Geometric tensors via spectral functionals

Ludwik Dąbrowski
SISSA, Trieste (I)

Given a Laplace operator we use the noncommutative residue to define certain functionals of vector fields which yield metric and Einstein tensors. Alternatively, given a Dirac operator we define dual metric and Einstein functionals of differential forms, and also Ricci and torsion functionals. We generalise these concepts in non-commutative geometry and show e.g. that for the conformally rescaled noncommutative 2-torus the Einstein and the torsion functionals vanish. Also the Hodge-de Rham, Einstein-Yang-Mills and quantum SU(2) group spectral triples are torsion free, while the quantum 2-sheeted space has torsion.
[Adv.Math. 427, 1091286, 2023; Commun.Math.Phys. 130, 2024 and DOI 10.4171/JNCG/573 (2024) with A. Sitarz and P. Zalecki].

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Spectral Geometry:

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An eminent spectral scheme that generates geometric objects on Riemannian manifolds (volume, scalar curvature . . .) is $t \searrow 0$ asymptotic expansion of the trace of heat kernel

$$\mathrm{Tr}\, e^{-t\Delta} \approx \sum_{\ell=0}^{\infty} t^{\frac{\ell-n}{2}} a_\ell .$$

Here the scalar laplacian Δ for metric $g = \{g_{jk}\}$ reads

$$\Delta = -\frac{1}{\sqrt{\det(g)}} \partial_j \big(\sqrt{\det(g)} g^{jk} \partial_k \big). \quad (1)$$

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The coefficients a_ℓ can be transmuted into some values or residues of the zeta function of Δ, and in turn expressed using the noncommutative (Wodzicki) residue $\mathcal{W}$

$$\mathcal{W}(P) := \frac{1}{vol(S^{n-1})} \int_M \left(\int_{|\xi|=1} tr\, \sigma_{-n}(P)(x,\xi)\ \mathcal{V}_\xi \right) d^n x. \quad (2)$$

♠

→ residues

Geometry from residues:

Then, for $P = \Delta$

$$\mathcal{W}(\Delta^{-m}) = vol(M),$$

and in the *localized* form (as a functional of $f \in C^\infty(M)$)

$$\mathcal{V}(f) := \mathcal{W}(f\Delta^{-m}) = \int_M f \; vol_g.$$

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A. Connes divulged in 90s a startling result, confirmed by Kastler and by Kalau-Walze:

$$\mathcal{W}(\Delta^{-m+1}) = \frac{n-2}{12} \int_M R \; vol_g , \tag{3}$$

which is $\propto$ the Einstein-Hilbert action functional (of g) for the Riemannian general relativity (in vacuum).
Here R is the scalar curvature

$$R = R(g) = g^{jk} R_{jk} = g^{jk} R_{\ell j \ell k} .$$

A localised form of (3) is the *scalar curvature* functional on $C^\infty(M)$

$$\mathcal{R}(f) := \mathcal{W}(f\Delta^{-m+1}) = \frac{n-2}{12} \int_M f R \, vol_g . \tag{4}$$

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↬ This is related to the asymptotic growth of eigenvalues of Δ; clear e.g. from the Connes "trace thm." that $\mathcal{W} = \mathrm{Tr}^+$. ↫

↬ We have uncovered few new spectral 'localised' functionals, by placing some differential operators in place of f.
Let's start e.g. with a pair of vector fields V and W on M, viewed as derivations of $C^\infty(M)$:

New functionals

Def/Thm: Metric functional

The functional

$$\mathrm{g}^{\Delta}(\mathrm{V},\mathrm{W}) := \mathcal{W}\big(\mathrm{V}\mathrm{W}\Delta^{-\mathrm{m}-1}\big)$$

is a bilinear, symmetric map, whose density is proportional to the metric g evaluated on V, W

$$\mathrm{g}^{\Delta}(V,W) = -\frac{1}{n}\int_M g(V,W)\, vol_g.$$

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Def/Thm: Einstein functional

The functional

$$\mathrm{G}^{\Delta}(V,W) := \mathcal{W}\big(VW\Delta^{-m}\big)\,, \tag{5}$$

is a bilinear, symmetric map, whose density is proportional to the Einstein tensor $G := Ric - \frac{1}{2}R\, g$ *evaluated on V, W*

$$\mathrm{G}^{\Delta}(V,W) = \frac{1}{6}\int_M G(V,W)\, vol_g.$$

→ pf

"Proof"

Algebra of symbols of pseudodifferential operators:

$$\sigma(PQ)(x,\xi) = \sum_\beta \frac{(-i)^{|\beta|}}{|\beta|!} \frac{\partial}{\partial \xi^\beta}\sigma(P)(x,\xi)\,\frac{\partial}{\partial x^\beta}\sigma(Q)(x,\xi). \qquad (6)$$

"Proof"

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Taylor expansion in <u>normal</u> coordinates x:

metric

$$g_{ab} = \delta_{ab} - \frac{1}{3} R_{acbd} x^c x^d + o(|x|^2), \quad (7)$$

volume element

$$\sqrt{\det(g)} = 1 - \frac{1}{6} R_{ab} x^a x^b + o(|x|^2), \quad (8)$$

and Levi-Civita symbol

$$\Gamma^a_{bc}(x) = -\frac{1}{3}(R_{abcd} + R_{acbd}) x^d + o(|x|^2). \quad (9)$$

where R_{acbd} and R_{ab} are the values at $x = 0$.

"Proof" 2

Consequently, $\sigma(\Delta) = \mathfrak{a}_2 + \mathfrak{a}_1$, where

$$\begin{aligned} \mathfrak{a}_2 =& \big(\delta_{ab} + \frac{1}{3} R_{acbd} x^c x^d\big) \xi_a \xi_b + o(|x|^2), \\ \mathfrak{a}_1 =& \frac{2i}{3} R_{ab} x^a \xi_b + o(|x|^2). \end{aligned} \tag{10}$$

Next we compute the first three leading symbols of Δ^{-1}, and then of Δ^{-k}, $k > 0$, up to order resp. $o(|x|^2), o(|x|), o(1)$:

$$\sigma(\Delta^{-k}) = \mathfrak{c}_{2k} + \mathfrak{c}_{2k+1} + \mathfrak{c}_{2k+2} + \dots,$$

$$\begin{aligned} \mathfrak{c}_{2k} &= ||\xi||^{-2k-2} \left(\delta_{ab} - \frac{k}{3} R_{acbd} x^c x^d \right) \xi_a \xi_b + o(|x|^2), \\ \mathfrak{c}_{2k+1} &= \frac{-2ki}{3||\xi||^{2k+2}} R_{ab} x^b \xi_a + o(|x|), \\ \mathfrak{c}_{2k+2} &= \frac{k(k+1)}{3||\xi||^{2k+4}} R_{ab} \xi_a \xi_b + o(1). \end{aligned} \tag{11}$$

Now the composition with $\sigma(VW)$ shows the statements. □

Laplace-type, Spin Laplacian, squared Dirac

More generally, we've treated Laplace-type operators

$$\Delta_{T,E} = -g^{ab}(\nabla_a \nabla_b - \Gamma^c_{ab}\nabla_c) + E$$

on a vector bundle Ξ with connection ∇ and $E \in End\ \Xi$.

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A particular interesting case is a $spin_c$ manifold M with Ξ a spinor bundle Σ of rank 2^m and the spin Laplacian

$$\Delta^{(s)} := \nabla^{(s)*}\nabla^{(s)} = -\nabla^{(s)}_{e_i}\nabla^{(s)}_{e_i} + \nabla^{(s)}_{\nabla_{e_i}e_i}, \tag{12}$$

where $\nabla^{(s)}$ is the spin connection and e_j is ON frame:

Proposition

$$\begin{aligned} \mathrm{g}^{\Delta^{(s)}}(V,W) &:= \mathcal{W}\big(\nabla^{(s)}_V \nabla^{(s)}_W (\Delta^{(s)})^{-m-1}\big) = 2^m \mathrm{g}^{\Delta}(V,W), \\ \mathrm{G}^{\Delta^{(s)}}(V,W) &:= \mathcal{W}\big(\nabla^{(s)}_V \nabla^{(s)}_W (\Delta^{(s)})^{-m}\big) = 2^m \mathrm{G}^{\Delta}(V,W) + 0. \end{aligned} \tag{13}$$

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or squared Dirac (coupled do $U(1)$-gauge 1-form A):

Proposition

$$\mathrm{g}^{D_A^2}(V,W) := \mathcal{W}(\nabla^{(s)}_V \nabla^{(s)}_W |D_A|^{-n-2}) = 2^m \mathrm{g}^{\Delta}(V,W),$$

$$\mathrm{G}^{D_A^2}(V,W) := \mathcal{W}(\nabla^{(s)}_V \nabla^{(s)}_W |D_A|^{-n})$$

$$= 2^m \Big(\mathrm{G}^{\Delta}(V,W) + 2^{-3}\int_M R\, g(V,W) vol_g\Big).$$

Go quantum (= noncommutative)

Noncommutative tori are prominent examples of quantum spaces. Their smooth algebra $A = C^\infty(\mathbb{T}^n_\theta)$, generated by n unitaries U_j,

$$U_j U_k = \delta_{jk} e^{i\theta} U_k U_j,$$

has a faithful state τ invariant under derivations δ_j, $\delta_j U_k = \delta_{jk} U_k$, which are interpreted as noncommutative vector fields.

One regards $\Delta = \sum_j \delta_j^2$ on $H = L^2(\mathbb{T}^2_\theta, \tau)$ as 'flat' Laplace operator, $D = \sum_j \gamma^j \delta_j$ on $H = L^2(\mathbb{T}^2_\theta, \tau) \otimes \mathbb{C}^{2^m}$ as 'flat' Dirac operator and the A-bimodule $\Omega_D(A)$ generated by $[D, A]$, as 1-forms. ♠
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They generalise to the (non-flat) conformally rescaled geometry:

For simplicity consider the *strictly irrational* $\mathbb{T}^n_\theta$ (i.e., $\mathcal{Z}(A) = \mathbb{C}$) with τ extended to $\hat{A} := A \otimes A^o$ as $\tau(a \otimes b^o) = \tau(a)\tau(b^o)$, where A^o is a copy of A in the commutant A' of A in $B(H)$. Such τ is still invariant under the extended derivations. We use it to define the tracial state $\mathcal{W}$ on $\hat{A}$-valued symbols $\sigma(\xi)$ (where $\delta_a \mapsto \xi_a$ much the same as for M).

Rescaled NC 2-torus: vector fields

Given $0 < h \in C^\infty(\mathbb{T}^2_\theta)$, by a conformally rescaled Δ on $\mathbb{T}^2_\theta$ we mean the selfadjoint operator on $H = L^2(\mathbb{T}^2_\theta, \tau)$:

♠
$$\Delta_h = h^{-1} \Delta\, h^{-1}.$$

Accordingly, as vector fields we take

$$V_h = \sum_{a=1,2} V^a h \delta_a h^{-1}, \quad V^a \in \mathbb{C}.$$

Proposition

$$\mathrm{g}^{\Delta_h}(V_h, W_h) = \mathcal{W}\big(V_h W_h \Delta_h^{-2}\big) = \pi\tau(h^4) V^a W^a,$$

whereas

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We have also computed $\mathbb{T}^4_\theta$.
Can do also θ-deformed spaces, or NC spaces with derivations.

Alternatively ...

Spectral functionals on 1-forms

Now use D on spinors in a two-fold way to get (in terms of $\mathcal{W}$) certain "dual functionals" which are bilinear on 1-forms (co-vectors) and yield contravariant tensors (with "raised indices").

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For that need to represent 1-forms v as differential operators.
On a spin$_c$c manifold M use the Clifford representation of v as 0-order differential operators $\hat{\nu} \in \mathrm{End}(\Sigma)$.
As known they form a $C^\infty(M)$-bimodule $\Omega^1_D \simeq \Omega^1(M)$ generated by commutators of D with functions.
Thus the spinorial Dirac operator is self-sufficient for our purposes (and NCG-ready when assembled to a spectral triple of A. Connes), so comes now in its grandeur

→ fnls

Metric and Einstein functionals on 1-forms

Thm

The spectral functionals of one-forms on M

$$\begin{aligned} \mathrm{g}_D(v,w) &:= \mathcal{W}\big(\hat{v}\hat{w}D^{-n}\big), \\ \mathrm{G}_D(v,w) &:= \mathcal{W}\big(\hat{v}(D\hat{w}+\hat{w}D)D^{-n+1}\big) \\ &= \mathcal{W}\big((D\hat{v}+\hat{v}D)\hat{w}D^{-n+1}\big), \end{aligned} \tag{14}$$

read

$$\begin{aligned} \mathrm{g}_D(v,w) &= 2^m \int_M g(v,w)\ vol_g, \\ \mathrm{G}_D(v,w) &= \frac{2^m}{6} \int_M G(v,w)\ vol_g, \end{aligned} \tag{15}$$

where $G = Ric - \frac{1}{2}Rg$ is the contravariant Einstein tensor.

They perfectly (dually) match g^{Δ} and G^{Δ} up to 2^m.

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where $G = Ric - \frac{1}{2}Rg$ is the contravariant Einstein tensor.

They perfectly (dually) match g^Δ and G^Δ up to 2^m.

Actually,

$$\mathrm{Ric}_D(v,w) := \mathcal{W}\big(\hat{v}(D\hat{w}+\tfrac{n-4}{n-2}\hat{w}D)D^{-n+1}\big) = \tfrac{2^m}{6}\int_M Ric(v,w)\ vol_g.$$

→ torus

Rescaled noncommutative 2-torus: 1-forms

The above functionals extend to NC spaces:
As the conformal rescaling of D on $\mathbb{T}^n_\theta$ we take on H

$$D_k = kDk,$$

following Connes-Moscovici, however with $0 < k \in A^o \subset A'$,
which assures that (A, D_k, H) is a spectral triple and $\exists\ \Omega^1_{D_k}(A)$. ♠
In effect, $\Omega^1_{D_k}(A)$ is freely generated by $k^2\gamma^j$.

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For $n = 2$, $\gamma^j = \sigma^j$, and for $\mathbb{T}^2_\theta$ we have

Proposition

For $v = k^2 v^j \sigma^j$ and $w = k^2 w^j \sigma^j$, $v^j, w^j \in A$,

$$\mathrm{g}_{D_k}(v, w) = \tau(v^j w^j),$$

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Spectral Torsion

In principle *connections* not needed for abstract Δ or D.
Thanks to our g_D we can now 'control' the *metricity* condition.
Instead what about the *zero-torsion* condition ?
Not clear if any (enigmatic & complicated) minimization procedure could be employed for that.
But the contribution of torsion can contaminate our g & G (!).

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But the contribution of torsion can contaminate our g & G (!).
Fortunately, for a n-summable regular $(\mathcal{A}, D, \mathcal{H})$, using $\mathcal{W}$ coming from the ΨDO calculus and tracial state by Connes-Moscovici'95, we found:

Def/Thm: Torsion functional

Torsion functional *is a trilinear functional of* $u, v, w \in \Omega^1_D(\mathcal{A})$,

$$\mathcal{T}_D(u, v, w) := \mathcal{W}(uvwD|D|^{-n}).$$

We say that D is torsion-free if $\mathcal{T}_D \equiv 0$. For the Dirac operator D_T with torsion T on a closed spin manifold of dimension n

$$\mathcal{T}_{D_T}(u, v, w) = -2^{[\frac{n}{2}]} i \int_M u_a v_b w_c T_{abc} vol_g. \tag{16}$$

Examples

$\boxed{\mathcal{T} = 0}$ for:

- Hodge-de Rham: $\big(C^\infty(M), L^2(\Omega^\bullet_M), d + d^*\big)$.
- Einstein-Yang-Mills: $\big(C^\infty(M) \otimes M_N(\mathbb{C}), L^2(\Sigma) \otimes M_N(\mathbb{C})\big), \widetilde{D})$, where $\widetilde{D} = D \otimes \mathrm{id}_N + A + JAJ^{-1}$ with $A = A^* \in \Omega^1_{\widetilde{D}}$ and $J = C \otimes *$, with C being the charge conjugation on spinors in Σ.
- conformally rescaled noncommutative tori.
- quantum $SU(2)$: $\big(\mathcal{A}(SU_q(2)), \mathcal{H}, D\big)$, where $\mathcal{H}$ and D are isomorphic to the classical case $q = 1$.

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$\boxed{\mathcal{T}\neq 0}$ for:

- almost commutative $M\times\mathbb{Z}_2$: $(C^\infty(M)\otimes\mathbb{C}^2, L^2(\Sigma)\otimes\mathbb{C}^2, \mathcal{D})$,

where $\mathcal{D} = \begin{pmatrix} D & \chi\phi \\ \chi\phi^* & D\end{pmatrix}$, with D on Σ graded by χ, and $\phi\in\mathbb{C}$.

Now, $\Omega^1_{\mathcal{D}} \ni \omega = \begin{pmatrix} w^+ & \phi\chi f^+ \\ \phi^*\chi f^- & w^-\end{pmatrix}$ for $w^\pm \in \Omega^1(M)$, $f^\pm \in C^\infty(M)$.

Then,
$$\mathcal{W}\big(\omega_1^o\omega_2^o\omega_3^o\mathcal{D}\mathcal{D}^{-2m}\big) = \mathcal{W}\big(|\phi|^4(f_1^+f_2^-f_3^+ + f_1^-f_2^+f_3^-)D^{-2m}\big)$$
$$= |\phi|^4\int_M (f_1^+f_2^-f_3^+ + f_1^-f_2^+f_3^-)vol_g.$$

Spectral vs. Algebraic Torsion

[L.D., Y. Liu, S. Mukhopadhyay] in preparation

The torsionful case $M \times \mathbb{Z}/2\mathbb{Z}$ requires some subtle adjustments, but can work out the inner spectral triple $\left(\mathbb{C}^2, \mathbb{C}^2, \begin{bmatrix} 0 & \phi \\ \phi^* & 0 \end{bmatrix}\right)$.

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Here $e = (1, 0) \in \mathbb{C}^2$ is represented on $\mathbb{C}^2$ as $\operatorname{diag}(1, 0)$,

$de := [D, e] = \begin{bmatrix} 0 & -\phi \\ \phi^* & 0 \end{bmatrix} \in \Omega^1 = \left\{ \begin{bmatrix} 0 & h^+\phi \\ h^-\phi^* & 0 \end{bmatrix} \middle| \, h^\pm \in \mathbb{C} \right\}$,

$(de)^2 = -\begin{bmatrix} \phi\phi^* & 0 \\ 0 & \phi^*\phi \end{bmatrix} \in \Omega^2$.

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$(de)^2 = -\begin{bmatrix} \phi\phi^* & 0 \\ 0 & \phi^*\phi \end{bmatrix} \in \Omega^2$. An arbitrary (left) connection reads

$$\nabla : \Omega^1 \to \Omega^1 \otimes_{\mathcal{A}} \Omega^1, \quad de \mapsto \begin{bmatrix} c^+ & 0 \\ 0 & c^- \end{bmatrix} de \otimes de, \quad c_\pm \in \mathbb{C}$$

and its torsion

$$T^\nabla := m \circ \nabla - d : \Omega^1 \to \Omega^2, \quad de \mapsto -\begin{bmatrix} c^+\phi\phi^* & 0 \\ 0 & c^-\phi^*\phi \end{bmatrix}.$$

Spectral vs. Algebraic Torsion

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The torsionful case $M \times \mathbb{Z}/2\mathbb{Z}$ requires some subtle adjustments, but <u>can</u> work out the inner spectral triple $\left(\mathbb{C}^2, \mathbb{C}^2, \begin{bmatrix} 0 & \phi \\ \phi^* & 0 \end{bmatrix}\right)$.

Here $e = (1,0) \in \mathbb{C}^2$ is represented on $\mathbb{C}^2$ as $\mathrm{diag}(1,0)$,
$de := [D,e] = \begin{bmatrix} 0 & -\phi \\ \phi^* & 0 \end{bmatrix} \in \Omega^1 = \left\{ \begin{bmatrix} 0 & h^+\phi \\ h^-\phi^* & 0 \end{bmatrix} \,\middle|\, h^\pm \in \mathbb{C} \right\}$,
$(de)^2 = -\begin{bmatrix} \phi\phi^* & 0 \\ 0 & \phi^*\phi \end{bmatrix} \in \Omega^2$. An arbitrary (left) connection reads

$$\nabla : \Omega^1 \to \Omega^1 \otimes_{\mathcal{A}} \Omega^1, \quad de \mapsto \begin{bmatrix} c^+ & 0 \\ 0 & c^- \end{bmatrix} de \otimes de, \quad c_\pm \in \mathbb{C}$$

and its torsion

$$T^\nabla := m \circ \nabla - d : \Omega^1 \to \Omega^2, \quad de \mapsto -\begin{bmatrix} c^+\phi\phi^* & 0 \\ 0 & c^-\phi^*\phi \end{bmatrix}.$$

Then, for $u, v, w \in \Omega^1$:
$\mathcal{T}^\nabla(u,v,w) := \mathrm{Tr}\left(uv\,T^\nabla(w)\right)$

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Then, for $u, v, w \in \Omega^1$:

$$\mathcal{T}^\nabla(u, v, w) := \operatorname{Tr}\left(uvT^\nabla(w)\right) = \operatorname{Tr}(uvwD) \qquad \text{for } c^\pm = \pm 1$$
$$= \mathcal{T}_D(u, v, w) \qquad \text{as } \lim_{n\to 0}\mathcal{W}(xD^{-n}) = \operatorname{Tr} x.$$

Outlook

• The spectral formulation of geometric objects g, G, Ric & T should be beneficial for global study on the analytic/operator level of manifolds as well as generalized geometries, like NCG.

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Conjecture: For a 2-dimensional regular spectral triple $\mathrm{G}_D = 0$.

- relation of $\mathcal{T}_D$ to other settings (algebraic, differential) for T and quantum analogues of Levi-Civita connection in the literature
- relation to W. Ugalde differential forms & conformal gometry

THANK YOU !

Let E be a finite dimensional complex vector bundle over a closed compact manifold M of dimension n. Recall that the non-commutative residue of a pseudo-differential operator $P \in \Psi\mathrm{DO}(E)$ can be defined by

$$\mathcal{W}(P) := (2\pi)^{-n} \int_{S^*M} tr\big(\sigma^P_{-n}(x,\xi)\big)\, dxd\xi,$$

where $S^*M \subset T^*M$ denotes the co-sphere bundle on M and σ^P_{-n} is the component of order $-n$ of the complete symbol $\sigma^P := \sum_i \sigma^P_i$ of P, cf [W]. In his thesis, Wodzicki has shown that the linear functional $res\colon \Psi\mathrm{DO}(E,F) \to \mathbb{C}$ is in fact the unique trace (up to multiplication by constants) on the algebra of pseudo-differential operators $\Psi\mathrm{DO}(E)$.

Now let $P \in \Psi DO(E)$ be elliptic with $\operatorname{ord} P = d > 0$.
It is well-known (cf. [Gi]) that its zeta function $\zeta(P,s)$ is holomorphic on the half-plane $\operatorname{Re} s > n/d$ with meromorphic continuation to $\mathbb{C}$ with simple poles at $\{ \frac{(n-k)}{d} \, | k \in \mathbb{N} \setminus \{n\} \}$.
For $n-k>0$ with $k \in \mathbb{N}$ one has [W]:

$$\mathcal{W}(P^{-(\frac{n-k}{d})}) = d \cdot Res_{s=\frac{n-k}{d}} \zeta(P,s),$$

and using the Mellin transform

$$\int_0^\infty t^{s-1} e^{-\lambda t} dt = \lambda^{-s} \int_0^\infty (\lambda t)^{s-1} e^{-\lambda t} d(\lambda t) = \lambda^{-s} \Gamma(s),$$

also [Gi]:

$$Res_{s=\frac{n-k}{d}} \zeta(P,s) = a_k(P) \cdot \Gamma(\tfrac{n-k}{d})^{-1}.$$

Here Γ is the gamma function and $a_k(P)$ denotes the the coefficient of $t^{\frac{k-n}{d}}$ in the asymptotic expansion of $Tr_{L^2} e^{-tP}$.
Consequently:

$$a_k(P) = d^{-1} \cdot \Gamma(\tfrac{n-k}{d}) \cdot \mathcal{W}(P^{-(\frac{n-k}{d})}).$$

Laplace-type operators

- ♠ More generally, let
$$\Delta_{T,E} = -g^{ab}(\nabla_a \nabla_b - \Gamma^c_{ab} \nabla_c) + E$$
be a Laplace-type operator on a vector bundle Ξ of rank r, where $\nabla_a = \partial_a - T$ with $T \in End\ \Xi$, and $E \in End\ \Xi$.
By a lengthy computation:

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Thm

The functional

$$\mathrm{g}^{\Delta_{T,E}}(V,W) := \mathcal{W}(\nabla_V \nabla_W \Delta_{T,E}^{-m-1})$$

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$$= r\, \mathrm{g}^{\Delta}(V,W).$$

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equals

$$= \frac{1}{6} \int_M \Big(rG(V,W) + 3F(V,W) + 3\mathrm{Tr} E\, g(V,W) \Big) vol_g,$$

where $F(V,W) = \mathrm{Tr}\ V^a W^b F_{ab}$ *and* F_{ab} *is the curvature of* ∇_a.

Spin Laplacian

A particular interesting case is a $spin_c$ manifold M with Ξ a spinor bundle Σ of rank 2^m. The spin Laplacian

$$\Delta^{(s)} := \nabla^{(s)*}\nabla^{(s)} = -\nabla^{(s)}_{e_i}\nabla^{(s)}_{e_i} + \nabla^{(s)}_{\nabla_{e_i}e_i}, \tag{17}$$

where $\nabla^{(s)}$ is the spin connection and e_j is ON frame; biexpands in the order/normal coordinates as

$$\begin{aligned}\Delta^{(s)} &= -\partial_i\partial_i + \frac{1}{3}R_{ijk\ell}\,x^j x^k \partial_i\partial_\ell + o(|x|^2) \\ &\quad + \frac{2}{3}R_{ij}\,x^i\partial_j + \frac{1}{4}R_{i\ell jk}\,x^\ell\gamma^j\gamma^k\partial_i + o(|x|) \\ &\quad + o(1),\end{aligned} \tag{18}$$

where $\gamma^j \in M_{2^m}(\mathbb{C})$ satisfy $\gamma^j\gamma^k + \gamma^k\gamma^j = \delta^{jk}$.

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where $\gamma^j \in M_{2^m}(\mathbb{C})$ satisfy $\gamma^j\gamma^k + \gamma^k\gamma^j = \delta^{jk}$.

Now, $\Delta^{(s)} = \Delta_{T,E}$ for $T = \frac{1}{8}R_{abjk}\gamma^j\gamma^k x^a x^b$ & $E = 0$. Hence,

Proposition

$$\begin{aligned}\mathrm{g}^{\Delta^{(s)}}(V,W) &:= \mathcal{W}\big(\nabla^{(s)}_V\nabla^{(s)}_W(\Delta^{(s)})^{-m-1}\big) = 2^m\mathrm{g}^{\Delta}(V,W),\\ \mathrm{G}^{\Delta^{(s)}}(V,W) &:= \mathcal{W}\big(\nabla^{(s)}_V\nabla^{(s)}_W(\Delta^{(s)})^{-m}\big) = 2^m\mathrm{G}^{\Delta}(V,W)+0.\end{aligned} \tag{19}$$

Hodge-de Rham spectral triple

.. back to M (not necess. spin$_c$).
Another well known classical Dirac-type operator is $D = d + d^*$ on the (rank 2^n) bundle $\Omega(M)$ of differential forms, where d is the exterior derivative and d^* is its (formal) adjoint.
In normal coordinates

$$\sigma(D) = i(\lambda_+^p - \lambda_-^p)\xi_p - \frac{i}{3}\lambda_-^p R_{sapb} x^a x^b \xi_s - \frac{1}{3}\lambda_-^p \lambda_+^r \lambda_-^s (R_{srpa} + R_{spra}) x^a + o(|x|^2),$$

where the matrices λ_+^p, λ_-^p satisfy

$$\lambda_+^p \lambda_+^r + \lambda_+^r \lambda_+^p = 0 = \lambda_-^p \lambda_-^r + \lambda_-^r \lambda_-^p, \quad \lambda_+^p \lambda_-^r + \lambda_-^r \lambda_+^p = \delta_{pr}\,\mathrm{id}.$$

Their components labelled by a pair of multi-indices $(\lambda_+^p)_J^I, (\lambda_-^p)_J^I$, are equal to $(-)^{|\pi|}$ if the juxtaposed index pJ (resp. pI) is a permutation π of I (resp. J) and 0 otherwise.

Squared Dirac operator

We already *spin*, so take on Σ the Dirac operator (coupled do $U(1)$-gauge 1-form A):

$$D_A = i\gamma^j \nabla^{(s)}_{e_j} + A,$$

and employ its square D_A^2, which by the Lichnerowicz thm

$$D_A^2 = \Delta^{(s)} + \frac{1}{4}R + F,$$

where $F = \gamma^j \gamma^k F_{jk}$, and F_{jk} is the curvature of A.

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Proposition

$$\mathrm{g}^{D_A^2}(V,W) := \mathcal{W}(\nabla^{(s)}_V \nabla^{(s)}_W |D_A|^{-n-2}) = 2^m \mathrm{g}^{\Delta}(V,W),$$

$$\mathrm{G}^{D_A^2}(V,W) := \mathcal{W}(\nabla^{(s)}_V \nabla^{(s)}_W |D_A|^{-n})$$

$$= 2^m \Big(\mathrm{G}^{\Delta}(V,W) + \frac{1}{8}\int_M R(g)g(V,W) vol_g\Big).$$

→ go quant

Hodge-de Rham^2: symbols

The 3 symbols of D^2:

$$\mathfrak{a}_2 = \big(\delta_{ab} + \frac{1}{3} R_{acbd} x^c x^d\big) \xi_a \xi_b + o(|x|^2),$$

$$\mathfrak{a}_1 = \frac{2}{3} i R_{ab} \xi_a x^b - \frac{2}{3} i \lambda_+^p \lambda_-^r (R_{rpab} + R_{rapb}) x^b \xi_a + o(|x|),$$

$$\mathfrak{a}_0 = \frac{2}{3} \lambda_+^a \lambda_-^b R_{ab} + \frac{1}{3} \lambda_+^p \lambda_+^r \lambda_-^s \lambda_-^t (R_{tsrp} + R_{trsp}) + o(1).$$

The 3 leading symbols of D^{-2k} up to the appropriate order in x:

$$\mathfrak{c}_{2k} = ||\xi||^{-2k-2} \left(\delta_{ab} - \frac{k}{3} R_{acbc} x^c x^d \right) \xi_a \xi_b + o(|x|^2),$$

$$\mathfrak{c}_{2k+1} = -\frac{2}{3} k i ||\xi||^{-2k-2} \mathrm{R}_{ab} x^b \xi_a + \frac{2}{3} k i ||\xi||^{-2k-2} \lambda_+^r \lambda_-^s \big(R_{srba} + R_{sbra} \big) x^a \xi_b + o(|x|$$

$$\begin{aligned} \mathfrak{c}_{2k+2} = {} & \frac{k(k+1)}{3} ||\xi||^{-2k-4} R_{ab} \xi_a \xi_b \\ & - \frac{2}{3} k(k+1) ||\xi||^{-2k-4} \lambda_+^r \lambda_-^s (R_{srab} + R_{sarb}) \xi_a \xi_b \\ & + \frac{1}{3} k ||\xi||^{-2k-2} \lambda_+^p \lambda_-^q \lambda_+^r \lambda_-^s (R_{sqrp} + R_{srqp}) + o(1). \end{aligned}$$

Proposition

For $v, w \in \Omega^1(M)$*,*

$$\mathrm{g}_{d+d^*}(v,w) := \mathcal{W}(uw|d+d^*|^{-n}) = 2^n \int_M g(v,w)\, vol_g,$$

where g *is the contravariant metric tensor,*

$$\mathrm{G}_{d+d^*}(v,w) := \mathcal{W}(u\{d+d^*, w\}(d+d^*)^{-n+1}) = \frac{2^n}{6} \int_M G(v,w)\, vol_g,$$

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Thus for the spin_c manifolds our spectral functionals for the Hodge-de Rham spectral triple are equal (up to the bundle rank) to those for the canonical spin_c spectral triple.