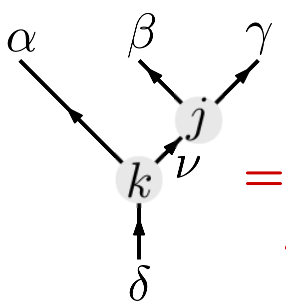
fusion category $\mathcal{D}$ with simple objects labeled by $\alpha, \beta, \ldots, A, B, \ldots$

and morphisms $j \in \mathrm{Hom}_{\mathcal{D}}(A \otimes \alpha, C)$

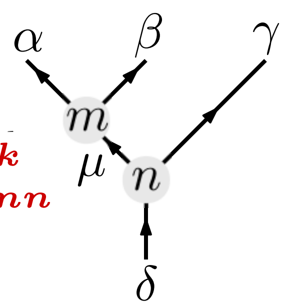
$n \in \mathrm{Hom}_{\mathcal{D}}(C \otimes \beta, B)$

$m \in \mathrm{Hom}_{\mathcal{D}}(A \otimes \gamma, B)$

$k \in \mathrm{Hom}_{\mathcal{D}}(\alpha \otimes \beta, \gamma)$

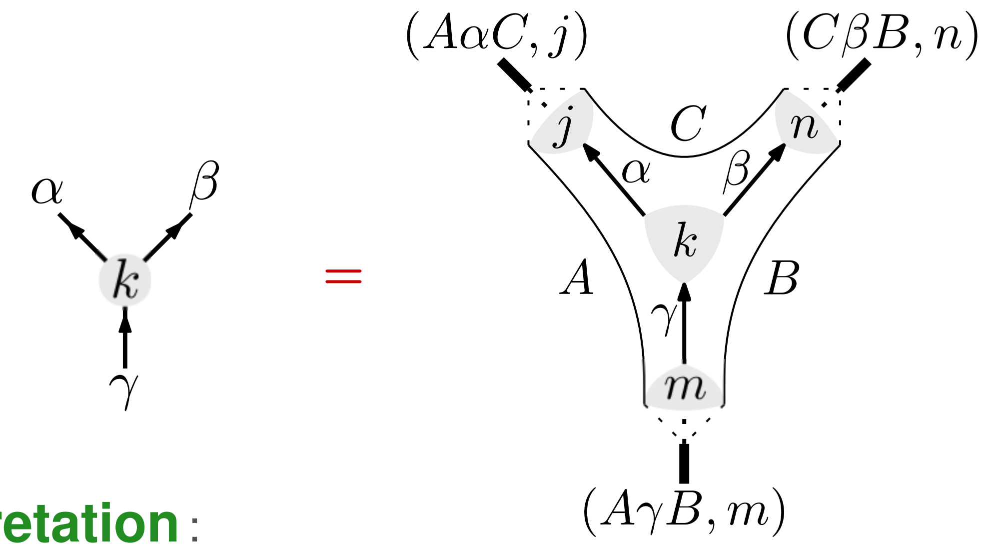


$$= \sum_{f,m,n} \left({}^4\mathbf{F}_\delta^{\alpha\beta\gamma}\right)_{\mu,mn}^{\nu,jk}$$



PEPS rep's of fusion categories

- consider PEPS description
  of ground states of string-net models on honeycomb lattice
- amounts to realization

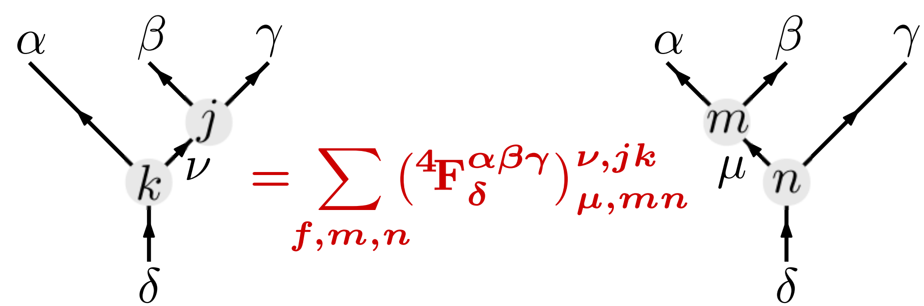


More general interpretation :

fusion category $\mathcal{D}$ with simple objects labeled by $\alpha, \beta, \ldots$

together with a "PEPS realization" of $\mathcal{D}$

with

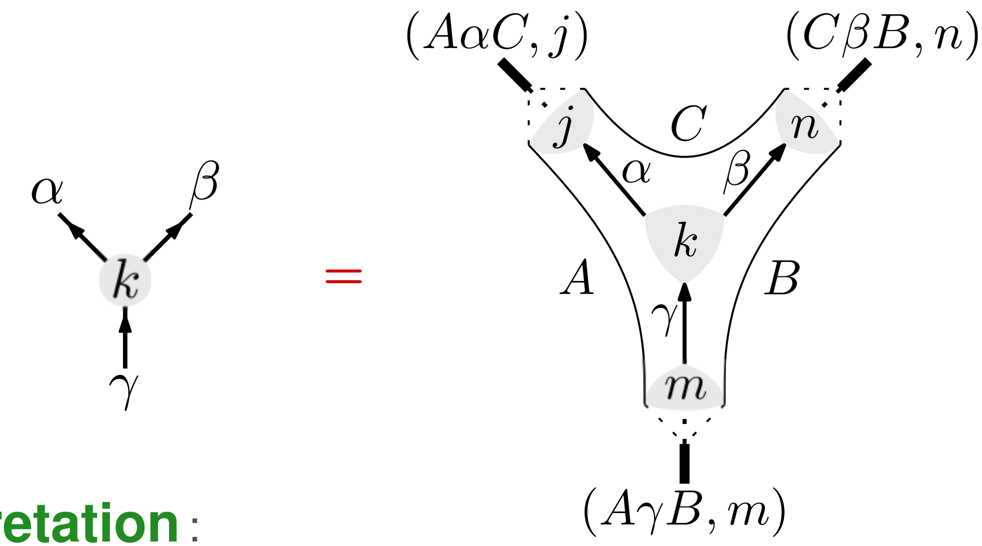


as recoupling identity
for PEPS tensor

PEPS module category

☞ consider PEPS description
of ground states of string-net models on honeycomb lattice

☞ amounts to realization



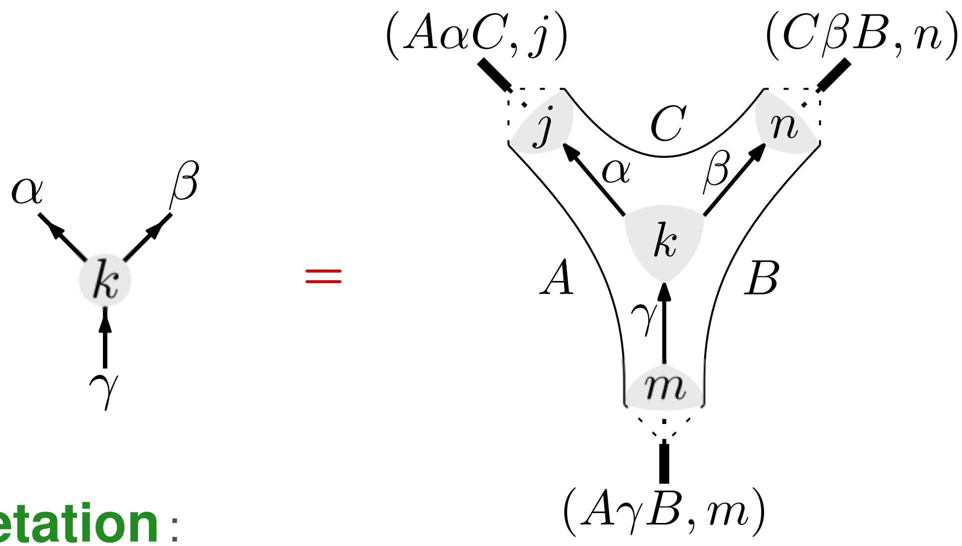
More general interpretation :

fusion category $\mathcal{D}$ with simple objects labeled by $\boldsymbol{\alpha, \beta, \ldots}$

together with right $\mathcal{D}$ - module category $\mathcal{M}$ with simple objects labeled by $\boldsymbol{A, B, \ldots}$
(“quantum subgroup”)

PEPS module category

- ☞ consider PEPS description
  of ground states of string-net models on honeycomb lattice
- ☞ amounts to realization



More general interpretation:

fusion category $\mathcal{D}$ with simple objects labeled by $\alpha, \beta, \ldots$

together with right $\mathcal{D}$-module category $\mathcal{M}$ with simple objects labeled by $A, B, \ldots$

and morphisms $j \in \mathrm{Hom}_{\mathcal{M}}(A \triangleleft \alpha, C)$

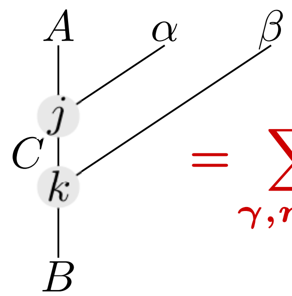
$n \in \mathrm{Hom}_{\mathcal{M}}(C \triangleleft \beta, B)$

$m \in \mathrm{Hom}_{\mathcal{M}}(A \triangleleft \gamma, B)$

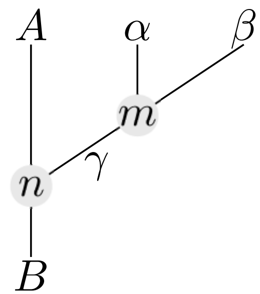
$k \in \mathrm{Hom}_{\mathcal{D}}(\alpha \otimes \beta, \gamma)$

PEPS module category

- recoupling identity for PEPS tensor $\Longrightarrow$ PEPS tensor is module 6j-symbol ${}^{3}\mathbf{F}$
- module 6j-symbol ${}^{3}\mathbf{F}$ expresses isomorphism $(A \triangleleft \alpha) \triangleleft \beta \xrightarrow{\cong} A \triangleleft (\alpha \otimes \beta)$:



$$= \sum_{\gamma,m,n} \left({}^{3}\mathbf{F}_{B}^{A\alpha\beta}\right)_{C,jk}^{\gamma,mn}$$



- recoupling identity for PEPS tensor then reads explicitly

$$\sum_{o} \left({}^{3}\mathbf{F}_{B}^{C\beta\gamma}\right)_{D,lm}^{\mu,no} \left({}^{3}\mathbf{F}_{B}^{A\alpha\mu}\right)_{C,ko}^{\nu,pq} = \sum_{\delta,rst} \left({}^{3}\mathbf{F}_{D}^{A\alpha\beta}\right)_{C,kl}^{\delta,rs} \left({}^{3}\mathbf{F}_{B}^{A\delta\gamma}\right)_{D,sm}^{\nu,tq} \left({}^{4}\mathbf{F}_{\nu}^{\alpha\beta\gamma}\right)_{\delta,rt}^{\mu,np}$$

$\Longrightarrow$ is mixed pentagon identity

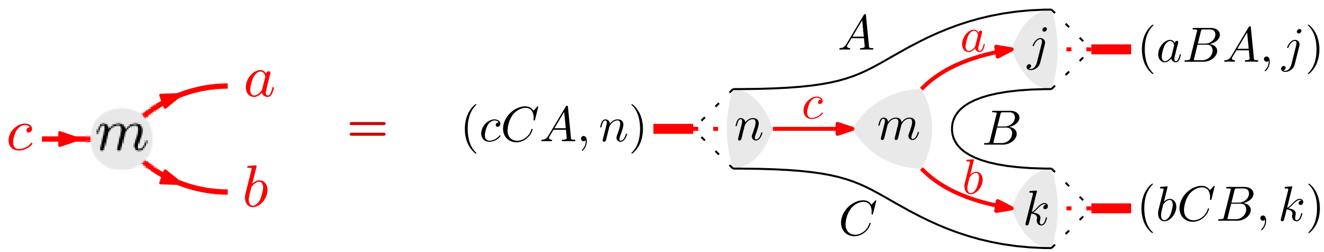
stating the equality of two distinguished isomorphisms

$$((A \triangleleft \alpha) \triangleleft \beta) \triangleleft \gamma \xrightarrow{\cong} A \triangleleft (\alpha \otimes (\beta \otimes \gamma))$$

PEPS/MPO module category

☞ recall : also a recoupling identity for MPO fusion tensor

which is now realized as



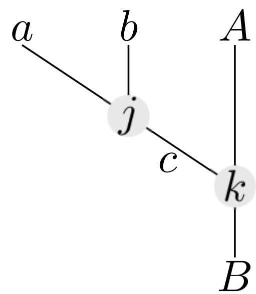
PEPS/MPO module category

- recall : also a recoupling identity for MPO fusion tensor
- interpretation :

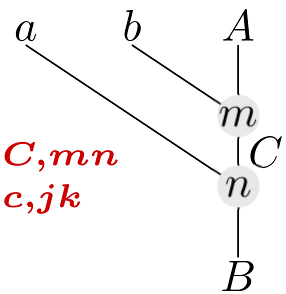
  $\mathcal{M}$ also has structure of a left module category over the fusion category $\mathcal{C}$

  and MPO fusion tensor is module 6j-symbol ${}^{1}\mathbf{F}$

- module 6j-symbol ${}^{1}\mathbf{F}$ describes isomorphism $(a \otimes b) \triangleright A \xrightarrow{\cong} a \triangleright (b \triangleright A)$



$$= \sum_{C,m,n} \left({}^{1}\mathbf{F}^{abA}_{B}\right)^{C,mn}_{c,jk}$$



- recoupling identity for MPO tensor reads explicitly

$$\sum_{o} \left({}^{1}\mathbf{F}^{fcA}_{B}\right)^{C,no}_{g,lm} \left({}^{1}\mathbf{F}^{abC}_{B}\right)^{D,pq}_{f,ko} = \sum_{j,rst} \left({}^{0}\mathbf{F}^{abc}_{g}\right)^{j,rs}_{f,kl} \left({}^{1}\mathbf{F}^{ajA}_{B}\right)^{D,tq}_{g,sm} \left({}^{1}\mathbf{F}^{bcA}_{D}\right)^{C,np}_{j,rt}$$

$\Longrightarrow$ is mixed pentagon identity

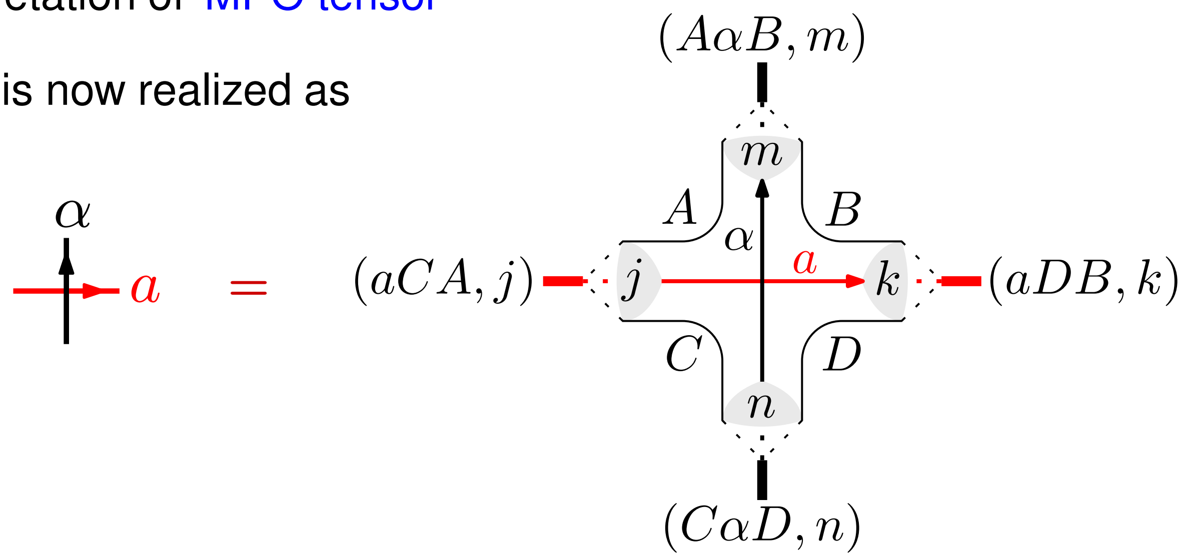
stating the equality of two distinguished isomorphisms

$$((a \otimes b) \otimes c) \triangleright A \xrightarrow{\cong} a \triangleright (b \triangleright (c \triangleright A))$$

PEPS/MPO bimodule category

☞ still lacking : interpretation of MPO tensor

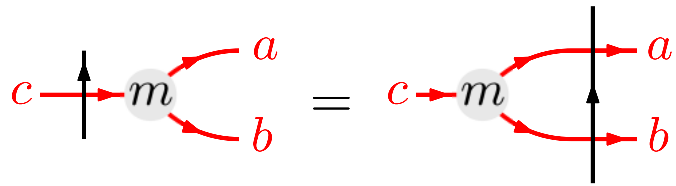
which is now realized as



☞ still lacking : interpretation of MPO tensor

☞ not yet used : MPO tensor satisfies two further consistency conditions :

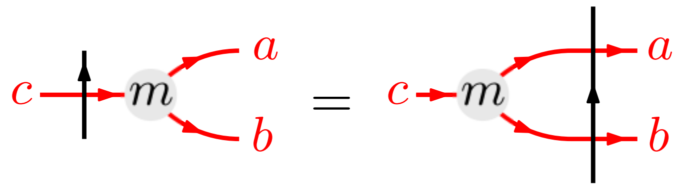
⇝ zipper condition :



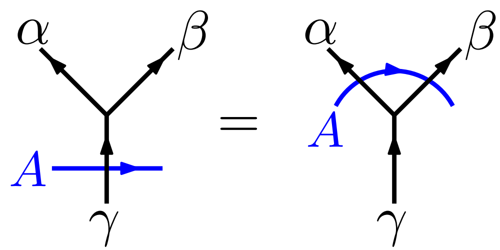
location of fusion process on the lattice does not matter

PEPS/MPO bimodule category

- still lacking : interpretation of MPO tensor
- not yet used : MPO tensor satisfies two further consistency conditions :
  - zipper condition :



  - pulling-through condition



MPOs pass freely through lattice of PEPS tensors

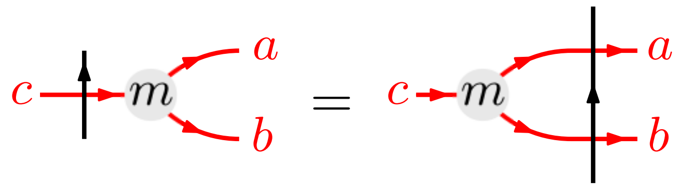
(can be interpreted as
RG transformation for scale invariant MPOs)

PEPS/MPO bimodule category

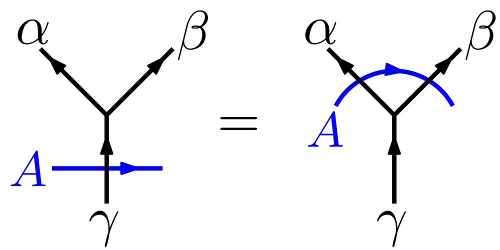
☞ still lacking : interpretation of MPO tensor

☞ not yet used : MPO tensor satisfies two further consistency conditions :

⇝ zipper condition :



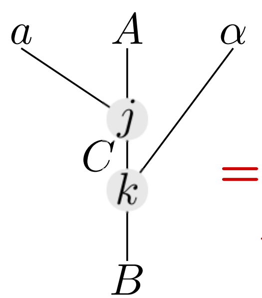
⇝ pulling-through condition



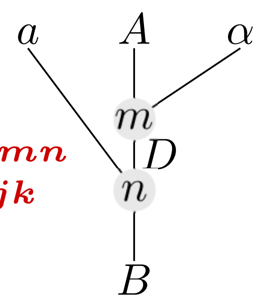
Interpretation : $\mathcal{M}$ is in fact a bimodule category

☞ MPO tensor is bimodule 6j-symbol ${}^2\mathbf{F}$

expressing isomorphism $a \triangleright (A \triangleleft \alpha) \xrightarrow{\cong} (a \triangleright A) \triangleleft \alpha$

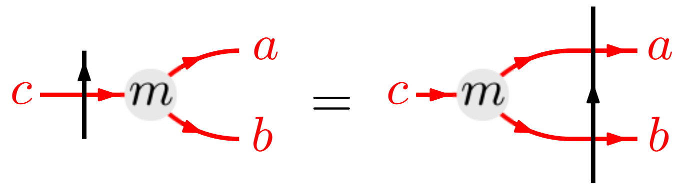


$$= \sum_{D,m,n} \left({}^2\mathbf{F}^{a\,A\,\alpha}_{B}\right)^{D,mn}_{C,jk}$$

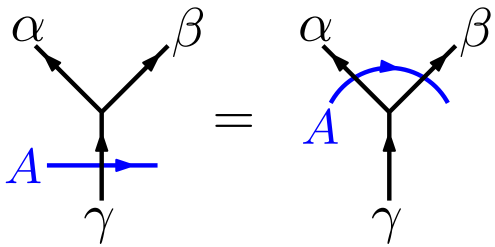


PEPS/MPO bimodule category

- still lacking : interpretation of MPO tensor
- not yet used : MPO tensor satisfies two further consistency conditions :
  - zipper condition :



  - pulling-through condition



Interpretation : $\mathcal{M}$ is in fact a bimodule category

- MPO tensor is bimodule 6j-symbol ${}^2\mathbf{F}$
- zipper and pulling-through conditions are mixed pentagon equations

$$\sum_{o} ({}^2\mathbf{F}^{fA\alpha}_{B})^{D,no}_{C,lm} \, ({}^1\mathbf{F}^{abD}_{B})^{E,pq}_{f,ko} = \sum_{F,rst} ({}^1\mathbf{F}^{abA}_{C})^{F,rs}_{f,kl} \, ({}^2\mathbf{F}^{aF\alpha}_{B})^{E,tq}_{C,sm} \, ({}^2\mathbf{F}^{bA\alpha}_{E})^{D,np}_{F,rt}$$

$$\sum_{o} ({}^3\mathbf{F}^{C\alpha\beta}_{B})^{\gamma,no}_{D,lm} \, ({}^2\mathbf{F}^{aA\gamma}_{B})^{E,pq}_{C,ko} = \sum_{F,rst} ({}^2\mathbf{F}^{aA\alpha}_{D})^{F,rs}_{C,kl} \, ({}^2\mathbf{F}^{aF\beta}_{B})^{E,tq}_{D,sm} \, ({}^3\mathbf{F}^{A\alpha\beta}_{E})^{\gamma,np}_{F,rt}$$

for $((a\otimes b)\triangleright A)\triangleleft\alpha \xrightarrow{\cong} a\triangleright(b\triangleright(A\triangleleft\alpha))$ / $((a\triangleright A)\triangleleft\alpha)\triangleleft\beta \xrightarrow{\cong} a\triangleright(A\triangleleft(\alpha\otimes\beta))$

Summary:

PEPS, MPO and MPO fusion tensors and their consistency relations amount to

- a fusion category $\mathcal{C}$ $+$ a fusion category $\mathcal{D}$ $+$ a $\mathcal{C}$-$\mathcal{D}$-bimodule category $\mathcal{M}$
- identifications

right module constraint	${}^3\mathbf{F}$	$\longleftrightarrow$	PEPS tensor
bimodule constraint	${}^2\mathbf{F}$	$\longleftrightarrow$	MPO tensor
left module constraint	${}^1\mathbf{F}$	$\longleftrightarrow$	MPO fusion tensor

- identifications

pentagon identity for $\mathcal{C}$	"$\mathbf{0\,0 = 0\,0\,0}$"		
left module mixed pentagon	"$\mathbf{1\,1 = 0\,1\,1}$"	$\longleftrightarrow$	recoupling MPO fusion tensor
bimodule mixed pentagon 1	"$\mathbf{2\,1 = 1\,2\,2}$"	$\longleftrightarrow$	zipper condition
bimodule mixed pentagon 2	"$\mathbf{3\,2 = 2\,2\,3}$"	$\longleftrightarrow$	pulling-through condition
right module mixed pentagon	"$\mathbf{3\,3 = 3\,3\,4}$"	$\longleftrightarrow$	recoupling PEPS tensor
pentagon identity for $\mathcal{D}$	"$\mathbf{4\,4 = 4\,4\,4}$"		

PEPS/MPO bicategory

Summary :

PEPS , MPO and MPO fusion tensors and their consistency relations amount to

- a fusion category $\mathcal{C}$ + a fusion category $\mathcal{D}$ + a $\mathcal{C}$-$\mathcal{D}$-bimodule category $\mathcal{M}$
- identifications

right module constraint	${}^{3}\mathbf{F}$	$\longleftrightarrow$	PEPS tensor
bimodule constraint	${}^{2}\mathbf{F}$	$\longleftrightarrow$	MPO tensor
left module constraint	${}^{1}\mathbf{F}$	$\longleftrightarrow$	MPO fusion tensor

- identifications of pentagon identities

a string-net PEPS satisfying a $\mathcal{D}$-type recoupling condition
has $\mathcal{C}$-type MPO symmetries
iff there exists a compatible $\mathcal{C}$-$\mathcal{D}$-bimodule category $\mathcal{M}$

Summary :

PEPS , MPO and MPO fusion tensors and their consistency relations amount to

- ☞ a fusion category $\mathcal{C}$ + a fusion category $\mathcal{D}$ + a $\mathcal{C}$-$\mathcal{D}$-bimodule category $\mathcal{M}$
- ☞ identifications

right module constraint	${}^{3}\mathbf{F}$	$\longleftrightarrow$	PEPS tensor
bimodule constraint	${}^{2}\mathbf{F}$	$\longleftrightarrow$	MPO tensor
left module constraint	${}^{1}\mathbf{F}$	$\longleftrightarrow$	MPO fusion tensor

Special case : $\mathcal{M}$ invertible bimodule category

as is arguably required for MPO injectivity

PEPS/MPO bicategory

Summary :

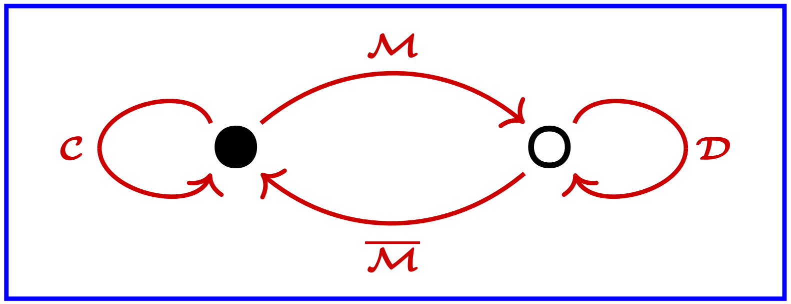
PEPS , MPO and MPO fusion tensors and their consistency relations amount to

- a fusion category $\mathcal{C}$ + a fusion category $\mathcal{D}$ + a $\mathcal{C}$-$\mathcal{D}$-bimodule category $\mathcal{M}$
- identifications

right module constraint	${}^{3}\mathbf{F}$	$\longleftrightarrow$	PEPS tensor
bimodule constraint	${}^{2}\mathbf{F}$	$\longleftrightarrow$	MPO tensor
left module constraint	${}^{1}\mathbf{F}$	$\longleftrightarrow$	MPO fusion tensor

Special case : $\mathcal{M}$ invertible bimodule category

$\Longrightarrow$ data fit into 2 - Morita context (or : 2 - object bicategory)



in particular : $\mathcal{D} = \mathcal{C}^{\star}_{\mathcal{M}} \equiv \mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{M})$ and $\mathcal{Z}(\mathcal{D}) \simeq \mathcal{Z}(\mathcal{C})$

PEPS/MPO from TFT

Tensor network data:

- PEPS tensor A and MPO tensor B
- oriented surface Σ with cell decomposition Δ

  for concreteness: Δ $=$ honeycomb lattice
- space $\mathcal{H}$ associated to physical leg of A : $\mathcal{H} = \bigoplus_{\alpha,\beta,\gamma \in I_\mathcal{D}} \mathrm{Hom}_\mathcal{D}(\alpha \otimes \beta, \gamma)$
- space associated to surface Σ : $\mathcal{H}_\Sigma = \bigotimes_{v \in \Delta_0} \mathcal{H}$ (a copy of $\mathcal{H}$ at each vertex)

  depends on cell decomposition

Tensor network data :

- PEPS tensor A and MPO tensor B
- oriented surface Σ with cell decomposition Δ

  for concreteness : Δ $=$ honeycomb lattice
- space $\mathcal{H}$ associated to physical leg of A : $\mathcal{H} = \bigoplus_{\alpha,\beta,\gamma \in I_\mathcal{D}} \mathrm{Hom}_\mathcal{D}(\alpha \otimes \beta, \gamma)$
- space associated to surface Σ : $\mathcal{H}_\Sigma = \bigotimes_{v \in \Delta_0} \mathcal{H}$ (a copy of $\mathcal{H}$ at each vertex)
- protected space $\mathcal{H}_\Sigma^0 \subseteq \mathcal{H}_\Sigma$

  obtained by contracting the virtual legs of the PEPS tensors along the edges of Δ

does not depend on cell decomposition

Tensor network data :

- PEPS tensor A and MPO tensor B
- oriented surface Σ with cell decomposition Δ
  for concreteness : Δ $=$ honeycomb lattice
- space $\mathcal{H}$ associated to physical leg of A : $\mathcal{H} = \bigoplus_{\alpha,\beta,\gamma \in I_{\mathcal{D}}} \mathrm{Hom}_{\mathcal{D}}(\alpha \otimes \beta, \gamma)$
- space associated to surface Σ : $\mathcal{H}_\Sigma = \bigotimes_{v \in \Delta_0} \mathcal{H}$ (a copy of $\mathcal{H}$ at each vertex)
- protected space $\mathcal{H}^0_\Sigma \subseteq \mathcal{H}_\Sigma$
  obtained by contracting the virtual legs of the PEPS tensors along the edges of Δ

Goal :

- obtain subspace $\mathcal{H}^0_\Sigma$ by a Turaev-Viro state-sum construction
- recover the PEPS/MPO bicategory from that construction

Tensor network data:

- PEPS tensor A and MPO tensor B
- oriented surface Σ with cell decomposition Δ
  for concreteness: Δ = honeycomb lattice
- space $\mathcal{H}$ associated to physical leg of A: $\mathcal{H} = \bigoplus_{\alpha,\beta,\gamma\in I_{\mathcal{D}}} \mathrm{Hom}_{\mathcal{D}}(\alpha\otimes\beta,\gamma)$
- space associated to surface Σ: $\mathcal{H}_\Sigma = \bigotimes_{v\in\Delta_0} \mathcal{H}$ (a copy of $\mathcal{H}$ at each vertex)
- protected space $\mathcal{H}^0_\Sigma \subseteq \mathcal{H}_\Sigma$
  obtained by contracting the virtual legs of the PEPS tensors along the edges of Δ

Goal:

- obtain subspace $\mathcal{H}^0_\Sigma$ by a Turaev-Viro state-sum construction

similar to 1-dim case KAPUSTIN-TURZILLO-YOU 2017
and $\mathcal{M} = \mathcal{D}$ - case LUO-LAKE-WU 2017

State sum construction

Strategy :

- Turaev-Viro TFT associated with spherical fusion category $\mathcal{D}$
  assigns to 3-manifold M a linear map T-V(M) : T-V$(\partial_- M) \to$ T-V$(\partial_+ M)$
- in particular for $M = M_\Sigma$ with $\partial_- M_\Sigma = \emptyset$
  and $\partial_+ M_\Sigma = \Sigma$: a linear map $\mathbb{C} \to$ T-V(Σ)
  and hence T-V$(M) \,.\, 1 \in$ T-V(Σ)
- show that in fact T-V$(M) \,.\, 1 = |\psi(A)\rangle$

$\Longrightarrow$ explicit construction of T-V on M_Σ provides a construction of $\mathcal{H}^0_\Sigma$

Prescription for 3-manifold M_Σ :

- 3-manifold : cylinder $M_\Sigma := \Sigma \times [0,1]$
  geometric boundary : $\partial M_\Sigma = \Sigma \times \{0\} \cup \Sigma \times \{1\}$
- take $\Sigma \times \{0\}$ to be a physical boundary ("brane boundary")
  and $\Sigma \times \{1\}$ to be a gluing boundary (microscopic degrees of freedom)

  $\Longrightarrow$ M_Σ of desired form

State sum construction

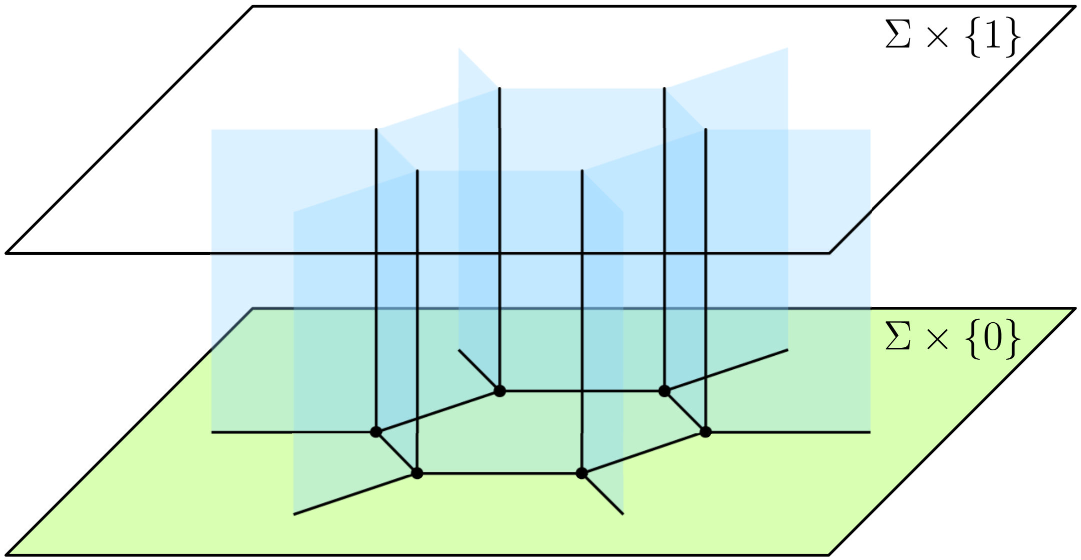
State sum variables :

☞ fix a skeleton P for M_{Σ} not having vertices or edges on $\Sigma \times \{1\}$

State sum variables :

☞ fix a skeleton P for M_{Σ}

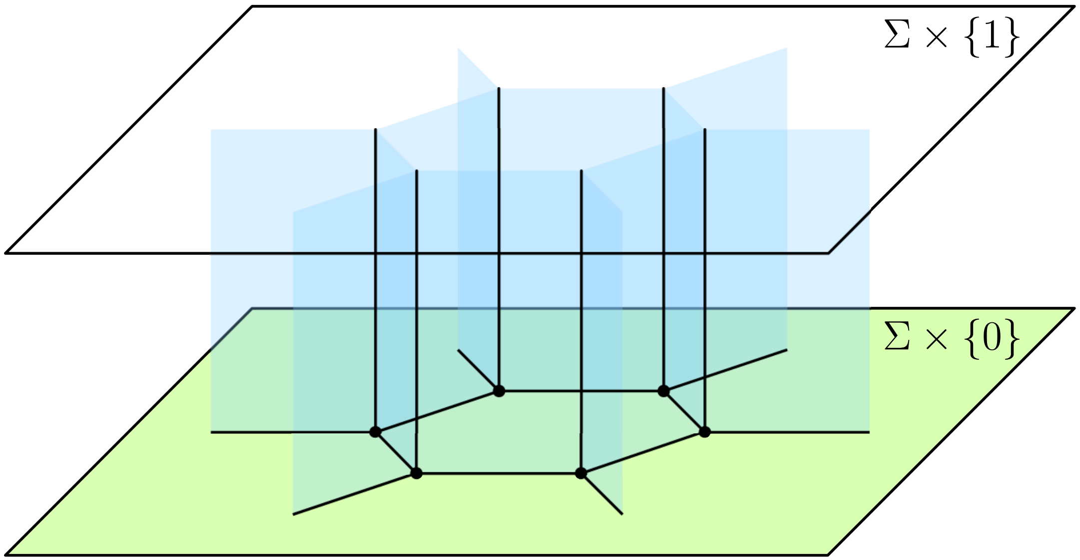
☞ for convenience take P to consist of prisms matching Δ :



(but results do not depend on choice of skeleton)

State sum variables:

- fix a skeleton P for M_{Σ}
- for convenience take P to consist of prisms matching Δ :

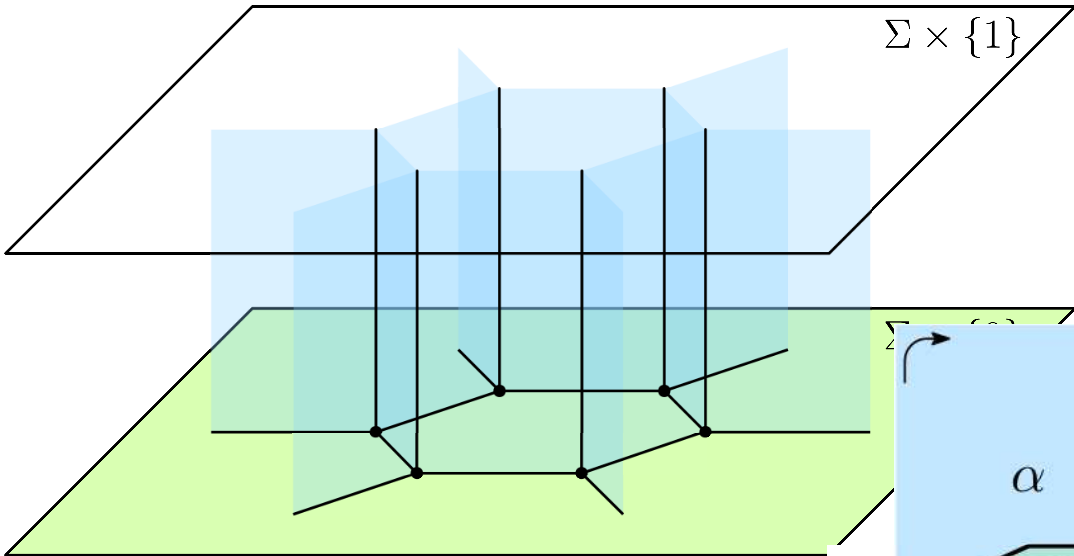


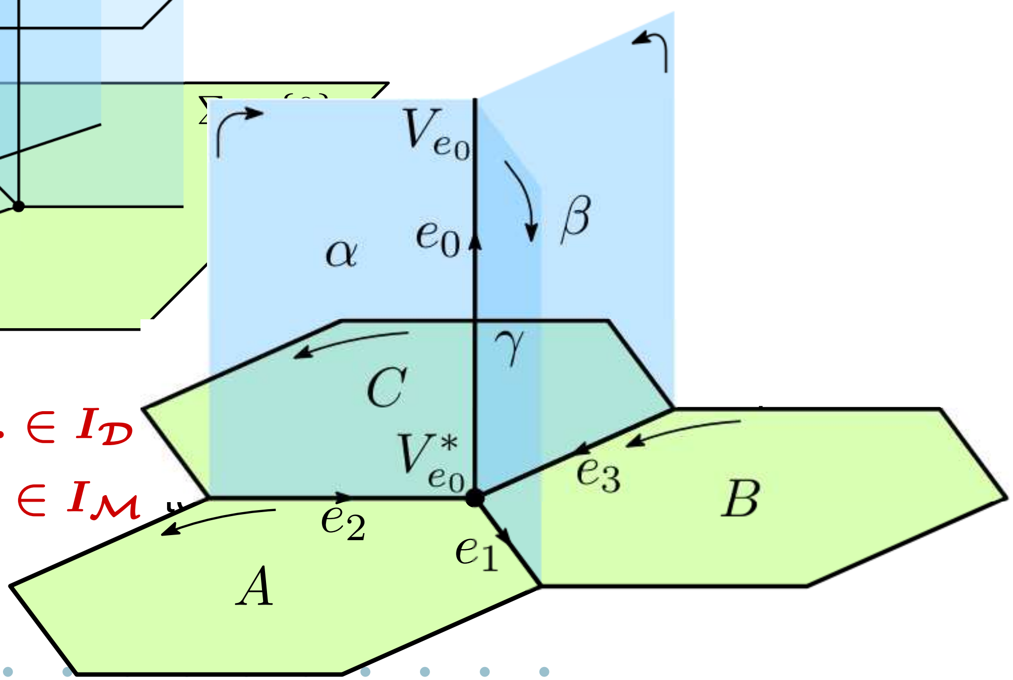
- attach state-sum variables $\alpha, \beta, \gamma, ... \in I_{\mathcal{D}}$ to the plaquettes of P in interior and state-sum variables $A, B, C, ... \in I_{\mathcal{M}}$ to the plaquettes of P on $\Sigma \times \{0\}$

State sum construction

State sum variables :

- fix a skeleton P for M_{Σ}
- for convenience take P to consist of prisms matching Δ :





- attach state-sum variables $\alpha, \beta, \gamma, \ldots \in I_{\mathcal{D}}$

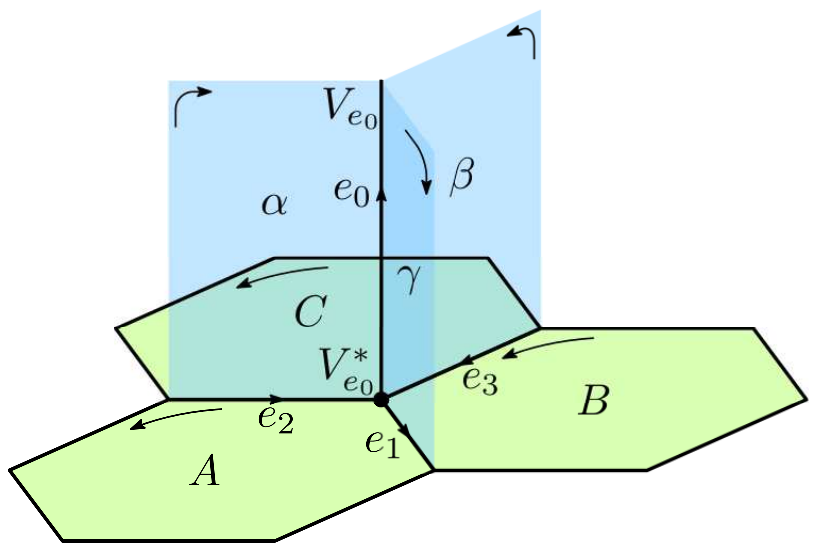
  and state-sum variables $A, B, C, \ldots \in I_{\mathcal{M}}$

Vector spaces:

☞ to every edge $e \in P$ associate vector space $\mathcal{H}_e = V_e \otimes V_e^*$ (two half-edges)

⇝ for edge in interior of M_Σ:

$$V_e = \mathrm{Hom}_{\mathcal{D}}(\alpha \otimes \beta, \gamma) \quad \text{and} \quad V_e^* = \mathrm{Hom}_{\mathcal{D}}(\alpha \otimes \beta, \gamma)^* \cong \mathrm{Hom}_{\mathcal{D}}(\gamma, \alpha \otimes \beta)$$



Vector spaces :

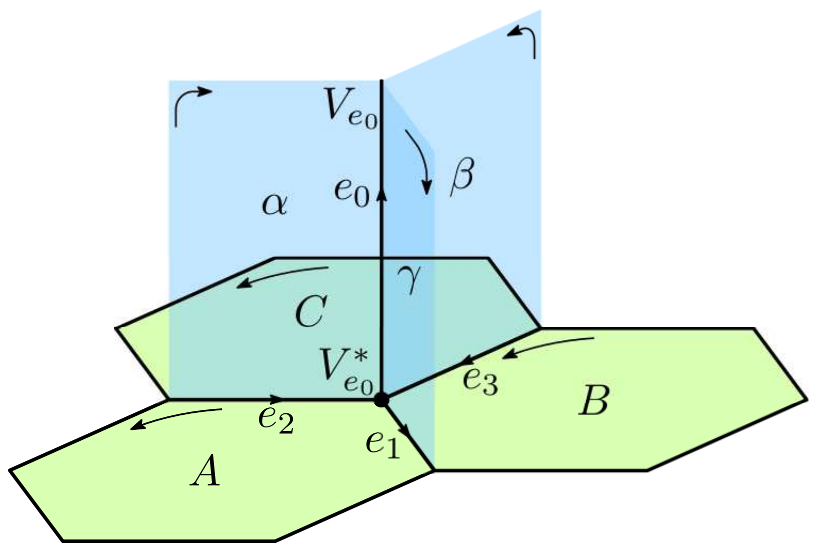
☞ to every edge $e \in P$ associate vector space $\mathcal{H}_e = V_e \otimes V_e^*$ (two half-edges)

- for edge in interior of M_Σ :

  $V_e = \mathrm{Hom}_{\mathcal{D}}(\alpha \otimes \beta, \gamma)$ and $V_e^* = \mathrm{Hom}_{\mathcal{D}}(\alpha \otimes \beta, \gamma)^*$

- for edge on $\Sigma \times \{0\}$:

  $V_e = \mathrm{Hom}_{\mathcal{M}}(A \triangleleft \gamma, B)$ and $V_e^* = \mathrm{Hom}_{\mathcal{M}}(A \triangleleft \gamma, B)^* \cong \mathrm{Hom}_{\mathcal{M}}(B, A \triangleleft \gamma)$

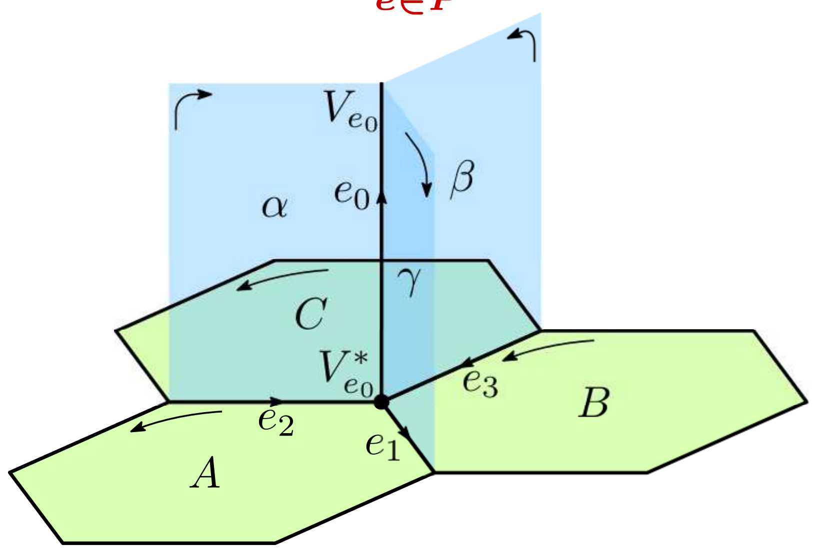


Vector spaces:

- to every edge $e \in P$ associate vector space $\mathcal{H}_e = V_e \otimes V_e^*$ (two half-edges)
  - for edge in interior of M_Σ:

    $V_e = \mathrm{Hom}_{\mathcal{D}}(\alpha \otimes \beta, \gamma)$ and $V_e^* = \mathrm{Hom}_{\mathcal{D}}(\alpha \otimes \beta, \gamma)^*$
  - for edge on $\Sigma \times \{0\}$:

    $V_e = \mathrm{Hom}_{\mathcal{M}}(A \triangleleft \gamma, B)$ and $V_e^* = \mathrm{Hom}_{\mathcal{M}}(A \triangleleft \gamma, B)^*$
- to M_Σ with skeleton P associate vector space $V_P = \bigotimes_{e \in P} \mathcal{H}_e$



Vector spaces :

- to every edge $e \in P$ associate vector space $\mathcal{H}_e = V_e \otimes V_e^*$ (two half-edges)
  - for edge in interior of M_Σ :

    $V_e = \mathrm{Hom}_{\mathcal{D}}(\alpha \otimes \beta, \gamma)$ and $V_e^* = \mathrm{Hom}_{\mathcal{D}}(\alpha \otimes \beta, \gamma)^*$
  - for edge on $\Sigma \times \{0\}$:

    $V_e = \mathrm{Hom}_{\mathcal{M}}(A \triangleleft \gamma, B)$ and $V_e^* = \mathrm{Hom}_{\mathcal{M}}(A \triangleleft \gamma, B)^*$
- to M_Σ with skeleton P associate vector space $V_P = \bigotimes_{e \in P} \mathcal{H}_e$

Canonical vectors :

- for each edge $e \in P$ canonical vector $v_e = \sum_i b_i \otimes b^i \in V_e \otimes V_e^*$

  independent of choice of bases $\{b_i\}$ and $\{b^i\}$
- thus canonical vector $$\boxed{v_P = \bigotimes_{e \in P} v_e \in V_P}$$

Evaluation map :

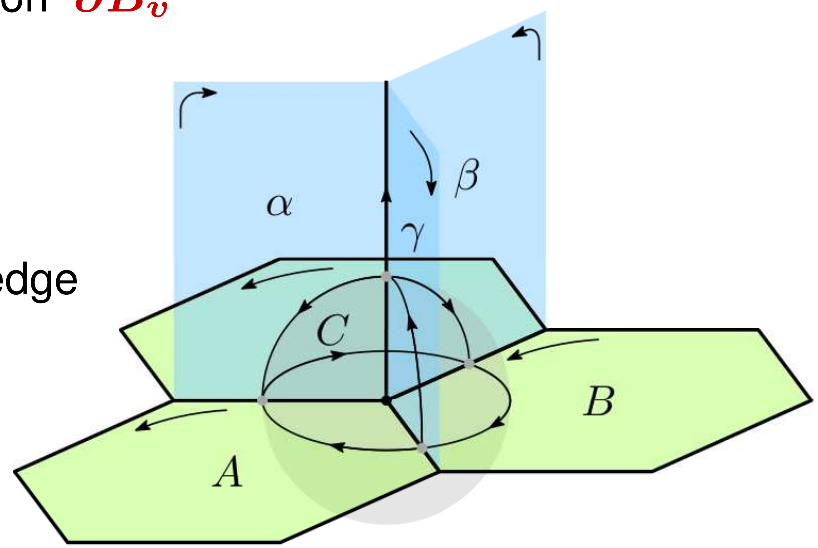
☞ at every vertex v of P have evaluation map ev_v

introduced by Turaev & Virelizier in absence of physical boundary

State sum construction

Evaluation map :

- ☞ at every vertex v of P have evaluation map ev_v
  - draw closed ball B_v around v
  - intersection of B_v gives graph Γ_v on ∂B_v
  - every edge of Γ_v inherits object label from plaquette
  - every vertex of Γ_v inherits vector space label from half-edge

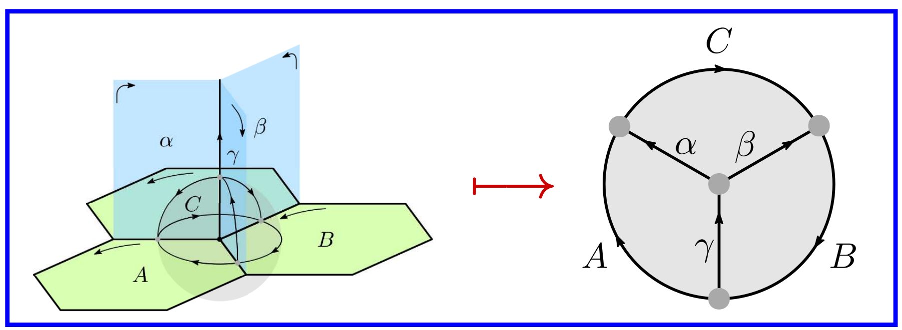


- ☞ evaluate Γ_v

  according to T-V′ rules

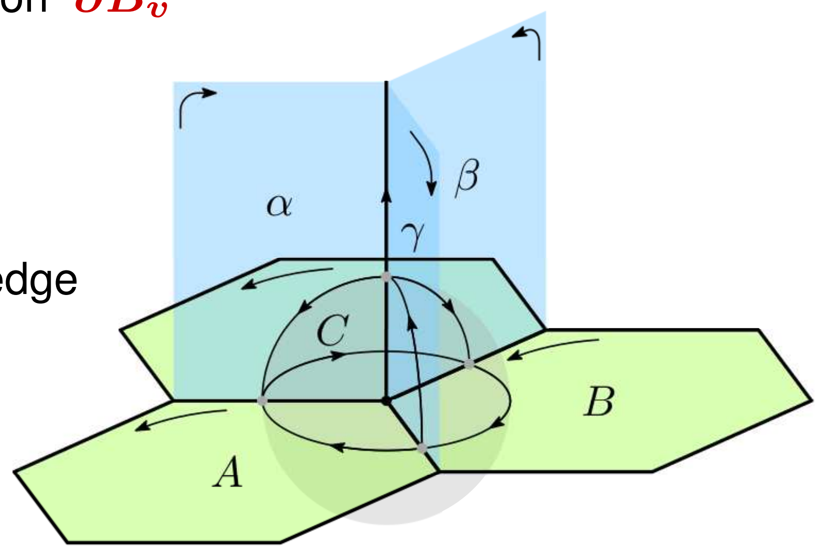
  of state-sum TFT

specifically :



Evaluation map:

- at every vertex v of P have evaluation map ev_v
  - draw closed ball B_v around v
  - intersection of B_v gives graph Γ_v on ∂B_v
  - every edge of Γ_v inherits object label from plaquette
  - every vertex of Γ_v inherits vector space label from half-edge

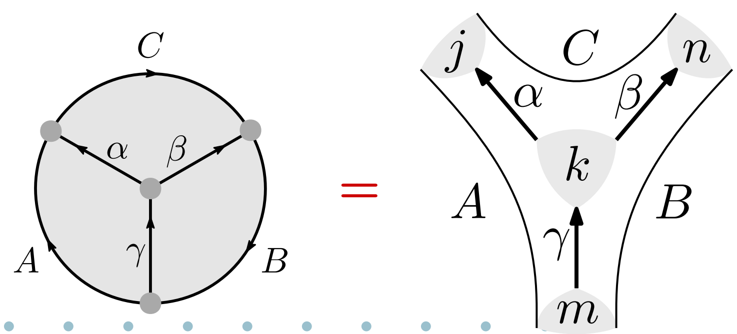


- evaluate Γ_v

  according to T-V′ rules

  of state-sum TFT

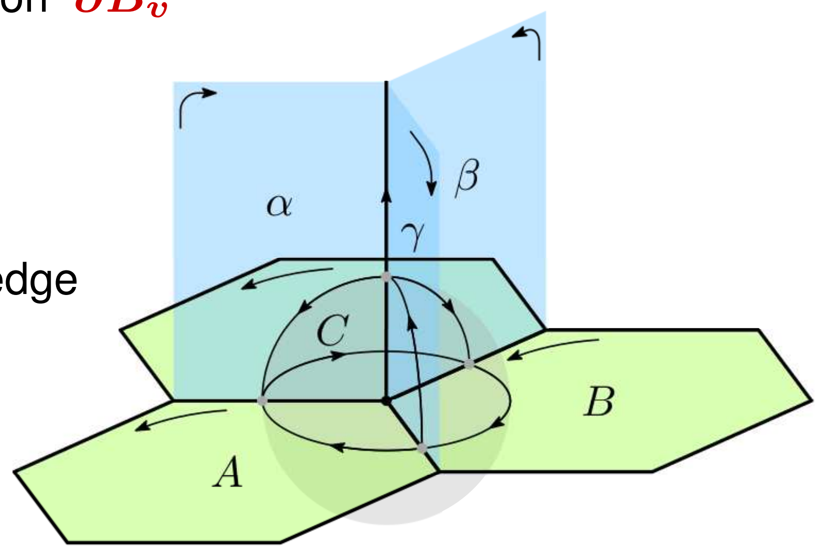
by inspection:



State sum construction

Evaluation map :

- at every vertex v of P have evaluation map ev_v
  - draw closed ball B_v around v
  - intersection of B_v gives graph Γ_v on ∂B_v
  - every edge of Γ_v inherits object label from plaquette
  - every vertex of Γ_v inherits vector space label from half-edge

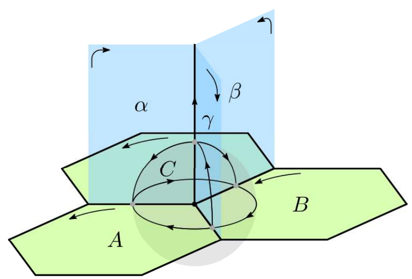


- evaluate Γ_v

  according to T-V$'$ rules

  of state-sum TFT

specifically :

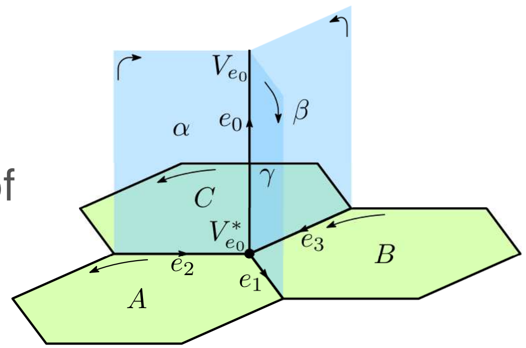


$$\longmapsto \;\sim\; ({}^{3}\mathbf{F}^{A\alpha\beta}_{B})^{\gamma}_{C}$$

Perform evaluation :

☞ ev_v is map from $\bigotimes_e V_e$ for the half-edges incident to v to $\mathbb{C}$

e.g. $\mathrm{ev}_v : V_{e_0}^* \otimes V_{e_1}^* \otimes V_{e_2} \otimes V_{e_3} \to \mathbb{C}$ in case of



Perform evaluation :

- ev_v is map from $\bigotimes_e V_e$ for the half-edges incident to v to $\mathbb{C}$
- combine the evaluations for all vertices of P to get linear map

$$\mathrm{ev}_P = \bigotimes_{v\in P} \mathrm{ev}_v : V_P \to \bigotimes_{\substack{e \text{ ending on} \\ \text{gluing bdy}}} V_e$$

State sum construction

Perform evaluation :

- ☞ ev_v is map from $\bigotimes_e V_e$ for the half-edges incident to v to $\mathbb{C}$
- ☞ combine the evaluations for all vertices of P to get linear map

$$\mathrm{ev}_P = \bigotimes_{v \in P} \mathrm{ev}_v : V_P \to \bigotimes_{\substack{e \text{ ending on} \\ \text{gluing bdy}}} V_e = \mathcal{H}_\Sigma$$

Perform evaluation :

- ev_v is map from $\bigotimes_e V_e$ for the half-edges incident to v to $\mathbb{C}$
- combine the evaluations for all vertices of P to get linear map

$$\mathrm{ev}_P = \bigotimes_{v \in P} \mathrm{ev}_v : V_P \to \bigotimes_{e \text{ ending on gluing bdy}} V_e = \mathcal{H}_\Sigma$$

- finally :

  apply evaluation map ev_P to canonical vector v_P

  by inspection : $\mathrm{ev}_P(v_P) = \mathrm{PEPS}_{\mathcal{D},\mathcal{M}}$

- note :

  by construction $\mathrm{ev}_P(v_P)$ lies in $\text{T-V}(\Sigma) = \mathcal{H}^0_\Sigma$

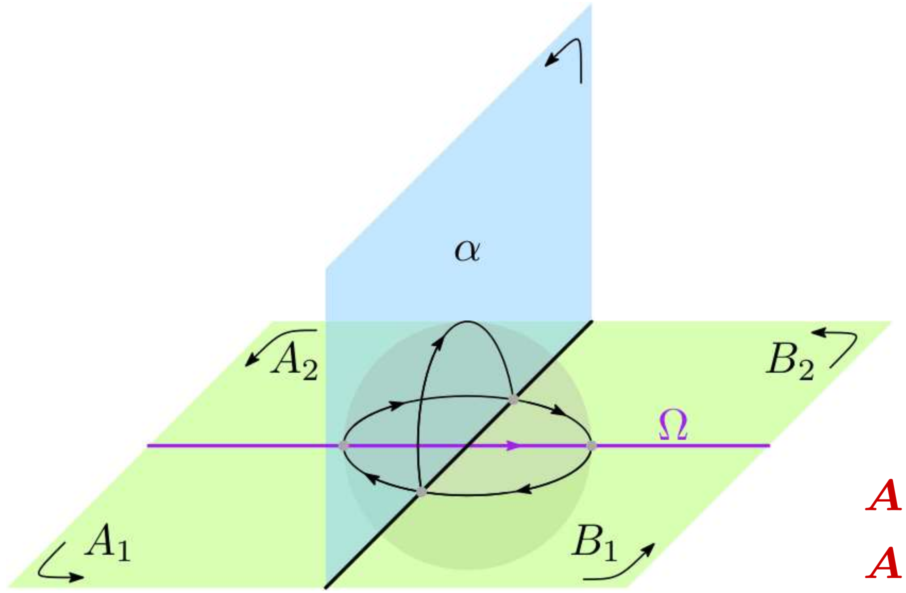
  so $\mathrm{PEPS}_{\mathcal{D},\mathcal{M}}$ lies in protected space as it should

Generalization : physical boundary containing Wilson lines

- ☞ boundary Wilson line $\boldsymbol{\Omega}$ separates regions with boundary conditions $\boldsymbol{\mathcal{M}_1}$ & $\boldsymbol{\mathcal{M}_2}$

  $\Longrightarrow$ $\boldsymbol{\Omega}$ is object in functor category $\mathrm{Fun}_{\mathcal{D}}(\boldsymbol{\mathcal{M}_1}, \boldsymbol{\mathcal{M}_2})$

- ☞ M_{Σ} looks locally as



$\boldsymbol{A_1, B_1 \in I_{\mathcal{M}_1}}$

$\boldsymbol{A_2, B_2 \in I_{\mathcal{M}_2}}$

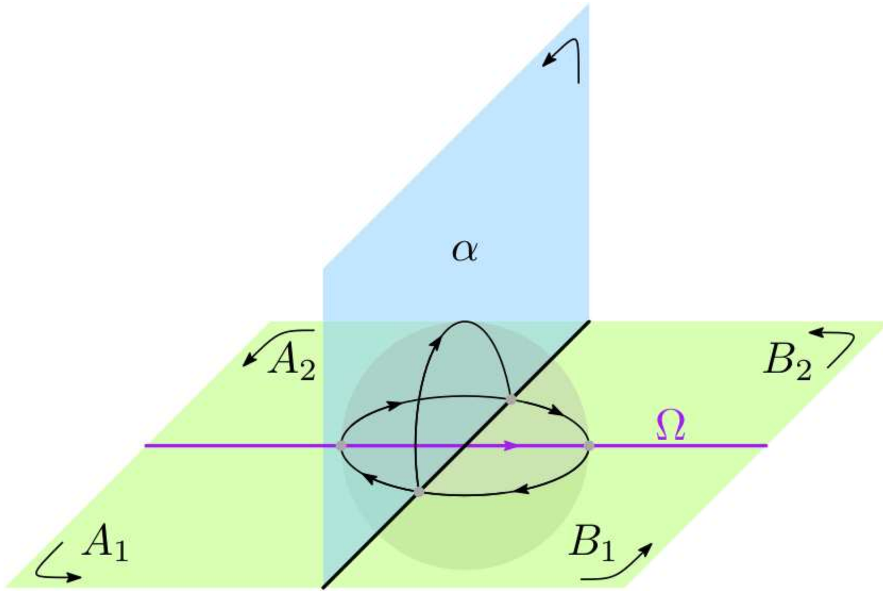
Boundary Wilson lines

Generalization : physical boundary containing Wilson lines

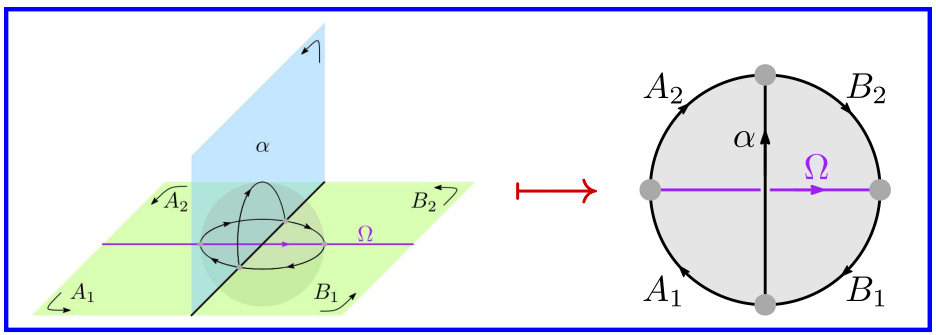
☞ boundary Wilson line $\boldsymbol{\Omega}$ separates regions with boundary conditions $\boldsymbol{\mathcal{M}_1}$ & $\boldsymbol{\mathcal{M}_2}$

$\Longrightarrow$ $\boldsymbol{\Omega}$ is object in functor category $\mathrm{Fun}_{\mathcal{D}}(\boldsymbol{\mathcal{M}_1}, \boldsymbol{\mathcal{M}_2})$

☞ M_{Σ} looks locally as

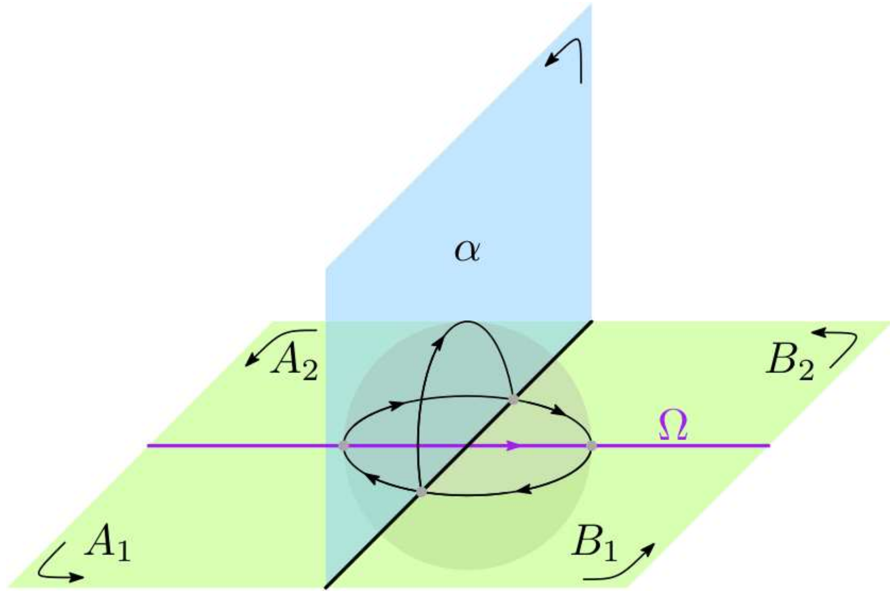


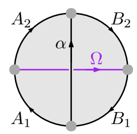
☞ again evaluation map



Generalization: physical boundary containing Wilson lines

- ☞ boundary Wilson line Ω separates regions with boundary conditions $\mathcal{M}_1$ & $\mathcal{M}_2$
  $\Longrightarrow$ Ω is object in functor category $\mathrm{Fun}_{\mathcal{D}}(\mathcal{M}_1, \mathcal{M}_2)$
- ☞ M_Σ looks locally as



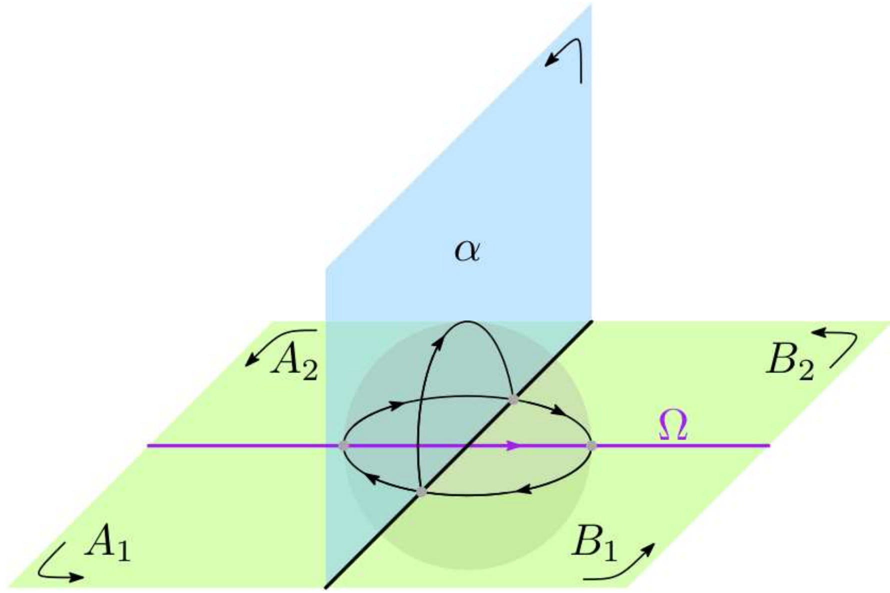
- ☞ can evaluate resulting graph  directly in special cases

Generalization : physical boundary containing Wilson lines

- boundary Wilson line Ω separates regions with boundary conditions $\mathcal{M}_1$ & $\mathcal{M}_2$

  $\Longrightarrow$ Ω is object in functor category $\mathrm{Fun}_{\mathcal{D}}(\mathcal{M}_1, \mathcal{M}_2)$

- M_Σ looks locally as



- special case : $\mathcal{M}_1 = \mathcal{M} = \mathcal{M}_2 \Longrightarrow \mathrm{Fun}_{\mathcal{D}}(\mathcal{M}, \mathcal{M}) = \mathcal{D}^\star_{\mathcal{M}} = \mathcal{C}$

  $\Longrightarrow$ get a ${}^2\mathbf{F}$ symbol (i.e. MPO tensor)

- special case : $\mathcal{M}_1 = \mathcal{D}$ & $\mathcal{M}_2 = \mathcal{M} \Longrightarrow \mathrm{Fun}_{\mathcal{D}}(\mathcal{D}, \mathcal{M}) = \mathcal{M}$

  $\Longrightarrow$ get a ${}^3\mathbf{F}$ symbol

Outlook :

☞ general boundary Wilson lines

using input from module Eilenberg-Watts calculus

☞ describe excitations (in $\mathcal{Z}(\mathcal{D})$)

☞ . . . . . .

☞ e.g. keep promises