# Continued fractions as induction process on a family of dynamical system : three examples

Pierre Arnoux

April 22, 2025

*Uniform distribution of sequences*
Erwin Schrödinger Institute, Wien

Joint work with

Valérie Berthé
Milton Minervino
Wolfgang Steiner
Jörg M. Thuswaldner

# Aim of the talk

- Continued fractions : a way to approximate numbers (or vectors) by a particular type of fractions;
- But also a way to zoom on a family of recurrent dynamical systems stable by induction,
- associated with a scenery flow and an explicit symbolic coding.
- Two classical examples : the geodesic flow on the modular surface and the Teichmüller geodesic flow
- A new example: Brun's continued fraction

# Aim of the talk

- Continued fractions : a way to approximate numbers (or vectors) by a particular type of fractions;
- But also a way to zoom on a family of recurrent dynamical systems stable by induction,
- associated with a scenery flow and an explicit symbolic coding.
- Two classical examples : the geodesic flow on the modular surface and the Teichmüller geodesic flow
- A new example: Brun's continued fraction

# Aim of the talk

- Continued fractions : a way to approximate numbers (or vectors) by a particular type of fractions;
- But also a way to zoom on a family of recurrent dynamical systems stable by induction,
- associated with a scenery flow and an explicit symbolic coding.
- Two classical examples : the geodesic flow on the modular surface and the Teichmüller geodesic flow
- A new example: Brun's continued fraction

# Aim of the talk

- Continued fractions : a way to approximate numbers (or vectors) by a particular type of fractions;
- But also a way to zoom on a family of recurrent dynamical systems stable by induction,
- associated with a scenery flow and an explicit symbolic coding.
- Two classical examples : the geodesic flow on the modular surface and the Teichmüller geodesic flow
- A new example: Brun's continued fraction

# Aim of the talk

- Continued fractions : a way to approximate numbers (or vectors) by a particular type of fractions;
- But also a way to zoom on a family of recurrent dynamical systems stable by induction,
- associated with a scenery flow and an explicit symbolic coding.
- Two classical examples : the geodesic flow on the modular surface and the Teichmüller geodesic flow
- A new example: Brun's continued fraction

# The traditional (projective) viewpoint

- Write $x$ as

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots}}}$$

- Hence $a_0 = [x]$, $x_0 = x - [x] = \{x\}$
- $a_n = [\frac{1}{x_{n-1}}]$ and $x_n = \{\frac{1}{x_{n-1}}\}$
- $T(x) = \{\frac{1}{x}\}$ Gauss map
- And lots of classical formulas

# The traditional (projective) viewpoint

- Write $x$ as

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots}}}$$

- Hence $a_0 = [x]$, $x_0 = x - [x] = \{x\}$
- $a_n = [\frac{1}{x_{n-1}}]$ and $x_n = \{\frac{1}{x_{n-1}}\}$
- $T(x) = \{\frac{1}{x}\}$ Gauss map
- And lots of classical formulas

# The traditional (projective) viewpoint

- Write $x$ as

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots}}}$$

- Hence $a_0 = [x]$, $x_0 = x - [x] = \{x\}$
- $a_n = [\frac{1}{x_{n-1}}]$ and $x_n = \{\frac{1}{x_{n-1}}\}$
- $T(x) = \{\frac{1}{x}\}$ Gauss map
- And lots of classical formulas

# The traditional (projective) viewpoint

- Write $x$ as

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots}}}$$

- Hence $a_0 = [x]$, $x_0 = x - [x] = \{x\}$
- $a_n = [\frac{1}{x_{n-1}}]$ and $x_n = \{\frac{1}{x_{n-1}}\}$
- $T(x) = \{\frac{1}{x}\}$ Gauss map
- And lots of classical formulas

# The traditional (projective) viewpoint

- Write $x$ as

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots}}}$$

- Hence $a_0 = [x]$, $x_0 = x - [x] = \{x\}$
- $a_n = [\frac{1}{x_{n-1}}]$ and $x_n = \{\frac{1}{x_{n-1}}\}$
- $T(x) = \{\frac{1}{x}\}$ Gauss map
- And lots of classical formulas

# The linear viewpoint

- Consider two lengths (or quantities) $u, v$, with $u > v$
- We want to find a common measure $m$ (if possible)
- Use Euclid's algorithm and measure $u$ by $v$:
- $a_0 = [u/v]$, $u_1 = u - a_0 v$, $v_1 = v$
- Exchange the role of $u$ and $v$ and iterate

## The linear viewpoint

- Consider two lengths (or quantities) $u, v$, with $u > v$
- We want to find a common measure $m$ (if possible)
- Use Euclid's algorithm and measure $u$ by $v$:
- $a_0 = [u/v]$, $u_1 = u - a_0 v$, $v_1 = v$
- Exchange the role of $u$ and $v$ and iterate

# The linear viewpoint

- Consider two lengths (or quantities) $u, v$, with $u > v$
- We want to find a common measure $m$ (if possible)
- Use Euclid's algorithm and measure $u$ by $v$:
- $a_0 = [u/v]$, $u_1 = u - a_0 v$, $v_1 = v$
- Exchange the role of $u$ and $v$ and iterate

# The linear viewpoint

- Consider two lengths (or quantities) $u, v$, with $u > v$
- We want to find a common measure $m$ (if possible)
- Use Euclid's algorithm and measure $u$ by $v$:
- $a_0 = [u/v]$, $u_1 = u - a_0 v$, $v_1 = v$
- Exchange the role of $u$ and $v$ and iterate

# The linear viewpoint

- Consider two lengths (or quantities) $u, v$, with $u > v$
- We want to find a common measure $m$ (if possible)
- Use Euclid's algorithm and measure $u$ by $v$:
- $a_0 = [u/v]$, $u_1 = u - a_0 v$, $v_1 = v$
- Exchange the role of $u$ and $v$ and iterate

# The linear viewpoint

- We obtain

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & a_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_3 & 1 \end{pmatrix} \begin{pmatrix} u_4 \\ v_4 \end{pmatrix}$$

- and so on
- projectivising these equalities, we recover the previous formula.
- Euclid's algorithm extended to the incommensurable case.
- We get the best approximations of the direction $\begin{pmatrix} u \\ v \end{pmatrix}$ by integer vectors

# The linear viewpoint

- We obtain

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & a_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_3 & 1 \end{pmatrix} \begin{pmatrix} u_4 \\ v_4 \end{pmatrix}$$

- and so on
- projectivising these equalities, we recover the previous formula.
- Euclid's algorithm extended to the incommensurable case.
- We get the best approximations of the direction $\begin{pmatrix} u \\ v \end{pmatrix}$ by integer vectors

# The linear viewpoint

- We obtain

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & a_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_3 & 1 \end{pmatrix} \begin{pmatrix} u_4 \\ v_4 \end{pmatrix}$$

- and so on
- projectivising these equalities, we recover the previous formula.
- Euclid's algorithm extended to the incommensurable case.
- We get the best approximations of the direction $\begin{pmatrix} u \\ v \end{pmatrix}$ by integer vectors

# The linear viewpoint

- We obtain

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & a_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_3 & 1 \end{pmatrix} \begin{pmatrix} u_4 \\ v_4 \end{pmatrix}$$

- and so on
- projectivising these equalities, we recover the previous formula.
- Euclid's algorithm extended to the incommensurable case.
- We get the best approximations of the direction $\begin{pmatrix} u \\ v \end{pmatrix}$ by integer vectors

# The linear viewpoint

- We obtain

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & a_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_3 & 1 \end{pmatrix} \begin{pmatrix} u_4 \\ v_4 \end{pmatrix}$$

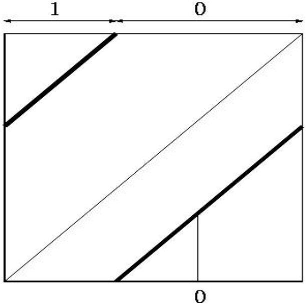
- and so on
- projectivising these equalities, we recover the previous formula.
- Euclid's algorithm extended to the incommensurable case.
- We get the best approximations of the direction $\begin{pmatrix} u \\ v \end{pmatrix}$ by integer vectors

# The dynamical viewpoint

- A rotation on the circle can be seen as an exchange of two intervals
-
- The first return map to the largest interval is again a rotation!
- We can zoom in the dynamics of the rotation
- and recover the previous formulas

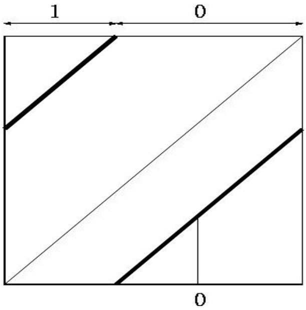
# The dynamical viewpoint

- A rotation on the circle can be seen as an exchange of two intervals



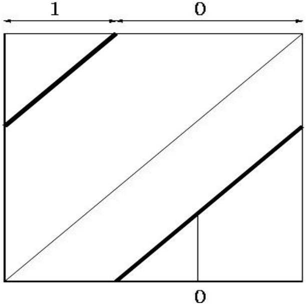

- 
- The first return map to the largest interval is again a rotation!
- We can zoom in the dynamics of the rotation
- and recover the previous formulas

# The dynamical viewpoint

- A rotation on the circle can be seen as an exchange of two intervals
- 

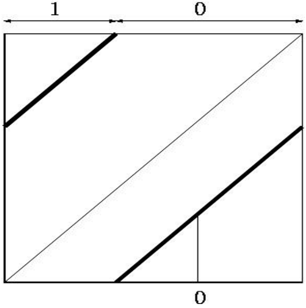
- The first return map to the largest interval is again a rotation!
- We can zoom in the dynamics of the rotation
- and recover the previous formulas

# The dynamical viewpoint

- A rotation on the circle can be seen as an exchange of two intervals
- 

- The first return map to the largest interval is again a rotation!
- We can zoom in the dynamics of the rotation
- and recover the previous formulas

# The dynamical viewpoint

- A rotation on the circle can be seen as an exchange of two intervals
- 

- The first return map to the largest interval is again a rotation!
- We can zoom in the dynamics of the rotation
- and recover the previous formulas

# The dynamical viewpoint

- Two very different dynamics:
- The rotation (one-to-one, invertible, entropy zero)
- The Gauss map, zooming in on the rotation
- many-to-one, not invertible, chaotic, unstable, positive entropy

# The dynamical viewpoint

- Two very different dynamics:
- The rotation (one-to-one, invertible, entropy zero)
- The Gauss map, zooming in on the rotation
- many-to-one, not invertible, chaotic, unstable, positive entropy

# The dynamical viewpoint

- Two very different dynamics:
- The rotation (one-to-one, invertible, entropy zero)
- The Gauss map, zooming in on the rotation
- many-to-one, not invertible, chaotic, unstable, positive entropy

# The dynamical viewpoint

- Two very different dynamics:
- The rotation (one-to-one, invertible, entropy zero)
- The Gauss map, zooming in on the rotation
- many-to-one, not invertible, chaotic, unstable, positive entropy

# Natural extension and suspension (1)

- We want to zoom out
- Add a second dimension, as suspension of the rotation
- and get a linear flow on a flat torus with a cutting and stacking dynamics
- We get a model of the natural extension of the Gauss map.
- Explicit formula : $(x, y) \mapsto (\{\frac{1}{x}\}, x - x^2 y)$
- which preserves Lebesgue measure

# Natural extension and suspension (1)

- We want to zoom out
- Add a second dimension, as suspension of the rotation
- and get a linear flow on a flat torus with a cutting and stacking dynamics
- We get a model of the natural extension of the Gauss map.
- Explicit formula : $(x, y) \mapsto (\{\frac{1}{x}\}, x - x^2 y)$
- which preserves Lebesgue measure

# Natural extension and suspension (1)

- We want to zoom out
- Add a second dimension, as suspension of the rotation
- and get a linear flow on a flat torus with a cutting and stacking dynamics
- We get a model of the natural extension of the Gauss map.
- Explicit formula : $(x, y) \mapsto (\{\frac{1}{x}\}, x - x^2 y)$
- which preserves Lebesgue measure

# Natural extension and suspension (1)

- We want to zoom out
- Add a second dimension, as suspension of the rotation
- and get a linear flow on a flat torus with a cutting and stacking dynamics
- We get a model of the natural extension of the Gauss map.
- Explicit formula : $(x, y) \mapsto (\{\frac{1}{x}\}, x - x^2 y)$
- which preserves Lebesgue measure

# Natural extension and suspension (1)

- We want to zoom out
- Add a second dimension, as suspension of the rotation
- and get a linear flow on a flat torus with a cutting and stacking dynamics
- We get a model of the natural extension of the Gauss map.
- Explicit formula : $(x, y) \mapsto (\{\frac{1}{x}\}, x - x^2 y)$
- which preserves Lebesgue measure

# Natural extension and suspension (1)

- We want to zoom out
- Add a second dimension, as suspension of the rotation
- and get a linear flow on a flat torus with a cutting and stacking dynamics
- We get a model of the natural extension of the Gauss map.
- Explicit formula : $(x, y) \mapsto (\{\frac{1}{x}\}, x - x^2 y)$
- which preserves Lebesgue measure

# Natural extension and suspension (2)

- We build a suspension of this natural extension
- Domain of the flow turns out to be $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$
- Unit tangent bundle of the modular surface
- The link between continued fraction and the modular surface has been known for at least one century:
- Artin (1924), followed by Adler, Flatto, Series and many others

# Natural extension and suspension (2)

- We build a suspension of this natural extension
- Domain of the flow turns out to be $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$
- Unit tangent bundle of the modular surface
- The link between continued fraction and the modular surface has been known for at least one century:
- Artin (1924), followed by Adler, Flatto, Series and many others

# Natural extension and suspension (2)

- We build a suspension of this natural extension
- Domain of the flow turns out to be $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$
- Unit tangent bundle of the modular surface
- The link between continued fraction and the modular surface has been known for at least one century:
- Artin (1924), followed by Adler, Flatto, Series and many others

# Natural extension and suspension (2)

- We build a suspension of this natural extension
- Domain of the flow turns out to be $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$
- Unit tangent bundle of the modular surface
- The link between continued fraction and the modular surface has been known for at least one century:
- Artin (1924), followed by Adler, Flatto, Series and many others

# Natural extension and suspension (2)

- We build a suspension of this natural extension
- Domain of the flow turns out to be $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$
- Unit tangent bundle of the modular surface
- The link between continued fraction and the modular surface has been known for at least one century:
- Artin (1924), followed by Adler, Flatto, Series and many others

# Symbolic dynamics : sturmian sequences

- One can build a completely symbolic model :
- coding by the two intervals = *sturmian sequences*
- sequences of minimal complexity, rotation sequences, square billiard sequences ...
- Any sturmian sequence has an isolated letter
- it can be recoded to a shorter sequence by removing the letter following the isolated letter

## Symbolic dynamics : sturmian sequences

- One can build a completely symbolic model :
- coding by the two intervals = *sturmian sequences*
- sequences of minimal complexity, rotation sequences, square billiard sequences ...
- Any sturmian sequence has an isolated letter
- it can be recoded to a shorter sequence by removing the letter following the isolated letter

# Symbolic dynamics : sturmian sequences

- One can build a completely symbolic model :
- coding by the two intervals = *sturmian sequences*
- sequences of minimal complexity, rotation sequences, square billiard sequences ...
- Any sturmian sequence has an isolated letter
- it can be recoded to a shorter sequence by removing the letter following the isolated letter

# Symbolic dynamics : sturmian sequences

- One can build a completely symbolic model :
- coding by the two intervals = *sturmian sequences*
- sequences of minimal complexity, rotation sequences, square billiard sequences ...
- Any sturmian sequence has an isolated letter
- it can be recoded to a shorter sequence by removing the letter following the isolated letter

# Symbolic dynamics : sturmian sequences

- One can build a completely symbolic model :
- coding by the two intervals = *sturmian sequences*
- sequences of minimal complexity, rotation sequences, square billiard sequences ...
- Any sturmian sequence has an isolated letter
- it can be recoded to a shorter sequence by removing the letter following the isolated letter

# Symbolic dynamics : adic systems

- Denote $\sigma_0 : 0 \mapsto 0 \quad 1 \mapsto 10$ and $\sigma_1 : 0 \mapsto 01 \quad 1 \mapsto 1$
- Any sturmian sequence $u$ can be (up to the first letter) written $u = \sigma_i v$, with $i = 0$ or $i = 1$.
- We can recode $v$, and so on, and get a sequence $\sigma_0^{a_0} \sigma_1^{a_1} \sigma_0^{a_2} \ldots$
- A discrete version of the usual continued fraction
- Here, we only recover all known classical results

# Symbolic dynamics : adic systems

- Denote $\sigma_0 : 0 \mapsto 0 \quad 1 \mapsto 10$ and $\sigma_1 : 0 \mapsto 01 \quad 1 \mapsto 1$
- Any sturmian sequence $u$ can be (up to the first letter) written $u = \sigma_i v$, with $i = 0$ or $i = 1$.
- We can recode $v$, and so on, and get a sequence $\sigma_0^{a_0} \sigma_1^{a_1} \sigma_0^{a_2} \ldots$
- A discrete version of the usual continued fraction
- Here, we only recover all known classical results

# Symbolic dynamics : adic systems

- Denote $\sigma_0 : 0 \mapsto 0 \quad 1 \mapsto 10$ and $\sigma_1 : 0 \mapsto 01 \quad 1 \mapsto 1$
- Any sturmian sequence $u$ can be (up to the first letter) written $u = \sigma_i v$, with $i = 0$ or $i = 1$.
- We can recode $v$, and so on, and get a sequence $\sigma_0^{a_0} \sigma_1^{a_1} \sigma_0^{a_2} \ldots$
- A discrete version of the usual continued fraction
- Here, we only recover all known classical results

# Symbolic dynamics : adic systems

- Denote $\sigma_0 : 0 \mapsto 0 \quad 1 \mapsto 10$ and $\sigma_1 : 0 \mapsto 01 \quad 1 \mapsto 1$
- Any sturmian sequence $u$ can be (up to the first letter) written $u = \sigma_i v$, with $i = 0$ or $i = 1$.
- We can recode $v$, and so on, and get a sequence $\sigma_0^{a_0}\sigma_1^{a_1}\sigma_0^{a_2}\ldots$
- A discrete version of the usual continued fraction
- Here, we only recover all known classical results

# Symbolic dynamics : adic systems

- Denote $\sigma_0 : 0 \mapsto 0 \quad 1 \mapsto 10$ and $\sigma_1 : 0 \mapsto 01 \quad 1 \mapsto 1$
- Any sturmian sequence $u$ can be (up to the first letter) written $u = \sigma_i v$, with $i = 0$ or $i = 1$.
- We can recode $v$, and so on, and get a sequence $\sigma_0^{a_0} \sigma_1^{a_1} \sigma_0^{a_2} \ldots$
- A discrete version of the usual continued fraction
- Here, we only recover all known classical results

# Some images

Three views of the geodesic flow on the modular surface, by Edmund Harriss

- As a cutting and stacking construction
- As a geodesic flow on the hyperbolic plane
- As a flow of deformation of lattices

# Some images

Three views of the geodesic flow on the modular surface, by Edmund Harriss

- As a cutting and stacking construction
- As a geodesic flow on the hyperbolic plane
- As a flow of deformation of lattices

# Some images

Three views of the geodesic flow on the modular surface, by Edmund Harriss

- As a cutting and stacking construction
- As a geodesic flow on the hyperbolic plane
- As a flow of deformation of lattices

# From rotation to IET

- Instead of permuting 2 intervals (rotations), why not permute $3, 4, \dots, k, \dots$ intervals?
- We get interval exchanges on $k$ intervals, ($k$-IET) studied since at least the 60s, first in Russia
- as an interesting generalisation of rotations AND first return map of the flow of a closed 1-form (already known to Poincaré)
- This is linked to the conformal structures on Riemann surface
- And the theory of deformations of flat singular surfaces

# From rotation to IET

- Instead of permuting 2 intervals (rotations), why not permute $3, 4, \ldots, k, \ldots$ intervals?
- We get interval exchanges on $k$ intervals, ($k$-IET) studied since at least the 60s, first in Russia
- as an interesting generalisation of rotations AND first return map of the flow of a closed 1-form (already known to Poincaré)
- This is linked to the conformal structures on Riemann surface
- And the theory of deformations of flat singular surfaces

# From rotation to IET

- Instead of permuting 2 intervals (rotations), why not permute $3, 4, \ldots, k, \ldots$ intervals?
- We get interval exchanges on $k$ intervals, ($k$-IET) studied since at least the 60s, first in Russia
- as an interesting generalisation of rotations AND first return map of the flow of a closed 1-form (already known to Poincaré)
- This is linked to the conformal structures on Riemann surface
- And the theory of deformations of flat singular surfaces

# From rotation to IET

- Instead of permuting 2 intervals (rotations), why not permute $3, 4, \ldots, k, \ldots$ intervals?
- We get interval exchanges on $k$ intervals, ($k$-IET) studied since at least the 60s, first in Russia
- as an interesting generalisation of rotations AND first return map of the flow of a closed 1-form (already known to Poincaré)
- This is linked to the conformal structures on Riemann surface
- And the theory of deformations of flat singular surfaces

# From rotation to IET

- Instead of permuting 2 intervals (rotations), why not permute $3, 4, \ldots, k, \ldots$ intervals?
- We get interval exchanges on $k$ intervals, ($k$-IET) studied since at least the 60s, first in Russia
- as an interesting generalisation of rotations AND first return map of the flow of a closed 1-form (already known to Poincaré)
- This is linked to the conformal structures on Riemann surface
- And the theory of deformations of flat singular surfaces

# Rauzy-Veech induction

- A fundamental observation is that the first return map on a suitable intervals of a $k$-IET is again a $k$-IET.
- This gives a continued fraction, linear in dimension $k$ (or projective in dimension $k-1$)
- This is the Rauzy - Veech induction (with all its many variants).
- Here, we start from a family stable by induction
- to build a Gauss map and a new continued fraction.

# Rauzy-Veech induction

- A fundamental observation is that the first return map on a suitable intervals of a $k$-IET is again a $k$-IET.
- This gives a continued fraction, linear in dimension $k$ (or projective in dimension $k-1$)
- This is the Rauzy - Veech induction (with all its many variants).
- Here, we start from a family stable by induction
- to build a Gauss map and a new continued fraction.

# Rauzy-Veech induction

- A fundamental observation is that the first return map on a suitable intervals of a $k$-IET is again a $k$-IET.
- This gives a continued fraction, linear in dimension $k$ (or projective in dimension $k-1$)
- This is the Rauzy - Veech induction (with all its many variants).
- Here, we start from a family stable by induction
- to build a Gauss map and a new continued fraction.

# Rauzy-Veech induction

- A fundamental observation is that the first return map on a suitable intervals of a $k$-IET is again a $k$-IET.
- This gives a continued fraction, linear in dimension $k$ (or projective in dimension $k-1$)
- This is the Rauzy - Veech induction (with all its many variants).
- Here, we start from a family stable by induction
- to build a Gauss map and a new continued fraction.

# Rauzy-Veech induction

- A fundamental observation is that the first return map on a suitable intervals of a $k$-IET is again a $k$-IET.
- This gives a continued fraction, linear in dimension $k$ (or projective in dimension $k-1$)
- This is the Rauzy - Veech induction (with all its many variants).
- Here, we start from a family stable by induction
- to build a Gauss map and a new continued fraction.

# Natural extension and suspensions: flat surfaces and the Teichmüller geodesic flow

- We can suspend the IET to obtain Zippered Rectangles and flat surfaces
- In this way, we obtain any conformal structure on an orientable surface, and any closed 1-form.
- acting by the diagonal flow, we can zoom in the IET
- and recover an explicit model of the elusive Teichmüller geodesic flow.
- The periodic orbits are of course specially interesting (pseudo-Anosov maps)

# Natural extension and suspensions: flat surfaces and the Teichmüller geodesic flow

- We can suspend the IET to obtain Zippered Rectangles and flat surfaces
- In this way, we obtain any conformal structure on an orientable surface, and any closed 1-form.
- acting by the diagonal flow, we can zoom in the IET
- and recover an explicit model of the elusive Teichmüller geodesic flow.
- The periodic orbits are of course specially interesting (pseudo-Anosov maps)

# Natural extension and suspensions: flat surfaces and the Teichmüller geodesic flow

- We can suspend the IET to obtain Zippered Rectangles and flat surfaces
- In this way, we obtain any conformal structure on an orientable surface, and any closed 1-form.
- acting by the diagonal flow, we can zoom in the IET
- and recover an explicit model of the elusive Teichmüller geodesic flow.
- The periodic orbits are of course specially interesting (pseudo-Anosov maps)

# Natural extension and suspensions: flat surfaces and the Teichmüller geodesic flow

- We can suspend the IET to obtain Zippered Rectangles and flat surfaces
- In this way, we obtain any conformal structure on an orientable surface, and any closed 1-form.
- acting by the diagonal flow, we can zoom in the IET
- and recover an explicit model of the elusive Teichmüller geodesic flow.
- The periodic orbits are of course specially interesting (pseudo-Anosov maps)

# Natural extension and suspensions: flat surfaces and the Teichmüller geodesic flow

- We can suspend the IET to obtain Zippered Rectangles and flat surfaces
- In this way, we obtain any conformal structure on an orientable surface, and any closed 1-form.
- acting by the diagonal flow, we can zoom in the IET
- and recover an explicit model of the elusive Teichmüller geodesic flow.
- The periodic orbits are of course specially interesting (pseudo-Anosov maps)

# Brun continued fraction: the linear version

- Brun continued fraction in dimension 3 acts on triple of positive numbers $(x, y, z)$
- It substracts the second largest from the largest; if $x > y > z$, then
- $T(x, y, z) = (x - y, y, z)$
- and similar formula in the 5 other cases.
- one can define an associated projective map on the simplex $x + y + z = 1$ by
- $(x, y, z) \mapsto (\frac{x-y}{1-y}, \frac{y}{1-y}, \frac{z}{1-y})$
- or many other possible formulas

# Brun continued fraction: the linear version

- Brun continued fraction in dimension 3 acts on triple of positive numbers $(x, y, z)$
- It substracts the second largest from the largest; if $x > y > z$, then
- $T(x, y, z) = (x - y, y, z)$
- and similar formula in the 5 other cases.
- one can define an associated projective map on the simplex $x + y + z = 1$ by
- $(x, y, z) \mapsto (\frac{x-y}{1-y}, \frac{y}{1-y}, \frac{z}{1-y})$
- or many other possible formulas

# Brun continued fraction: the linear version

- Brun continued fraction in dimension 3 acts on triple of positive numbers $(x, y, z)$
- It substracts the second largest from the largest; if $x > y > z$, then
- $T(x, y, z) = (x - y, y, z)$
- and similar formula in the 5 other cases.
- one can define an associated projective map on the simplex $x + y + z = 1$ by
- $(x, y, z) \mapsto (\frac{x-y}{1-y}, \frac{y}{1-y}, \frac{z}{1-y})$
- or many other possible formulas

# Brun continued fraction: the linear version

- Brun continued fraction in dimension 3 acts on triple of positive numbers $(x, y, z)$
- It substracts the second largest from the largest; if $x > y > z$, then
- $T(x, y, z) = (x - y, y, z)$
- and similar formula in the 5 other cases.
- one can define an associated projective map on the simplex $x + y + z = 1$ by
- $(x, y, z) \mapsto (\frac{x-y}{1-y}, \frac{y}{1-y}, \frac{z}{1-y})$
- or many other possible formulas

# Brun continued fraction: the linear version

- Brun continued fraction in dimension 3 acts on triple of positive numbers $(x, y, z)$
- It substracts the second largest from the largest; if $x > y > z$, then
- $T(x, y, z) = (x - y, y, z)$
- and similar formula in the 5 other cases.
- one can define an associated projective map on the simplex $x + y + z = 1$ by
- $(x, y, z) \mapsto (\frac{x-y}{1-y}, \frac{y}{1-y}, \frac{z}{1-y})$
- or many other possible formulas

# Brun continued fraction: the linear version

- Brun continued fraction in dimension 3 acts on triple of positive numbers $(x, y, z)$
- It substracts the second largest from the largest; if $x > y > z$, then
- $T(x, y, z) = (x - y, y, z)$
- and similar formula in the 5 other cases.
- one can define an associated projective map on the simplex $x + y + z = 1$ by
- $(x, y, z) \mapsto (\frac{x-y}{1-y}, \frac{y}{1-y}, \frac{z}{1-y})$
- or many other possible formulas

# Brun continued fraction: the linear version

- Brun continued fraction in dimension 3 acts on triple of positive numbers $(x, y, z)$
- It substracts the second largest from the largest; if $x > y > z$, then
- $T(x, y, z) = (x - y, y, z)$
- and similar formula in the 5 other cases.
- one can define an associated projective map on the simplex $x + y + z = 1$ by
- $(x, y, z) \mapsto (\frac{x-y}{1-y}, \frac{y}{1-y}, \frac{z}{1-y})$
- or many other possible formulas

# Sequences of matrices (1)

- To any initial triple $P_0 = (x_0, y_0, z_0)$, one can associate a sequence of points $P_n = (x_n, y_n, z_n) = T^n(P_0)$
- and a sequence of matrices $M_1, \dots, M_n$ such that $P_n = M_n P_{n+1}$ (subject to a Markov condition)
- The $M_n$ are positive elementary matrices, hence the products $M_{j,n} = M_j M_{j+1} \dots M_{n-1}$ are growing for fixed $j$
- For almost any $P_0$, the sequence $M_n$ is primitive: for any $j$, $M_{j,n}$ is strictly positive for $n$ large enough.

# Sequences of matrices (1)

- To any initial triple $P_0 = (x_0, y_0, z_0)$, one can associate a sequence of points $P_n = (x_n, y_n, z_n) = T^n(P_0)$
- and a sequence of matrices $M_1, \ldots, M_n$ such that $P_n = M_n P_{n+1}$ (subject to a Markov condition)
- The $M_n$ are positive elementary matrices, hence the products $M_{j,n} = M_j M_{j+1} \ldots M_{n-1}$ are growing for fixed $j$
- For almost any $P_0$, the sequence $M_n$ is primitive: for any $j$, $M_{j,n}$ is strictly positive for $n$ large enough.

# Sequences of matrices (1)

- To any initial triple $P_0 = (x_0, y_0, z_0)$, one can associate a sequence of points $P_n = (x_n, y_n, z_n) = T^n(P_0)$
- and a sequence of matrices $M_1, \dots, M_n$ such that $P_n = M_n P_{n+1}$ (subject to a Markov condition)
- The $M_n$ are positive elementary matrices, hence the products $M_{j,n} = M_j M_{j+1} \dots M_{n-1}$ are growing for fixed $j$
- For almost any $P_0$, the sequence $M_n$ is primitive: for any $j$, $M_{j,n}$ is strictly positive for $n$ large enough.

# Sequences of matrices (1)

- To any initial triple $P_0 = (x_0, y_0, z_0)$, one can associate a sequence of points $P_n = (x_n, y_n, z_n) = T^n(P_0)$
- and a sequence of matrices $M_1, \ldots, M_n$ such that $P_n = M_n P_{n+1}$ (subject to a Markov condition)
- The $M_n$ are positive elementary matrices, hence the products $M_{j,n} = M_j M_{j+1} \ldots M_{n-1}$ are growing for fixed $j$
- For almost any $P_0$, the sequence $M_n$ is primitive: for any $j$, $M_{j,n}$ is strictly positive for $n$ large enough.

# Sequences of matrices (2)

- Hence, we have a generalised Perron-Frobenius theorem:
- for almost all $P_0$, the image by $M_{0,n}$ of the positive cone is a decreasing sequence of cones converging to a line
- This defines the generalised Perron eigenvector associated with the sequence $M_n$.
- A small miracle: under weak conditions, this eigenvector is totally irrational!

# Sequences of matrices (2)

- Hence, we have a generalised Perron-Frobenius theorem:
- for almost all $P_0$, the image by $M_{0,n}$ of the positive cone is a decreasing sequence of cones converging to a line
- This defines the generalised Perron eigenvector associated with the sequence $M_n$.
- A small miracle: under weak conditions, this eigenvector is totally irrational!

# Sequences of matrices (2)

- Hence, we have a generalised Perron-Frobenius theorem:
- for almost all $P_0$, the image by $M_{0,n}$ of the positive cone is a decreasing sequence of cones converging to a line
- This defines the generalised Perron eigenvector associated with the sequence $M_n$.
- A small miracle: under weak conditions, this eigenvector is totally irrational!

# Sequences of matrices (2)

- Hence, we have a generalised Perron-Frobenius theorem:
- for almost all $P_0$, the image by $M_{0,n}$ of the positive cone is a decreasing sequence of cones converging to a line
- This defines the generalised Perron eigenvector associated with the sequence $M_n$.
- A small miracle: under weak conditions, this eigenvector is totally irrational!

# A family of dynamical systems: the problem

- We would like to associate a family of dynamical systems which is stable by induction;
- The simplest idea would be to consider the rotations of the 2-torus
- But there is a problem: Any periodic trajectory for Brun's algorithm will define a self-induced translation on the 2-torus
- And by a simple construction, a Markov partition for a linear automorphism of the 3-torus
- But a theorem of Bowen says that any such partition has fractal boundary
- So, any such domain cannot be defined by a simple explicit formula.

# A family of dynamical systems: the problem

- We would like to associate a family of dynamical systems which is stable by induction;
- The simplest idea would be to consider the rotations of the 2-torus
- But there is a problem: Any periodic trajectory for Brun's algorithm will define a self-induced translation on the 2-torus
- And by a simple construction, a Markov partition for a linear automorphism of the 3-torus
- But a theorem of Bowen says that any such partition has fractal boundary
- So, any such domain cannot be defined by a simple explicit formula.

# A family of dynamical systems: the problem

- We would like to associate a family of dynamical systems which is stable by induction;
- The simplest idea would be to consider the rotations of the 2-torus
- But there is a problem: Any periodic trajectory for Brun's algorithm will define a self-induced translation on the 2-torus
- And by a simple construction, a Markov partition for a linear automorphism of the 3-torus
- But a theorem of Bowen says that any such partition has fractal boundary
- So, any such domain cannot be defined by a simple explicit formula.

# A family of dynamical systems: the problem

- We would like to associate a family of dynamical systems which is stable by induction;
- The simplest idea would be to consider the rotations of the 2-torus
- But there is a problem: Any periodic trajectory for Brun's algorithm will define a self-induced translation on the 2-torus
- And by a simple construction, a Markov partition for a linear automorphism of the 3-torus
- But a theorem of Bowen says that any such partition has fractal boundary
- So, any such domain cannot be defined by a simple explicit formula.

# A family of dynamical systems: the problem

- We would like to associate a family of dynamical systems which is stable by induction;
- The simplest idea would be to consider the rotations of the 2-torus
- But there is a problem: Any periodic trajectory for Brun's algorithm will define a self-induced translation on the 2-torus
- And by a simple construction, a Markov partition for a linear automorphism of the 3-torus
- But a theorem of Bowen says that any such partition has fractal boundary
- So, any such domain cannot be defined by a simple explicit formula.

# A family of dynamical systems: the problem

- We would like to associate a family of dynamical systems which is stable by induction;
- The simplest idea would be to consider the rotations of the 2-torus
- But there is a problem: Any periodic trajectory for Brun's algorithm will define a self-induced translation on the 2-torus
- And by a simple construction, a Markov partition for a linear automorphism of the 3-torus
- But a theorem of Bowen says that any such partition has fractal boundary
- So, any such domain cannot be defined by a simple explicit formula.

# A family of dynamical systems: a possible solution

- The solution is known for some particular matrices in the periodic case: use symbolic dynamics (Rauzy fractal).
- To any elementary matrice $M$, associate the unique substitution $\sigma_M$ which fixes the first letter
- To a sequence $M_n$ of matrices given by Brun's algorithm, one can associate a sequence of substitution $\sigma_n$
- By primitivity, this defines a unique generalised fixed point starting with a given letter $a$, and a unique minimal symbolic system.
- To the fixed point, one can associate a stepped line in $\mathbb{R}^3$
- Which admits the generalised eigenvector as asymptotic direction.

# A family of dynamical systems: a possible solution

- The solution is known for some particular matrices in the periodic case: use symbolic dynamics (Rauzy fractal).
- To any elementary matrice $M$, associate the unique substitution $\sigma_M$ which fixes the first letter
- To a sequence $M_n$ of matrices given by Brun's algorithm, one can associate a sequence of substitution $\sigma_n$
- By primitivity, this defines a unique generalised fixed point starting with a given letter $a$, and a unique minimal symbolic system.
- To the fixed point, one can associate a stepped line in $\mathbb{R}^3$
- Which admits the generalised eigenvector as asymptotic direction.

# A family of dynamical systems: a possible solution

- The solution is known for some particular matrices in the periodic case: use symbolic dynamics (Rauzy fractal).
- To any elementary matrice $M$, associate the unique substitution $\sigma_M$ which fixes the first letter
- To a sequence $M_n$ of matrices given by Brun's algorithm, one can associate a sequence of substitution $\sigma_n$
- By primitivity, this defines a unique generalised fixed point starting with a given letter $a$, and a unique minimal symbolic system.
- To the fixed point, one can associate a stepped line in $\mathbb{R}^3$
- Which admits the generalised eigenvector as asymptotic direction.

# A family of dynamical systems: a possible solution

- The solution is known for some particular matrices in the periodic case: use symbolic dynamics (Rauzy fractal).
- To any elementary matrice $M$, associate the unique substitution $\sigma_M$ which fixes the first letter
- To a sequence $M_n$ of matrices given by Brun's algorithm, one can associate a sequence of substitution $\sigma_n$
- By primitivity, this defines a unique generalised fixed point starting with a given letter $a$, and a unique minimal symbolic system.
- To the fixed point, one can associate a stepped line in $\mathbb{R}^3$
- Which admits the generalised eigenvector as asymptotic direction.

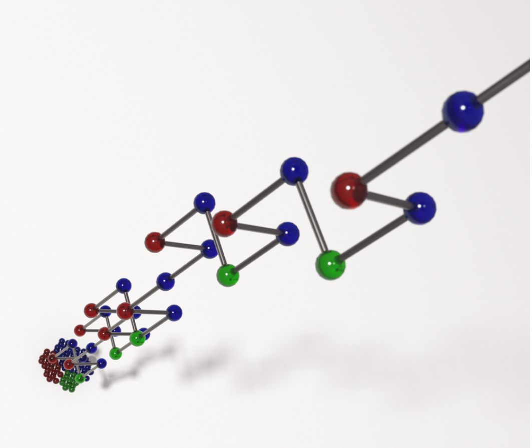
# A family of dynamical systems: a possible solution

- The solution is known for some particular matrices in the periodic case: use symbolic dynamics (Rauzy fractal).
- To any elementary matrice $M$, associate the unique substitution $\sigma_M$ which fixes the first letter
- To a sequence $M_n$ of matrices given by Brun's algorithm, one can associate a sequence of substitution $\sigma_n$
- By primitivity, this defines a unique generalised fixed point starting with a given letter $a$, and a unique minimal symbolic system.
- To the fixed point, one can associate a stepped line in $\mathbb{R}^3$
- Which admits the generalised eigenvector as asymptotic direction.

# A family of dynamical systems: a possible solution

- The solution is known for some particular matrices in the periodic case: use symbolic dynamics (Rauzy fractal).
- To any elementary matrice $M$, associate the unique substitution $\sigma_M$ which fixes the first letter
- To a sequence $M_n$ of matrices given by Brun's algorithm, one can associate a sequence of substitution $\sigma_n$
- By primitivity, this defines a unique generalised fixed point starting with a given letter $a$, and a unique minimal symbolic system.
- To the fixed point, one can associate a stepped line in $\mathbb{R}^3$
- Which admits the generalised eigenvector as asymptotic direction.

# A family of dynamical systems: a possible solution



# Sequences of matrices (3) : Pisot condition

- the sequence $M_n$ satisfies a much stronger condition : the generalised Pisot condition
- for such a sequence $M_n$, one can define the Lyapunov exponents, constant a.e. by ergodicity
- for Brun's algorithm in projective dimension 2 and 3, the second exponent is strictly negative a.e.
- This is the generalised Pisot condition
- It implies that the stepped line stays within bounded distance from the bounded line
- Hence its projection on a transversal plane along the asymptotic line is a bounded set with compact closure

# Sequences of matrices (3) : Pisot condition

- the sequence $M_n$ satisfies a much stronger condition : the generalised Pisot condition
- for such a sequence $M_n$, one can define the Lyapunov exponents, constant a.e. by ergodicity
- for Brun's algorithm in projective dimension 2 and 3, the second exponent is strictly negative a.e.
- This is the generalised Pisot condition
- It implies that the stepped line stays within bounded distance from the bounded line
- Hence its projection on a transversal plane along the asymptotic line is a bounded set with compact closure

# Sequences of matrices (3) : Pisot condition

- the sequence $M_n$ satisfies a much stronger condition : the generalised Pisot condition
- for such a sequence $M_n$, one can define the Lyapunov exponents, constant a.e. by ergodicity
- for Brun's algorithm in projective dimension 2 and 3, the second exponent is strictly negative a.e.
- This is the generalised Pisot condition
- It implies that the stepped line stays within bounded distance from the bounded line
- Hence its projection on a transversal plane along the asymptotic line is a bounded set with compact closure

# Sequences of matrices (3) : Pisot condition

- the sequence $M_n$ satisfies a much stronger condition : the generalised Pisot condition
- for such a sequence $M_n$, one can define the Lyapunov exponents, constant a.e. by ergodicity
- for Brun's algorithm in projective dimension 2 and 3, the second exponent is strictly negative a.e.
- This is the generalised Pisot condition
- It implies that the stepped line stays within bounded distance from the bounded line
- Hence its projection on a transversal plane along the asymptotic line is a bounded set with compact closure

# Sequences of matrices (3) : Pisot condition

- the sequence $M_n$ satisfies a much stronger condition : the generalised Pisot condition
- for such a sequence $M_n$, one can define the Lyapunov exponents, constant a.e. by ergodicity
- for Brun's algorithm in projective dimension 2 and 3, the second exponent is strictly negative a.e.
- This is the generalised Pisot condition
- It implies that the stepped line stays within bounded distance from the bounded line
- Hence its projection on a transversal plane along the asymptotic line is a bounded set with compact closure

# Sequences of matrices (3) : Pisot condition

- the sequence $M_n$ satisfies a much stronger condition : the generalised Pisot condition
- for such a sequence $M_n$, one can define the Lyapunov exponents, constant a.e. by ergodicity
- for Brun's algorithm in projective dimension 2 and 3, the second exponent is strictly negative a.e.
- This is the generalised Pisot condition
- It implies that the stepped line stays within bounded distance from the bounded line
- Hence its projection on a transversal plane along the asymptotic line is a bounded set with compact closure

# Sequences of matrices (4) : irrationality

- Since the generalised eigenvector is totally irrational
- the projection $\pi$ of $\mathbb{Z}^3$ along this vector on any transverse plane is dense

# Sequences of matrices (4) : irrationality

- Since the generalised eigenvector is totally irrational
- the projection $\pi$ of $\mathbb{Z}^3$ along this vector on any transverse plane is dense

## $S$-adic sequences: covering property

- Let $\Delta$ the diagonal subgroup of $\mathbb{Z}^3$
- Let $L$ be the stepped line associated to a fixed point
- The stepped lines $L+\delta$, for $\delta \in \Delta$, form a partition of $\mathbb{Z}^3$
- Hence the closure of the projection $\mathcal{R} = \overline{\pi(L)}$ has a nonempty interior,
- and the $\mathcal{R}+\delta$ cover the plane with a finite degree

# $S$-adic sequences: covering property

- Let $\Delta$ the diagonal subgroup of $\mathbb{Z}^3$
- Let $L$ be the stepped line associated to a fixed point
- The stepped lines $L + \delta$, for $\delta \in \Delta$, form a partition of $\mathbb{Z}^3$
- Hence the closure of the projection $\mathcal{R} = \overline{\pi(L)}$ has a nonempty interior,
- and the $\mathcal{R} + \delta$ cover the plane with a finite degree

## $S$-adic sequences: covering property

- Let $\Delta$ the diagonal subgroup of $\mathbb{Z}^3$
- Let $L$ be the stepped line associated to a fixed point
- The stepped lines $L + \delta$, for $\delta \in \Delta$, form a partition of $\mathbb{Z}^3$
- Hence the closure of the projection $\mathcal{R} = \overline{\pi(L)}$ has a nonempty interior,
- and the $\mathcal{R} + \delta$ cover the plane with a finite degree

# $S$-adic sequences: covering property

- Let $\Delta$ the diagonal subgroup of $\mathbb{Z}^3$
- Let $L$ be the stepped line associated to a fixed point
- The stepped lines $L + \delta$, for $\delta \in \Delta$, form a partition of $\mathbb{Z}^3$
- Hence the closure of the projection $\mathcal{R} = \overline{\pi(L)}$ has a nonempty interior,
- and the $\mathcal{R} + \delta$ cover the plane with a finite degree

# $S$-adic sequences: covering property

- Let $\Delta$ the diagonal subgroup of $\mathbb{Z}^3$
- Let $L$ be the stepped line associated to a fixed point
- The stepped lines $L + \delta$, for $\delta \in \Delta$, form a partition of $\mathbb{Z}^3$
- Hence the closure of the projection $\mathcal{R} = \overline{\pi(L)}$ has a nonempty interior,
- and the $\mathcal{R} + \delta$ cover the plane with a finite degree

# Rotations (1)

- Let $H$ be the diagonal plane $x + y + z = 0$
- $H/\Delta$ is a torus
- All the basis vectors project to the same element $v$ of $H/Delta$
- Hence the projection of the stepped line is the orbit of the rotation by $v$ on this torus.

# Rotations (1)

- Let $H$ be the diagonal plane $x + y + z = 0$
- $H/\Delta$ is a torus
- All the basis vectors project to the same element $v$ of $H/Delta$
- Hence the projection of the stepped line is the orbit of the rotation by $v$ on this torus.

# Rotations (1)

- Let $H$ be the diagonal plane $x + y + z = 0$
- $H/\Delta$ is a torus
- All the basis vectors project to the same element $v$ of $H/Delta$
- Hence the projection of the stepped line is the orbit of the rotation by $v$ on this torus.

# Rotations (1)

- Let $H$ be the diagonal plane $x + y + z = 0$
- $H/\Delta$ is a torus
- All the basis vectors project to the same element $v$ of $H/Delta$
- Hence the projection of the stepped line is the orbit of the rotation by $v$ on this torus.

# Rotations (2)

- In the case of Brun algorithm,
- One can prove that $\mathcal{R}$ is a fundamental domain for the action of $\Delta$ on $H$
- Hence we can identify $\mathcal{R}$ to the torus $H/Delta$
- and the symbolic system defined by the adic sequence is isomorphic to a torus rotation, with discrete spectrum.

# Rotations (2)

- In the case of Brun algorithm,
- One can prove that $\mathcal{R}$ is a fundamental domain for the action of $\Delta$ on $H$
- Hence we can identify $\mathcal{R}$ to the torus $H/Delta$
- and the symbolic system defined by the adic sequence is isomorphic to a torus rotation, with discrete spectrum.

# Rotations (2)

- In the case of Brun algorithm,
- One can prove that $\mathcal{R}$ is a fundamental domain for the action of $\Delta$ on $H$
- Hence we can identify $\mathcal{R}$ to the torus $H/Delta$
- and the symbolic system defined by the adic sequence is isomorphic to a torus rotation, with discrete spectrum.

# Rotations (2)

- In the case of Brun algorithm,
- One can prove that $\mathcal{R}$ is a fundamental domain for the action of $\Delta$ on $H$
- Hence we can identify $\mathcal{R}$ to the torus $H/Delta$
- and the symbolic system defined by the adic sequence is isomorphic to a torus rotation, with discrete spectrum.

# Induction of rotation

- This rotation is given with a specific coding partition in 3 components,
- whose measure is the vector $(x, y, z)$
- Induction on a subset gives the next rotation
- We can build a natural extension of the system
- by considering Rauzy boxes
- We show some pictures

# Induction of rotation

- This rotation is given with a specific coding partition in 3 components,
- whose measure is the vector $(x, y, z)$
- Induction on a subset gives the next rotation
- We can build a natural extension of the system
- by considering Rauzy boxes
- We show some pictures

# Induction of rotation

- This rotation is given with a specific coding partition in 3 components,
- whose measure is the vector $(x, y, z)$
- Induction on a subset gives the next rotation
- We can build a natural extension of the system
- by considering Rauzy boxes
- We show some pictures

# Induction of rotation

- This rotation is given with a specific coding partition in 3 components,
- whose measure is the vector $(x, y, z)$
- Induction on a subset gives the next rotation
- We can build a natural extension of the system
- by considering Rauzy boxes
- We show some pictures

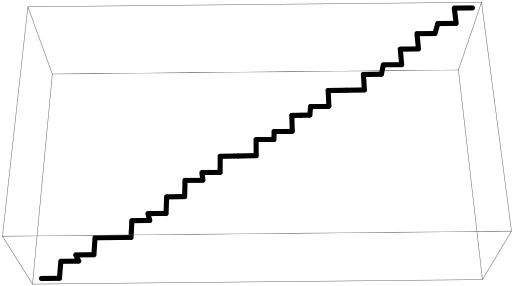
# Induction of rotation

- This rotation is given with a specific coding partition in 3 components,
- whose measure is the vector $(x, y, z)$
- Induction on a subset gives the next rotation
- We can build a natural extension of the system
- by considering Rauzy boxes
- We show some pictures

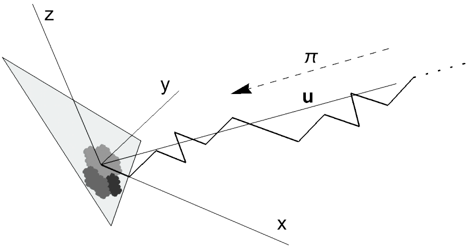
# Induction of rotation

- This rotation is given with a specific coding partition in 3 components,
- whose measure is the vector $(x, y, z)$
- Induction on a subset gives the next rotation
- We can build a natural extension of the system
- by considering Rauzy boxes
- We show some pictures

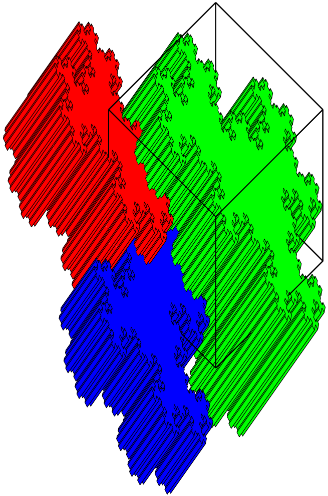
# Some images



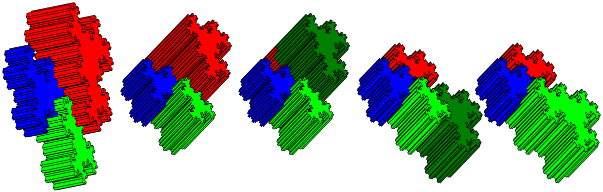
# Some images



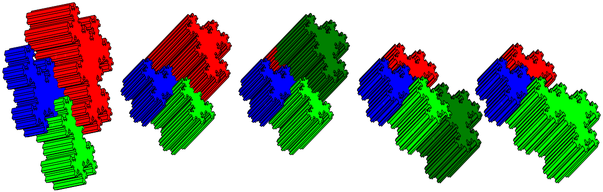
# Some images



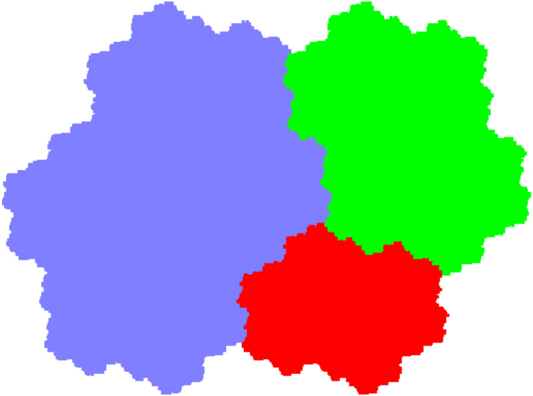
# Some images



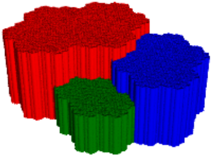
# Some images



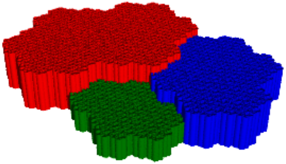
# Some images



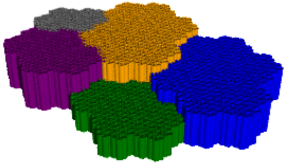
# Some images



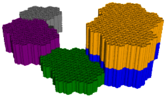
# Some images



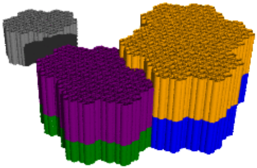
# Some images



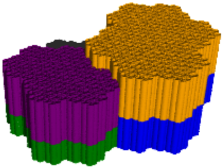
# Some images



# Some images



# Some images



# Some images