QUANTUM FIELD THEORY AND VERTEX ALGEBRAS WITHIN 2-MONOIDAL CATEGORIES

§1. MOTIVATION: ↳ IN MANY FORMULATIONS OF QFT, PROPAGATORS ARE GIVEN BY

$$\Delta \in \mathrm{Hom}_{\underset{\text{spacetime}\uparrow}{C^\infty(M)\otimes C^\infty(M)}}\big(\Gamma(\underset{\substack{\uparrow \\ \text{vector bundle over } M}}{J'E})\otimes\Gamma(J'E),\ \mathcal{D}'(M)\otimes\mathcal{D}'(M)\big)$$

AND WE WANT TO EXTEND Δ TO A LAPLACE PAIRING ON SOMETHING LIKE $T(S_{C^\infty(M)}\Gamma(J'E))$ [BORCHERDS, BROUDER-FAUSER-FRABETTI-OECKL, ...]

AS I NOTED THIS IS A BIALGEBRA IN A 2-MONOIDAL CATEGORY.

↳ IN ANOTHER ARTICLE, BORCHERDS GAVE ANOTHER DEFINITION OF VERTEX ALGEBRA, AS CERTAIN ALGEBRA IN A 2-MONOIDAL CATEGORY (WELL, THIS NOTION DIDN'T EXIST AT THE TIME: HE PROPOSED GENERAL PROPERTIES)

⤳ THESE TWO CONSTRUCTIONS SHARE MANY PROPERTIES/CHARACTERISTICS (IN THIS TALK I WILL FOCUS ON THE CONSTRUCTIONS ON THE VERTEX ALGEBRA SIDE, ANSWERING A PROBLEM IN BORCHERDS' ARTICLE)

↳ PROBLEM 5.12

§2. 2-MONOIDAL CATEGORIES [AGUIAR-MAHAJAN '10; BATANIN-MARKL '12; STREET '12]

(OR DUOIDAL CATEGORIES)

DEF A 2-MONOIDAL CATEGORY IS A CATEGORY $\mathcal{C}$ WITH TWO MONOIDAL STRUCTURES $(\mathcal{C},\otimes_{\mathcal{C}},I_{\otimes})$ AND $(\mathcal{C},\boxtimes_{\mathcal{C}},I_{\boxtimes})$, AS WELL AS

$$(A\otimes_{\mathcal{C}}B)\boxtimes_{\mathcal{C}}(C\otimes_{\mathcal{C}}D)\xrightarrow{sh}(A\boxtimes_{\mathcal{C}}C)\otimes_{\mathcal{C}}(B\boxtimes_{\mathcal{C}}D) \qquad \text{[INTERCHANGE LAW]}$$

AND

$$I_{\otimes}\boxtimes_{\mathcal{C}}I_{\otimes}\xrightarrow{\mu}I_{\otimes}\xleftarrow{\nu}I_{\boxtimes}\xrightarrow{\Delta}I_{\boxtimes}\otimes_{\mathcal{C}}I_{\boxtimes}$$

SATISFYING

(1) COHERENCE AXIOMS ON ASSOCIATIVITY [TWO HEXAGONS]

(2) COMPATIBILITY AXIOMS BETWEEN sh AND THE UNITS $I_{\boxtimes}$ AND $I_{\otimes}$ [FOUR SQUARES]

(3) $(I_{\otimes},\mu,\nu)$ IS A (UNITARY) ALG IN $(\mathcal{C},\boxtimes_{\mathcal{C}},I_{\boxtimes})$

$(I_{\boxtimes},\Delta,\nu)$ IS A (COUNITARY) COALG IN $(\mathcal{C},\otimes_{\mathcal{C}},I_{\otimes})$.

ONE MAY IMPOSE $\otimes_{\mathcal{C}}$ AND/OR $\boxtimes_{\mathcal{C}}$ TO BE SYMMETRIC AND A COMPATIBILITY WITH sh AND THE (CO)ALGEBRA IN (3)

RK A 2-MON. CAT IS THE SAME AS A PSEUDOMONOID IN THE MONOIDAL 2-CATEGORY $\ell(Cat)$.

PROP IF $\mathcal{C}$ IS A 2-MONOIDAL CATEGORY, A AND A' ARE ALGEBRAS IN $(\mathcal{C},\boxtimes_{\mathcal{C}},I_{\boxtimes})$ THEN $A\otimes_{\mathcal{C}}A'$ IS NATURALLY AN ALGEBRA IN $(\mathcal{C},\boxtimes_{\mathcal{C}},I_{\boxtimes})$.

(ANALOGOUS RESULT FOR COALGEBRAS, AND, IF $\mathcal{C}$ IS SYMMETRIC, FOR (CO)COMMUTATIVE (CO)ALG!)

§3. THE SET UP

$I_{\mathcal{F}}$ IS THE INITIAL OBJECT OF $\mathcal{F}$

(VA.1) A SEMICOCARTESIAN MONOIDAL CATEGORY $(\mathcal{F},\Box,I_{\mathcal{F}})$

(VA.2) A SYM. MON. CATEGORY $(\mathcal{C},\otimes_{\mathcal{C}},I_{\mathcal{C}},\tau^{\mathcal{C}})$, THAT IS COCOMPLETE AND S.TH. $\otimes_{\mathcal{C}}$ COMMUTES WITH COLIMITS ON EACH SIDE

(VA.3) AN OPLAX COCOMM. BIALGEBRA B IN THE SYM. MON. CATEGORY $\mathrm{Fun}(\mathcal{F}^{op},\mathcal{C})$

↳ THE UNDERLYING FUNCTOR $B:\mathcal{F}^{op}\to\mathcal{C}$ IS OPLAX MONOIDAL WITH STR. MORPH. OF BIALGEBRAS

(VA.4) A LAX COMM. ALG $A:\mathcal{F}\to\mathcal{C}$ IN THE SYM. MON. CATEGORY

$\boxed{\mathcal{A} = {}_{B}\mathrm{ExMod}(\mathcal{F}, \mathcal{C})}$ (CATEGORY OF $\boxed{\text{EXTERNAL}}$ MODULES) ⟵ $\boxed{\text{FACT}}$ TENSOR PRODUCT $\otimes_{\mathcal{C}}$ AS IN MODULES OVER BIALG!

${}_{B}\mathrm{ExMod}(\mathcal{F}, \mathcal{C})$ IS THE CATEGORY OF FUNCTORS $M : \mathcal{F} \to \mathcal{C}$ S.TH $M(I) \in {}_{B(I)}\mathrm{Mod}$ VIA $\rho_I : B(I) \otimes_{\mathcal{C}} M(I) \to M(I)$ AND

$$\begin{tikzcd}
 & B(I) \otimes_{\mathcal{C}} M(I) \arrow[dr, "\rho_I"] & \\
B(J) \otimes_{\mathcal{C}} M(I) \arrow[ur, "B(f)\otimes id"] \arrow[dr, "id \otimes M(f)"'] & \circlearrowright & M(I) \arrow[d, "M(f)"] \\
 & B(J) \otimes_{\mathcal{C}} M(J) \arrow[r, "\rho_J"] & M(J)
\end{tikzcd}$$

$\forall f : I \to J$ IN $\mathcal{F}$

$\boxed{\text{RK}}$ AN EXTERNAL MODULE IS THE SAME AS AN h-MODULE, WHERE $h : \mathcal{F} \to \mathcal{C}at$ IS THE STRICT MORPHISM OF BICATEGORIES GIVEN BY COMPOSING $B : \mathcal{F} \to \mathrm{Alg}(\mathcal{C})^{op}$ AND THE RESTRICTION MODULE FUNCTOR $R : \mathrm{Alg}(\mathcal{C})^{op} \to \mathcal{C}at$ SENDING A TO ${}_{A}\mathrm{Mod}(\mathcal{C})$ AND $f : A' \to A$ TO THE RESTRICTION OF SCALARS FUNCTOR ${}_{A}\mathrm{Mod}(\mathcal{C}) \to {}_{A'}\mathrm{Mod}(\mathcal{C})$.

(VA, 5) THE SYM. MONOIDAL CATEGORY $C = {}_{A}\mathrm{Mod}(\mathcal{A})$ WITH USUAL TENSOR PRODUCT $\otimes_A$.

WE WILL EVENTUALLY ASSUME:

(VA.6) $\mathcal{F}$ HAS ASYMMETRIC BRAIDING $\zeta^{\mathcal{F}}$.

$\boxed{\text{EXAMPLES}}$

- $\mathcal{F} = \mathrm{FinSet}^{eq}$: OBJ: $(X, \sim_X)$ WITH X FINITE SET AND $\sim_X$ EQUIV. REL

  MOR: $(X, \sim_X) \xrightarrow{f} (Y, \sim_Y)$ IS A MAP $X \xrightarrow{f} Y$ S.TH $f(x) \sim_Y f(x') \Rightarrow x \sim_X x'$

  THEN $(\mathrm{FinSet}^{eq}, \square, J)$ IS A SYM. MON. CATEGORY WITH $(X, \sim_X) \square (Y, \sim_Y) = (X \sqcup Y, \sim_X \sqcup \sim_Y)$ AND $J = \emptyset$

  $\boxed{\text{FACT}}$ $U : \mathrm{FinSet}^{eq} \to \mathrm{FinSet}$, $(X, \sim_X) \longmapsto X$ IS BRAIDED STRONG MONOIDAL.

- IF H IS A COCOMMUTATIVE BIALGEBRA IN $\mathcal{C}$, DEFINE $T^*B : \mathrm{FinSet}^{op} \to \mathcal{C}$ BY

  $T^*(H)(I) = \bigotimes_{i \in I} H$, $T^*(H)(f)\left(\bigotimes_{j \in J} x_j\right) = \prod_{j \in J \setminus f(I)} \epsilon(x_j) \bigotimes_{i \in I} y_i$, WITH $\bigotimes_{i \in f^{-1}(\{j\})} y_i = \Delta^{[\# f^{-1}(\{j\})]}(x_j)$

  FOR $I, J \in \mathrm{FinSet}$, $f : I \to J$.

  $\boxed{\text{FACT}}$ T^*H IS AN OPLAX COCOMM. BIALGEBRA IN $\mathrm{Fun}(\mathrm{FinSet}^{op}, \mathcal{C})$.

  HENCE $B = T^*H \circ U$ IS A OPLAX COCOMM. BIALGEBRA IN $\mathrm{Fun}((\mathrm{FinSet}^{eq})^{op}, \mathcal{C})$.

§4. MAIN RESULTS

$\boxed{\text{DEF}}$ [BORCHERDS, -] GIVEN M AND N IN $C = {}_{A}\mathrm{Mod}(\mathcal{A})$, DEFINE THE SINGULAR TENSOR PRODUCT

$$(M \boxtimes_C N)(I) = \operatorname*{colim}_{I_1 \square I_2 \to I} \underbrace{\left(M(I_1) \otimes_{\mathcal{C}} N(I_2)\right) \otimes_{A(I_1) \otimes A(I_2)} A(I)}_{\mathrm{Ind}_{A(I_1)\otimes A(I_2)}^{A(I)}\left(M(I_1) \otimes_{\mathcal{C}} N(I_2)\right)}$$

THIS DEFINITION GIVES INDEED A TENSOR PRODUCT, SINCE ASSUMING (VA.1-VA.5):

PROP THE SINGULAR TENSOR PRODUCT $\boxtimes_C$ IS A MONOIDAL TENSOR PRODUCT OVER $C = {}_A\mathrm{Mod}(\mathcal{A})$, WITH UNIT GIVEN BY A REGARDED AS A MODULE OVER ITSELF WITH THE REGULAR ACTION.

MOREOVER, IF WE ASSUME (VA.6) WE ALSO HAVE

PROP THE SYMMETRIC BRAIDINGS OF $\mathcal{H}$ AND $\mathcal{C}$ INDUCE A SYMMETRIC BRAIDING OVER $(C, \boxtimes_C, A)$.

FINALLY, WE HAVE THE MAIN RESULT OF THIS TALK, WHICH WAS

THM CONSIDER THE CATEGORY $C = {}_A\mathrm{Mod}(\mathcal{A})$ ENDOWED WITH THE TENSOR PRODUCTS $\otimes_A$ AND $\boxtimes_C$, WITH UNIT A FOR BOTH. THEN $(C, \otimes_A, \boxtimes_C)$ IS A SYMMETRIC 2-MONOIDAL CATEGORY, WITH $\mu = \Delta = \nu = id_A$.

EXAMPLE/DEF CONSIDER THE SAME EXAMPLES AS BEFORE AND TAKE

- $\mathcal{C} = {}_R\mathrm{Mod}$ FOR SOME COMMUTATIVE RING R
- $H = \bigoplus_{p \in \mathbb{N}_0} R.D^{(p)}$ WITH $D^{(p)} D^{(q)} = \binom{p+q}{p} D^{(p+q)}$, $\Delta(D^{(p)}) = \sum_{q=0}^{p} D^{(q)} \otimes D^{(p-q)}$ (DIVIDED POWER BIALGEBRA)
- A THE ALGEBRA SENDING $(X, \sim_X)$ TO $R[(x_i - x_j)^{\pm 1};\ i,j \in X,\ i \not\sim_X j]$.

THEN AN ALGEBRA M IN $(C, \boxtimes_C, A)$ IS A GENERALIZATION OF A VERTEX ALGEBRA SINCE $M(*, \{(*,*)\})$ IS AN USUAL VERTEX ALGEBRA [BORCHERDS]

NEW DIRECTIONS THE CONSTRUCTIONS OF pQFT IN MY WORK CAN BE OBTAINED USING THESE IDEAS BUT $\mathcal{C}$ SHOULD BE GENERALIZED TO BE A 2-MONOIDAL CATEGORY.