On double Lie bialgebroids

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based on joint work with
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Higher Structures and Field Theory
ESI, Vienna, August 2022

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1.1 Lie algebras and exterior algebras

A *Lie algebra* $\mathfrak{g}$ is a vector space endowed with a Lie bracket, an operation $[,] : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$ satisfying the *Jacobi* identity:

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A *differential graded algebra* A is a graded algebra plus a degree 1 differential $d : A \to A$ satisfying $d^2 = 0$ and *Leibniz*:

$$d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b)$$

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Proposition: Given V a vector space, the following are equivalent:

- a Lie bracket $[,] : V \wedge V \to V$
- a differential $d : \Lambda^\bullet V^* \to \Lambda^\bullet V^*$
- a Gerstenhaber bracket $\{,\} : \Lambda^\bullet V \wedge \Lambda^\bullet V \to \Lambda^\bullet V$

1.2 Lie bialgebras

A *Lie bialgebra* $(\mathfrak{g}, \mathfrak{g}^*)$ consists of a vector space $\mathfrak{g}$ and Lie algebra structures on $\mathfrak{g}, \mathfrak{g}^*$ which are *compatible* (best seen in $\Lambda^\bullet \mathfrak{g}^*$):

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- Denoting $x \triangleright \phi$ for the coadjoint representation, we have:

$$\begin{aligned}\langle [\phi, \psi], [x, y] \rangle = &- \langle x \triangleright \psi, \phi \triangleright y \rangle + \langle x \triangleright \phi, \psi \triangleright y \rangle \\ &+ \langle y \triangleright \psi, \phi \triangleright x \rangle - \langle y \triangleright \phi, \psi \triangleright x \rangle\end{aligned}$$

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Example: $sl(2, \mathbb{R})$

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, X^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, X^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$[H, X^\pm] = \pm 2X^\pm \quad [X^+, X^-] = H \qquad \delta(H) = 0 \quad \delta(X^\pm) = X^\pm \wedge H$$

1.3 Manin triple, Drinfeld double and matched pairs

A *Manin triple* consists of a Lie bialgebra $\mathfrak{g}$ with:

- an invariant non-degenerate symmetric bilinear form $\langle , \rangle$ on $\mathfrak{g}$, and
- isotropic subalgebras $p, q \subset \mathfrak{g}$ such that $\mathfrak{g} = p \oplus q$.

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Given $(\mathfrak{g}, \mathfrak{g}^*)$ a Lie bialgebra, its *Drinfeld double* $\mathfrak{g} \oplus \mathfrak{g}^*$ is a Manin triple:

$$[x \oplus \phi, y \oplus \psi] = ([x, y] + \phi \triangleright y - \psi \triangleright x) \oplus ([\phi, \psi] + y \triangleright \phi - x \triangleright \psi)$$

Proposition: Lie bialgebras $\underset{\text{Drinfeld double}}{\Longleftrightarrow}$ Manin triples

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Proposition: Lie bialgebras $\underset{\text{Drinfeld double}}{\Longleftrightarrow}$ Manin triples

A *matched pair* consists of Lie algebras $(\mathfrak{g}, \mathfrak{m})$ and actions $\mathfrak{m} \curvearrowleft \mathfrak{g}$, $\mathfrak{m} \curvearrowright \mathfrak{g}$

$$\phi \triangleright [x, y] = [\phi \triangleright x, y] + [x, \phi \triangleright y] + (\phi \triangleleft x) \triangleright y - (\phi \triangleleft y) \triangleright x$$

$$[\phi, \psi] \triangleleft x = [\phi \triangleleft x, \psi] + [\phi, \psi \triangleleft x] + \phi \triangleleft (\psi \triangleright x) - \psi \triangleleft (\phi \triangleright x)$$

Proposition: $\mathfrak{g}, \mathfrak{g}^*$ Lie algebras are bialgebra iff they form a matched pair

1.4 Linear Poisson structures

Given V a vector space, a *linear Poisson structure* $\{,\} : C^\infty(V) \times C^\infty(V) \to C^\infty(V)$ is a Poisson bracket (Lie+Leibniz) for which linear functions are closed under bracket.

$$\{\ell_\xi, \ell_\eta\} = \ell_{[\xi,\eta]}$$

Characterization: $\{,\}$ linear $\iff$ $(T^*V \to V^*) \xrightarrow{\pi^\#} (TV \to *)$ linear
Proposition: $(V, \{,\})$ linear Poisson $\iff$ $(V^*, [,])$ Lie algebra

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Straightforward extension to vector bundles $V \to M$, linear Poisson structures and *Lie algebroids* $(A, [,], \rho : A \to TM)$:

$$\{\ell_\alpha, \ell_\beta\} = \ell_{[\alpha,\beta]} \qquad \{\ell_\alpha, \pi^*(f)\} = \rho(\alpha)(f) \qquad \{\pi^*(f), \pi^*(g)\} = 0$$

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Proposition: A Poisson structure π on a Lie algebra $\mathfrak{g}$ defines a Lie bialgebra iff $\pi^\#$ is a Lie algebroid morphism

$$(T^*\mathfrak{g} \Rightarrow \mathfrak{g}^*) \xrightarrow{\pi^\#} (T\mathfrak{g} \Rightarrow *)$$

1.5 Integration of Lie bialgebras

A *Poisson group* (G, π) is a Lie group coupled with a Poisson structure π on G that is *multiplicative*, meaning that (tfae):

- $m : G \times G \to G$ is Poisson
- $m \subset G \times G \times \bar{G}$ coisotropic
- $(T^*G \rightrightarrows \mathfrak{g}^*) \xrightarrow{\pi^\#} (TG \rightrightarrows *)$ Lie groupoid morphism

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Theorem (Drinfeld)

Any Lie bialgebra $(\mathfrak{g}, \pi_{\mathfrak{g}})$ integrates to a Poisson group (G, π_G).
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- (Alt) Proof: Use Lie2 to integrate $(T^*\mathfrak{g} \Rightarrow \mathfrak{g}^*) \xrightarrow{\pi^\#} (T\mathfrak{g} \Rightarrow *)$
- Theory naturally extends to correspondence between Poisson groupoids and Lie bialgebroids (Weinstein, Mackenzie, Xu).

Lie 2-algebras

2.1 2-vector spaces

A *2-vector space* $V = (V_1 \rightrightarrows V_0)$ is a groupoid object in vector spaces:

- V_1, V_0 vector spaces (over $\mathbb{R}$, $\dim < \infty$)
- $s, t : V_1 \to V_0$ source and target linear maps
- a linear multiplication $m : V_1 \times_{V_0} V_1 \to V_1$ admitting unit $u : V_0 \to V_1$ and inverse $i : V_1 \to V_1$

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Proposition: A 2-vector space V is the same as a 2-term complex:

$$2vect \underset{Dold-Kan}{\cong} Ch_{1,0}(\mathbb{R})$$

$$\ker s = V_1' \xrightarrow{t} V_0 \qquad V = (V_1' \ltimes V_0 \rightrightarrows V_0)$$

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(Shifted) Duality: Given V a 2-vector space, its *dual* V^* is $V^* = Hom(V, \mathbb{R}[1])$, it corresponds via Dold-Kan to $V_0^* \xrightarrow{t^*} (V_1')^*$.

2.2 Tensor products

- 2-vector spaces are simplicial vector spaces and chain complexes
- both simplicial vector spaces and chain complexes have natural $\otimes$
- None of them restricts to 2-vector spaces! (mistake in literature)

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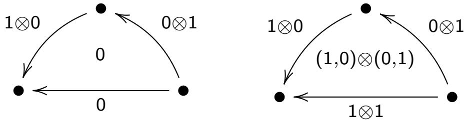
Example:

$(\mathbb{R} \rightrightarrows *)$ is a 2-vector space. Its *nerve* is the simplicial vector space $N\mathbb{R}_k = \mathbb{R}^k$. The levelwise tensor product $N\mathbb{R} \otimes N\mathbb{R}$ is a well-defined simplicial vector space. But it does not satisfy the ! horn-filling condition



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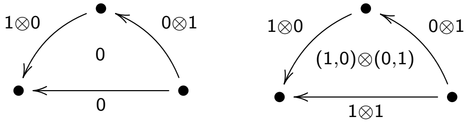

To define 2-algebras we need to deal with bilinearity.
But the category of 2vect is not a monoidal category...

2.3 Lie 2-algebras and crossed modules

A *Lie 2-algebra* $\mathfrak{g} = (\mathfrak{g}_1 \rightrightarrows \mathfrak{g}_0)$ is a groupoid object in Lie algebras:

- $\mathfrak{g}_1, \mathfrak{g}_0$ Lie algebras (over $\mathbb{R}$, dim $< \infty$)
- $s, t : \mathfrak{g}_1 \to \mathfrak{g}_0$ source and target morphisms
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Given a Lie 2-algebra $\mathfrak{g}$, the action $\mathfrak{g} \triangleright \ker s$, $v \triangleright c = [u(v), c]$ is by derivations and satisfy for all $v \in \mathfrak{g}$ and $c \in \ker s$

$$\text{a) } t(v \triangleright c) = [v, t(c)] \qquad \text{b) } t(c_1) \triangleright c_2 = [c_1, c_2]$$

In particular, the bracket in $\mathfrak{g}_1 = \ker s \oplus \mathfrak{g}_0$ can be recovered from the map $\ker s \xrightarrow{t} \mathfrak{g}_0$ and the action $\mathfrak{g} \triangleright \ker s$.

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A pair $(\mathfrak{h} \xrightarrow{\partial} \mathfrak{g}, \mathfrak{g} \triangleright \mathfrak{h})$ of a Lie algebra morphism and an action by derivations satisfying a) and b) is a *Lie algebra crossed module*.

Proposition: Lie 2-algebras $\iff$ Lie algebra crossed module

2.4 Weil algebra perspective

$V = V_1 \oplus V_0 \in Gr(Vect)$. Its *Weil algebra* $W(V) = \Lambda^\bullet(V_0^*) \otimes S^\bullet(V_1^*)$ has a bigrading $|\xi| = (1,0)$ for $\xi \in V_0^*$ and $|\mu| = (1,1)$ for $\mu \in V_1^*$.

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$$
(W(V), d) = \begin{array}{ccccccccc}
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & S^3 V_1^* & \longrightarrow & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & S^2 V_1^* & \longrightarrow & S^2 V_1^* \otimes V_0^* & \longrightarrow & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & V_1^* & \xrightarrow{d_h} & V_1^* \otimes V_0^* & \longrightarrow & V_1^* \otimes \Lambda^2 V_0^* & \longrightarrow & \\
\uparrow & & {\scriptstyle d_v}\uparrow & & \uparrow & {\scriptstyle d_J} & \uparrow & & \\
\mathbb{R} & \longrightarrow & V_0^* & \xrightarrow[d_h]{} & \Lambda^2 V_0^* & \longrightarrow & \Lambda^3 V_0^* & \longrightarrow &
\end{array}
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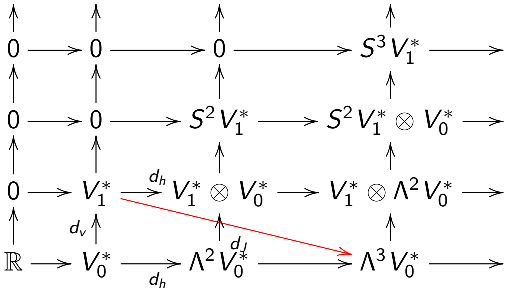
$(W(V), d) =$



Proposition:

2-vector space	$\Longleftrightarrow$	$(W(V), d_v)$	$\Longleftrightarrow$	$(W(V^*), d_v)$
Lie 2-algebra	$\Longleftrightarrow$	$(W(V), d_v + d_h)$	$\Longleftrightarrow$	$(W(V^*), d_v, \{,\})$
weak Lie 2-algebra	$\Longleftrightarrow$	$(W(V), d)$	$\Longleftrightarrow$	...

2.5 Integration to Lie 2-groups

2-vector spaces V are VB-groupoids / VB-algebroids over a point:

$$\begin{array}{ccc} V_1 & \rightrightarrows & V_0 \\ \downarrow & & \downarrow \\ * & \rightrightarrows & * \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} V_1 & \Rightarrow & V_0 \\ \downarrow & & \downarrow \\ * & \Rightarrow & * \end{array}$$

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Lie 2-algebras $\mathfrak{g}$ are LA-groupoids / double Lie algebroids over a point:

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A *Lie 2-group* is a groupoid object in Lie groups.

Proposition: Lie 2-groups $\iff$ Lie group crossed modules.

Theorem: Every Lie 2-algebra integrates to a Lie 2-group.

(Alt) Proof: Integrate vertically to get a group object in Lie algebroids. The orbits are Lie groups, so they don't have π_2, no obstructions!

Lie 2-bialgebras

3.1 Double vector bundles

A *double vector bundle* D consists of horizontal and vertical vector bundle structures which are compatible, in the sense that m^h_λ and m^v_μ commute $\forall \lambda, \mu \in \mathbb{R}$:

$$\begin{array}{ccc} D & \rightarrow & B \\ \downarrow & C & \downarrow \\ A & \rightarrow & M \end{array}$$

The *core* $C \rightarrow M$ is $\ker(D \rightarrow B) \cap \ker(D \rightarrow A)$.

3.1 Double vector bundles

A *double vector bundle* D consists of horizontal and vertical vector bundle structures which are compatible, in the sense that m^h_λ and m^v_μ commute $\forall \lambda, \mu \in \mathbb{R}$:

$$\begin{array}{ccc} D & \to & B \\ \downarrow & C & \downarrow \\ A & \to & M \end{array}$$

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Example

Given $E \to M$ a vector bundle, the tangent and cotanget:

$$\begin{array}{ccc} TE & \to & TM \\ \downarrow & E & \downarrow \\ E & \to & M \end{array} \qquad \begin{array}{ccc} T^*E & \to & E^* \\ \downarrow & T^*M & \downarrow \\ E & \to & M \end{array}$$

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Proposition: Every double vector bundle splits (though non-canonically)

$$D \cong A \oplus B \oplus C \qquad m^v_\lambda(a,b,c) = (a, \lambda b, \lambda c) \quad m^h_\mu(a,b,c) = (\mu a, b, \mu c)$$

3.2 Duality and triality

$$
\begin{array}{ccc}
D^{\bullet} & \rightarrow & B \\
\downarrow & A^{*} & \downarrow \\
C^{*} & \rightarrow & M
\end{array}
\quad \underset{dual}{\overset{horizontal}{\leftsquigarrow}} \quad
\begin{array}{ccc}
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\quad \underset{dual}{\overset{vertical}{\rightsquigarrow}} \quad
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Proposition: There is a non-degenerate pairing

$$\mathrm{fl}(D^*) \times_{C^*} D^{\bullet} \to \mathbb{R} \qquad \langle \alpha, \beta \rangle = \langle \alpha, d \rangle - \langle \beta, d \rangle$$

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The group of dualizing operations for double vector bundles has order 6:

$$D \quad \mapsto \quad D, D^\bullet, D^*, \mathrm{fl}(D), \mathrm{fl}(D^\bullet), \mathrm{fl}(D^*)$$

For triple vector bundles this group was studied by Mackenzie, Gracia-Saz, Metha, and it has order 96!

3.3 Compatible Poisson structures

One Poisson str: VB-algebroids / double linear Poisson structure

$$\begin{array}{ccc} D^\bullet & \Rightarrow & B \\ \downarrow & A^* & \downarrow \\ C^* & \Rightarrow & M \end{array} \quad \leftsquigarrow \quad \begin{array}{ccc} (D, \pi) & \to & B \\ \downarrow & C & \downarrow \\ A & \to & M \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} D^* & \to & C^* \\ \Downarrow & B^* & \Downarrow \\ A & \to & M \end{array}$$

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Three Poisson str: double Lie bialgebroids [BCdH, J. Geom. Mech. 2022]

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3.4 Main Theorems

Theorem (BCdH, J. Geom. Mech. 2022)

If (D, π) is a double Lie bialgebroid and π is symplectic then $D \cong T^*A$ is the cotangent of a Lie bialgebroid (Drinfeld double).

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Poisson double groupoids Σ (introduced by Mackenzie) differentiate to double Lie bialgebroids $\mathrm{Lie}^2(\Sigma)$.
Integration is possible under topological assumptions.

Theorem (Meinrenken-Pike, IMRN 2021)

Given a double vector bundle D, a double Lie bialgebroid structure on D is the same as a pair $(W(D), d_h + d_v, \{\})$ where $d = d_h + d_v$ is a differential and $\{,\}$ is a deg $(-1, -1)$ bracket satisfying Leibniz.

3.5 Application to Lie 2-bialgebra

A *Lie 2-bialgebra* is a Poisson LA-groupoid over a point:

$$\begin{array}{ccc} (\mathfrak{g}_1, \pi) & \rightrightarrows & \mathfrak{g}_0 \\ \Downarrow & & \Downarrow \\ * & \rightrightarrows & * \end{array}$$

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Thm: Lie 2-bialgebras can be integrated to Poisson 2-groups. [Chen-Stienon-Xu, J. Differential Geom. 2013]

Thanks!