Pure spinor string

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Pure spinor sigma-model in $AdS_5 \times S^5$: Main Action

Let $G = PSU(2,2|4)$ be the superconformal group, $\mathbf{g} = \mathrm{Lie}G$, it has some $\mathbf{Z}_4$ grading:

$$\mathbf{g} = \mathbf{g}_{\bar{0}} + \mathbf{g}_{\bar{1}} + \mathbf{g}_{\bar{2}} + \mathbf{g}_{\bar{3}}$$

The zero grading part $\mathbf{g}_{\bar{0}}$ is the Lie algebra of:

$$SO(1,4) \times SO(5)$$

The $AdS_5 \times S^5$ is a coset space:

$$\frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$$

$$\Big(\underset{\substack{\cap \\ \mathbf{g}_{\bar{3}}}}{\text{Cone}} \times \underset{\substack{\cap \\ \mathbf{g}_{\bar{1}}}}{\text{Cone}}\Big) \times_{SO(1,4)\times SO(5)} PSU(2,2|4)$$

Consider the cone in the spin bundle over $AdS_5 \times S^5$, parametrized by λ_3 and λ_1 satisfying the pure spinor constraints:

$$\lambda_3^{\alpha} \Gamma^m_{\alpha\beta} \lambda_3^{\beta} = \lambda_1^{\hat{\alpha}} \Gamma^m_{\hat{\alpha}\hat{\beta}} \lambda_1^{\hat{\beta}} = 0 \qquad (1)$$

We will call this space M . It is parametrized by coordinates $g \in PSU(2,2|4)$ modulo the equivalence relation $g \simeq hg$ (the $AdS_5 \times S^5$) and λ_3, λ_1 .

The coordinates of the base (*i.e.* g, λ_3, λ_1) are “fields”. The coordinates of the fiber are “antifields”. The BV Main Action is a functional on this space of maps. It consists of two terms:

$$S_{BV} = S_0 + \widehat{Q}$$

where S_0 depends on fields only, while $\widehat{Q}$ is a function of the fields and antifields, linear in the antifields.

The structure of $\widehat{Q}$

We think of $\widehat{Q}$ as a generating function of a nilpotent vector field Q on the space of maps:

$$i \ : \ \Sigma \to M$$

In fact, this vector field comes from a vector field on M which we also call Q. It is:

$$\begin{aligned} Q &\in \mathrm{Vect}(M) \\ Qg &= (\lambda_3 + \lambda_1) g \end{aligned} \tag{2}$$

The structure of S_0

The S_0 depends on fields only, it is often denoted “ S_{cl} ”. In our “minimalistic” sigma-model it is equal to the pullback to Σ of some two-form $\mathcal{B}$ on M:

$$S_0 = \int_\Sigma i^* \mathcal{B}$$

This two-form is given by the formula:

$$\begin{aligned} \mathcal{B} &\in \Omega^2(M) \\ \mathcal{B} &= \mathrm{STr}\left((dgg^{-1})_{\bar{3}} \wedge (\mathbf{1} - 2\mathbf{P}_{13})(dgg^{-1})_{\bar{1}} \right) \end{aligned} \tag{3}$$

where $\mathbf{P}_{13}$ is a projector on the tangent space to the pure spinor cone at the point λ_1 along the subspace generated by expressions of the form $[v_2, \lambda_3]$ when $v_2 \in \mathbf{g}_2$.

The main property of $\mathcal{B}$ is that $d\mathcal{B}$ is Q-base:

$$\iota_Q d\mathcal{B} = 0$$

Perhaps we can say that $d\mathcal{B}$ represents a cohomology class of $M/\mathbf{R}^{0|1}$ where $\mathbf{R}^{0|1}$ is generated by Q.

On the other hand we can consider the restriction to the fiber:

$$V = \mathcal{B}|_{T\mathbf{R}^{0|1}}$$

This is called “unintegrated vertex operator”.

Global symmetries

Global symmetries act on g by constant right shifts:

$$R_a g = g t_a$$

Additional fields and regularization

Problems with S_{BV}

We observe the following problems:

1. The ghost field λ lives on a cone — singular target space!
2. Action is not a polynomial function of λ
3. The ghost field λ enters without derivatives (no kinetic term?)
4. The action for g is weird (only contains a 2-form, no usual kinetic term)

We will not repair **item 1**.

But we will repair **items 2,3,4** by introducing extra fields and choosing an appropriate Lagrangian submanifold.

Structure of denominators in $\mathcal{B}$

This $\mathcal{B}$ has denominator, but only of a very special kind. The denominator only enters through:

$$\mathbf{P}_{13}J_{\bar{1}}$$
$$\text{where } J = -dgg^{-1}$$

and a similar expression with $1 \leftrightarrow 3$. These expressions have one crucial property: the BRST variation of them does not have denominators. Namely:

$$Q\mathbf{P}_{13}J_{\bar{1}} = -D_0\lambda_1$$

This hints at how the denominators can be actually removed. Let us first discuss some general construction. Suppose I have a BV action S_{BV} whose expansion in powers of antifields terminates at the linear terms. Suppose that we are given a set of local operators $\{\mathcal{O}_1, \ldots, \mathcal{O}_N\}$, which are built only from fields (*i.e.* do not contain antifields). Suppose that we can construct out of them a volume element $\mu(\mathcal{O}_1, \ldots, \mathcal{O}_N)$ on Σ. (For example, if $\mathcal{O}_1$ and $\mathcal{O}_2$ are one-forms, we may take $\mu(\mathcal{O}_1, \mathcal{O}_2) = \mathcal{O}_1 \wedge \mathcal{O}_2$.) Then, consider the following half-density:

$$\rho_{\mathrm{new}} \;=\; \exp\left(\mathrm{S}_{\mathrm{BV}} + \int_\Sigma \mu(\mathcal{O}_1, \ldots, \mathcal{O}_{\mathrm{N}})\right) \prod_{\mathrm{i}=1}^{\mathrm{N}} \delta(\mathrm{Q}\mathcal{O}_{\mathrm{i}}) \tag{4}$$

(while ρ_{old} was just $\mathrm{e}^{\mathrm{S}_{\mathrm{BV}}}$). Then ρ_{new} satisfies the Main Equation.

Adding w and $\mathrm{w}^\star$

General construction

We will now interpret Eq. (4) as a field theory by introducing the Lagrange multipliers. Namely, we represent:

$$\delta(\mathrm{Q}\mathcal{O}_{\mathrm{i}}) = \int [\mathrm{dw}^{\mathrm{i}}] \exp\left(\int_\Sigma \mathrm{w}^{\mathrm{i}}\, \mathrm{Q}\mathcal{O}_{\mathrm{i}}\right)$$

This is a different theory. Moreover, different sets $\{\mathcal{O}_1, \ldots, \mathcal{O}_{\mathrm{N}}\}$ and choices of $\mu(\mathcal{O}_1, \ldots, \mathcal{O}_{\mathrm{N}})$ give different theories. We can think of it as introducing an extra field-antifield pair $\mathrm{w}^{\mathrm{i}}, \mathrm{w}^\star_{\mathrm{i}}$ for each $\mathcal{O}_{\mathrm{i}}$ with $\mathrm{S}^{\{\mathrm{w}\}}_{\mathrm{BV}} = \int_\Sigma \mu(\mathrm{w}^\star_1, \ldots, \mathrm{w}^\star_{\mathrm{N}})$ and then taking the Lagrangian submanifold where $\mathrm{w}^\star_{\mathrm{i}} = 0$ and deforming it with the gauge fermion:

$$\Psi = \int_\Sigma \mathrm{w}^{\mathrm{i}} \mathcal{O}_{\mathrm{i}}$$

Applying to the pure spinor sigma-model

Now, we choose:

$$\text{1st approx: } \{\mathcal{O}_{\mathrm{i}}\} = \{\mathbf{P}_{31}\mathrm{J}_3 \;\text{ and }\; \mathbf{P}_{13}\mathrm{J}_1\}$$

And the construction of μ is the following. As we explained, all denominators come either *via* $\mathbf{P}_{31}\mathrm{J}_3$ or *via* $\mathbf{P}_{13}\mathrm{J}_1$. We just replace:

$$\mathbf{P}_{31}\mathrm{J}_3 \;\longrightarrow\; \omega^\star_3 \tag{5}$$
$$\mathbf{P}_{13}\mathrm{J}_1 \;\longrightarrow\; \omega^\star_1 \tag{6}$$

Literally doing this is wrong, because the kinetic terms becomes:

$$\mathrm{STr}(\omega_1 \wedge \mathrm{d}\lambda_3) + \mathrm{STr}(\omega_3 \wedge \mathrm{d}\lambda_1)$$

But I want **chiral** kinetic terms: left-moving λ_3 and right-moving λ_1 . I could have said, let us restrict $\omega^\star_3$ to only have $\overline{\mathrm{dz}}$ -component, and $\omega^\star_1$ to only have dz -component. But I dont want to break diffeomorphism invariance at the level of the BV Main Action. In string theory the Main Action should be invariant under diffeomorphisms. Therefore, I will leave

ω and $\omega^\star$ generic 1-forms on Σ.

Just to completely fix the notations, the odd sympectic form is:

$$\omega_{\mathrm{BV}} = \int_\Sigma \mathrm{STr}\left(\delta\omega_3^\star \wedge \delta\omega_1 + \delta\omega_1^\star \wedge \delta\omega_3\right)$$

The full Main Action

We define the BV Main Action as follows:

$$S_{\mathrm{BV}}^{+} = \int_\Sigma i^*\mathcal{B} + \widehat{Q} + \int_\Sigma \mathrm{STr}(\omega_3^\star \wedge \omega_1^\star) \tag{7}$$

and gauge fermion:

$$\text{(tentative)}\ \Psi \ = \ \int_\Sigma \mathrm{STr}\left(\omega_3 \wedge \mathbf{P}_{13} J_1 + \omega_1 \wedge \mathbf{P}_{31} J_3\right)$$

Naively this seems to be the sum of two non-interacting theories (one for g, λ and another for w), but there is an important subtlety. I want w and $w^\star$ to live in a nontrivial vector bundle over AdS, namely in $\mathbf{g}_{\mathrm{odd}}$.

Gluing charts

Vector bundle

Let M be a manifold, parameterized by ϕ, and S_{BV} on ΠTM^* of the form:

$$S_{\mathrm{BV}}(\phi, \phi^\star) = S_{\mathrm{cl}}(\phi) + Q^\mu(\phi)\phi^\star_\mu$$

Let H be a Lie group. Suppose that we are given some vector bundle over M with a fiber W — a symplectic linear space with the action of H. Consider the action which in a local trivialization looks like:

$$S_{\mathrm{tot}} \ = \ S_{(\phi)} + S_{(w)} = S_{\mathrm{cl}}(\phi) + Q^\mu(\phi)\phi^\star_\mu \ + \ \frac{1}{2} w_a^\star (\Omega^{-1})^{ab} w_b^\star$$

where Ω is the symplectic form of W.

Transition functions

We want the transition functions to be canonical transformations preserving S_{tot}. We can choose the transition functions to be $\{S_{\mathrm{tot}}, \_\}$-exact:

$$\chi_\alpha \ = \ \{S_{\mathrm{tot}}, F_\alpha\} \tag{8}$$

$$\texttt{where}\ F_\alpha \ = \ -\frac{1}{2} w^b \rho_*(\alpha(\phi))^a_b\, \Omega_{ac}\, w^c \tag{9}$$

$$\chi_\alpha \;=\; \rho_*(\alpha(\phi))^{\mathsf{a}}_{\mathsf{b}}\mathsf{w}^{\mathsf{b}}\;\mathsf{w}^\star_{\mathsf{a}} \;-\; \frac{1}{2}\mathsf{w}^{\mathsf{b}}\rho_*(\mathsf{Q}\alpha(\phi))^{\mathsf{a}}_{\mathsf{b}}\Omega_{\mathsf{ac}}\mathsf{w}^{\mathsf{c}}$$

Notice that:

$$\{\mathsf{F}_\alpha, \mathsf{F}_\beta\} = 0 \tag{10}$$

$$\{\chi_{\alpha_1}\,,\;\mathsf{F}_{\alpha_2}\} \;=\; -\mathsf{F}_{[\alpha_1,\alpha_2]} \tag{11}$$

This canonical transformation does not touch ϕ^μ, it only acts on $\phi^\star, \mathsf{w}, \mathsf{w}^\star$. We identify $(\phi, \phi^\star_{\mathsf{i}}, \mathsf{w}_{\mathsf{i}}, \mathsf{w}^\star_{\mathsf{i}})$ on chart $\mathsf{U}_{(\mathsf{i})}$ with $(\phi, \phi^\star_{\mathsf{j}}, \mathsf{w}_{\mathsf{j}}, \mathsf{w}^\star_{\mathsf{j}})$ on chart $\mathsf{U}_{(\mathsf{j})}$ when $(\phi^\star_{\mathsf{j}}, \mathsf{w}_{\mathsf{j}}, \mathsf{w}^\star_{\mathsf{j}})$ is the flux of $(\phi^\star_{\mathsf{i}}, \mathsf{w}_{\mathsf{i}}, \mathsf{w}^\star_{\mathsf{i}})$ by the time 1 along the vector field $\{\chi_{\alpha_{\mathsf{ji}}}, \_\}$ where α_{ji} is the log of u_{ji}, *i.e.* $\mathsf{u}_{\mathsf{ji}} = \mathsf{e}^{\alpha_{\mathsf{ji}}}$. Explicitly:

$$\mathsf{w}^{\mathsf{a}}_{\mathsf{j}} \;=\; \rho\left(\mathsf{u}_{\mathsf{ji}}\right)^{\mathsf{a}}_{\mathsf{b}}\mathsf{w}^{\mathsf{b}}_{\mathsf{i}} \tag{12}$$

$$\mathsf{w}^\star_{\mathsf{ja}} \;=\; \rho\left(\mathsf{u}_{\mathsf{ji}}^{-1}\right)^{\mathsf{b}}_{\mathsf{a}}\;\mathsf{w}^\star_{\mathsf{ib}} \;-\; \Omega_{\mathsf{ab}}\,\mathsf{Q}\rho\left(\mathsf{u}_{\mathsf{ji}}\right)^{\mathsf{b}}_{\mathsf{c}}\;\mathsf{w}^{\mathsf{c}}_{\mathsf{i}} \tag{13}$$

$$\phi^\star_{\mathsf{j}\mu} \;=\; \phi^\star_{\mathsf{i}\mu} - \mathsf{w}^\star_{\mathsf{ja}}\rho_*\left(\mathsf{u}_{\mathsf{ji}}\;\frac{\overleftarrow{\partial}}{\partial\phi^\mu}\;\mathsf{u}_{\mathsf{ji}}^{-1}\right)^{\mathsf{a}}_{\mathsf{b}}\mathsf{w}^{\mathsf{b}}_{\mathsf{j}} - \frac{1}{2}\mathsf{w}^{\mathsf{a}}_{\mathsf{j}}\Omega_{\mathsf{ab}}\frac{\partial}{\partial\phi^\mu}\rho_*\left(\mathsf{Q}\mathsf{u}_{\mathsf{ji}}\mathsf{u}_{\mathsf{ji}}^{-1}\right)^{\mathsf{b}}_{\mathsf{c}}\mathsf{w}^{\mathsf{c}}_{\mathsf{j}} \tag{14}$$

Given these transition functions, how can we construct a Lagrangian submanifold? The "standard" construction $\phi^\star = \mathsf{w}^\star = 0$ does not work because $\mathsf{w}^\star = 0$ is not invariant under transition functions. On every chart, let us pass to a new set of Darboux coordinates, by doing the canonical transformation with the following gauge fermion:

$$\Psi_{\mathsf{i}} = \frac{1}{2}\mathsf{w}^{\mathsf{a}}_{\mathsf{i}}\;\Omega_{\mathsf{ab}}\;\mathsf{Q}^\mu(\phi)\rho_*(\mathsf{A}_{\mathsf{i}\mu}(\phi))^{\mathsf{b}}_{\mathsf{c}}\;\mathsf{w}^{\mathsf{c}}_{\mathsf{i}}$$

The new S_{BV} will contain the term $\tilde{\mathsf{w}}^\star\mathsf{Q}^\mu\rho_*(\mathsf{A}_{\mathsf{i}\mu})\tilde{\mathsf{w}}$, which means that the action of the BRST operator on $\tilde{\mathsf{w}}$ involves the connection. On the other hand, the transition functions simplify:

$$\tilde{\mathsf{w}}^{\mathsf{a}}_{\mathsf{j}} \;=\; \rho\left(\mathsf{u}_{\mathsf{ji}}(\phi)\right)^{\mathsf{a}}_{\mathsf{b}}\tilde{\mathsf{w}}^{\mathsf{b}}_{\mathsf{i}} \tag{15}$$

$$\tilde{\mathsf{w}}^\star_{\mathsf{ja}} \;=\; \rho\left(\mathsf{u}_{\mathsf{ji}}(\phi)^{-1}\right)^{\mathsf{b}}_{\mathsf{a}}\;\tilde{\mathsf{w}}^\star_{\mathsf{ib}} \tag{16}$$

$$\tilde{\phi}^\star_{\mathsf{j}\mu} \;=\; \tilde{\phi}^\star_{\mathsf{i}\mu} - \tilde{\mathsf{w}}^\star_{\mathsf{ic}}\rho\left(\mathsf{u}_{\mathsf{ji}}(\phi)^{-1}\right)^{\mathsf{c}}_{\mathsf{a}}\left(\rho\left(\mathsf{u}_{\mathsf{ji}}(\phi)\right)^{\mathsf{a}}_{\mathsf{b}}\;\frac{\overleftarrow{\partial}}{\partial\phi^\mu}\right)\tilde{\mathsf{w}}^{\mathsf{b}}_{\mathsf{i}} \tag{17}$$

These are the usual transition functions of the odd cotangent bundle $\Pi\mathsf{T}^*\mathcal{W}$, where $\mathcal{W}$ is the vector bundle with the fiber W, associated to the principal vector bundle $\mathsf{E} \xrightarrow{\mathsf{H}} \mathsf{B}$.

In particular, the "standard" Lagrangian submanifold $\tilde{\mathsf{w}}^\star = \tilde{\phi}^\star = 0$ is compatible with gluing. The corresponding BRST operator is defined by the part of the BV action linear in the antifields:

$$Q_{BRST} = Q^{\mu}\frac{\partial}{\partial\phi^{\mu}} + Q^{\nu}\rho_{*}(A_{\nu})^{a}_{b}\tilde{w}^{b}\frac{\partial}{\partial\tilde{w}^{a}}$$

Lagrangian submanifold mixes w with λ and breaks $\mathrm{Diff}(\Sigma)$

Let us choose vector fields ∇ , $\overline{\nabla}$ — some sections of $\mathbf{C}\otimes T^{1}\Sigma$ (*i.e.* complex vector fields on the worldsheet).

We assume that ∇ and $\overline{\nabla}$ form a basis. In other words exist complex 1-forms α and $\overline{\alpha}$ such that:

$$\nabla\otimes\alpha + \overline{\nabla}\otimes\overline{\alpha} = \mathbf{1} \;:\; T\Sigma \longrightarrow T\Sigma \tag{18}$$
$$\text{that is: } \iota(\nabla)\overline{\alpha} = \iota(\overline{\nabla})\alpha = 0 \;\text{ and }\; \iota(\nabla)\alpha = \iota(\overline{\nabla})\overline{\alpha} = 1$$

(Example: $\nabla = \partial_z$ and $\alpha = dz$.)

We define the Lagrangian submanifold as the odd conormal bundle of the following constraint surface:

$$\begin{aligned}
(\mathbf{1} - \mathbf{P}_{13})\omega_1 &= 0 && (19)\\
(\mathbf{1} - \mathbf{P}_{31})\omega_3 &= 0 && (20)\\
\iota(\overline{\nabla})\omega_1 &= 0 && (21)\\
\iota(\nabla)\omega_3 &= 0 && (22)
\end{aligned}$$

> The last two break the diffeomorphism invariance. The purpose of this constraint is to kill half of the components of ω in the direction tangent to the cone.

In other words:

$$\begin{aligned}
\omega_1 &= \mathbf{P}_{13}w_{1+}\alpha\\
\omega_3 &= \mathbf{P}_{31}w_{3-}\overline{\alpha}
\end{aligned}$$

The fiber of the conormal bundle can be parameterized by fermionic 1-form fields $v_3^{\star}$, $v_1^{\star}$, and fermionic scalar fields $w_3^{\star}$ and $w_1^{\star}$:

$$\begin{aligned}
\omega_3^{\star} &= (\mathbf{1} - \mathbf{P}_{31})v_3^{\star} + \mathbf{P}_{31}w_{3+}^{\star}\alpha\\
\lambda_3 &= \cdots\\
\omega_1^{\star} &= (\mathbf{1} - \mathbf{P}_{13})v_1^{\star} + \mathbf{P}_{13}w_{1-}^{\star}\overline{\alpha}\\
\lambda_1 &= \cdots
\end{aligned}$$

We need to remove the denominator due to $\mathbf{P}_{31}$. This is done by the following gauge fermion:

$$\Psi = \int_{\Sigma} \mathrm{STr}\left(\omega_3\wedge\mathbf{P}_{13}(dgg^{-1})_1 + \omega_1\wedge\mathbf{P}_{31}(dgg^{-1})_3\right)$$

This generates the kinetic term for λ s:

$$\int_\Sigma \mathrm{STr}\left(w_{3-}\overline{\alpha}\wedge d\lambda_1 + w_{1+}\alpha\wedge d\lambda_3\right) \;=\; \int_\Sigma \alpha\wedge\overline{\alpha}\,\mathrm{STr}\left(w_{3-}\nabla\lambda_1 + w_{1+}\overline{\nabla}\lambda_3\right)$$

(pops up when we hit dgg^{-1} with Q). At the same time $\int_\Sigma \mathrm{STr}\left(\omega_3^\star\wedge\omega_1^\star\right)$ gives:

$$\int_\Sigma \mathrm{STr}\left(\left(\omega_3^\star + \mathbf{P}_{31}(dgg^{-1})_3\right)\wedge\left(\omega_1^\star + \mathbf{P}_{13}(dgg^{-1})_1\right)\right)\mapsto$$

$$\mapsto \int_\Sigma \mathrm{STr}\left(\left((\mathbf{1}-\mathbf{P}_{31})v_3^\star + \mathbf{P}_{31}(w_{3+}^\star\alpha + (dgg^{-1})_3)\right)\wedge\left((\mathbf{1}-\mathbf{P}_{13})v_1^\star + \mathbf{P}_{13}(w_{1+}^\star\alpha + (dgg^{-1})_1)\right)\right)$$

Integration over $v^\star$ decouples. Integration over $w_{3+}^\star$ projects $\mathbf{P}_{31}(dgg^{-1})_3$ to $\mathbf{P}_{31}(\overline{\nabla}gg^{-1})_3\overline{\alpha}$. Integration over $w_{1-}^\star$ projects $\mathbf{P}_{13}(dgg^{-1})_1$ to $\mathbf{P}_{13}(\nabla gg^{-1})_1\alpha$. We are left with:

$$\int_\Sigma \alpha\wedge\overline{\alpha}\,\mathrm{STr}\left((\nabla gg_{-1})_1\wedge\mathbf{P}_{31}(\overline{\nabla}gg_{-1})_3\right) \tag{23}$$

This term should cancel the denominators — see Eq. (3). But it is not antisymmetric under $+\leftrightarrow-$.

It turns out that we can further deform the Lagrangian submanifold, so that effectively

$$\mathcal{B}\mapsto\mathcal{B}+\mathcal{G}$$

where $\mathcal{G}$ is worldsheet parity even. (See ⟦The b -ghost is a target space symmetric tensor⟧.) It is important for us that the term in $\mathcal{G}$ containing $\mathbf{P}_{31}$ combine with the term in $\mathcal{B}$ containing $\mathbf{P}_{31}$ into the expression given by Eq. (23). This expression is rather special. Indeed, it has the form:

$$\int_\Sigma dz^\alpha\wedge dz^\beta\; Y_\alpha^{\alpha'}(z,\bar z)\overline{Y}_\beta^{\beta'}(z,\bar z)\; A_{\mu\nu}(x)\;\partial_{\alpha'}x^\mu\partial_{\beta'}x^\nu$$

where

$$\det Y = \det\overline{Y} = 0$$

and

$$\det(Y\otimes\overline{Y}) = \mathbf{1}\;:\;\Lambda^2\mathbf{C}^2\longrightarrow\Lambda^2\mathbf{C}^2$$

(Indeed, $Y=\nabla\otimes\alpha$ and $\overline{Y}=\overline{\nabla}\otimes\overline{\alpha}$.) This implies that $Y_{[\alpha}^{(\alpha'}\overline{Y}_{\beta]}^{\beta')}$ is of the form:

$$Y_{[\alpha}^{(\alpha'}\overline{Y}_{\beta]}^{\beta')} = \epsilon_{\alpha\beta}\sqrt{h}h^{\alpha'\beta'}$$

In other words, the parity-even part of Eq. (23) is of the form:

$$\int d^2z\sqrt{h}h^{\alpha\beta}\mathcal{G}_{\mu\nu}\partial_\alpha x^\mu\partial_\beta x^\nu$$

As in the case of bosonic string this is **not** of the most general form $\int d^2z a^{\alpha\beta}\mathcal{G}_{\mu\nu}\partial_\alpha x^\mu\partial_\beta x^\nu$, as $a^{\alpha\beta}$ is restricted to be of the form $\sqrt{h}h^{\alpha\beta}$ — a nonlinear constraint. The mechanism, however, is quite different from what it was in bosonic string.

The b-ghost is a target space symmetric tensor

First of all, sigma-models whose action is an integral of a two-form over the worldsheet (of the type $\int \mathcal{B}_{IJ}(x)\partial_+x^I\partial_-x^J$ with antisymmetric $\mathcal{B}_{IJ}(x)$) are degenerate and cannot be immediately quantized. We need a term symmetric under $+\leftrightarrow -$. Such a term is generated by the shift of the standard Lagrangian submanifold by a gauge fermion of the form:

$$\int_\Sigma d^2z\,\mathbf{b}_{IJ}(x)a^{\alpha\beta}\partial_\alpha x^I\partial_\beta x^J$$

where $a^{\alpha\beta}$ is a symmetric tensor-density on Σ and $\mathbf{b}_{IJ}$ is a **symmetric** tensor on the target space. It is unfortunate that we must call it $\mathbf{b}$ because letter B usually suggests the Kalb-Ramond B-field, an **antisymmetric** tensor. But we do insist on calling it $\mathbf{b}$ because it is actually the BV prototype of the pure spinor b-ghost. (For an antisymmetric tensor, we use $\mathcal{B}$.)

> The BV ``origin'' of the b-ghost is a fermionic symmetric tensor field $\mathbf{b}$ on the target space

(See: Target space $\mathbf{b}$) We introduce

$$Qi^*\mathbf{b} = i^*\left(\mathcal{L}_Q\mathbf{b}\right)$$

Then the deformation of the action is:

$$Q\int_\Sigma \langle a\,,\,i^*\mathbf{b}\rangle \;=\; \int_\Sigma \left\langle a\,,\,\mathrm{STr}\left(\frac{1}{2}i^*J_2\otimes i^*J_2 + i^*J_1\otimes i^*(\mathbf{1}-\mathbf{P}_{31})J_3\right)\right\rangle \qquad (24)$$

Finally

$$\begin{aligned} S &= \int d^2z\,L \\ L &= \mathrm{STr}\left(\frac{1}{2}J_{2+}J_{2-} + \frac{3}{4}J_{1+}J_{3-} + \frac{1}{4}J_{3+}J_{1-} + w_{1+}D_{0-}\lambda_3 + w_{3-}D_{0+}\lambda_1 - N_{0+}N_{0-}\right) \end{aligned}$$

where $N_{0+} = \{w_{1+},\lambda_3\}$ and $N_{0-} = \{w_{3-},\lambda_1\}$.

Diffeomorphisms

Example when diffeomorphisms are exact

Let M be some manifold. Consider the space of maps:

$$\mathrm{Map}(\Sigma, \Pi TM)$$

An element of $\mathrm{Map}(\Sigma, \Pi TM)$ is a map

$$\Sigma \times \mathbf{R}^{0|1} \to M$$

We parameterize $\mathbf{R}^{0|1}$ by ζ, so elements of $\mathrm{Map}(\Sigma, \Pi TM)$ are functions $\phi(z, \zeta)$ with values in M. There is a cohomological vector field Q induced by d on ΠTM. The flux by odd time θ of Q is:

$$(e^{\theta Q}\phi)(z, \zeta) = \phi(z, \zeta + \theta)$$

There is a canonical map:

$$v \; : \; \mathrm{Vect}(\Sigma) \to \mathrm{Vect}\left(\mathrm{Map}(\Sigma, \Pi TM)\right)$$

For any $\xi \in \mathrm{Vect}(\Sigma)$, by definition:

$$(e^{tv\langle\xi\rangle}\phi)(z, \zeta) = \phi(e^{tv}z, \zeta)$$

We observe that v is covariantly Q-exact, in the following sense. Exists a map

$$i \; : \; \mathrm{Vect}(\Sigma) \longrightarrow \mathrm{Vect}\left(\mathrm{Map}(\Sigma, \Pi TM)\right)$$

such that:

$$v(\xi) = [Q, i(\xi)] \tag{25}$$

$$[v(\xi), i(\eta)] = i([\xi, \eta]) \tag{26}$$

This map is defined as follows:

$$(e^{\theta i\langle\xi\rangle}\phi)(z, \zeta) = \phi(e^{\theta\zeta v\langle\xi\rangle}z, \zeta)$$

Differential ideals in PDFs

Suppose that we are given a submanifold (possibly singular):

$$C \subset \Pi TM$$

such that the vector field d is tangent to it. Then d induces on C a nilpotent vector field Q. This is same as specifying a differential ideal in the supercommutative algebra of PDFs.

We can then consider the space of maps $\mathrm{Map}(\Sigma, C)$. It still has Q and v. But does it have i such that Eq. (25) is satisfied?

Example: pure spinors in AdS

Consider, again, the pure spinor cone over $AdS_5 \times S^5$:

$$\left(\underset{\mathfrak{g}_3}{\underset{\cap}{\mathrm{Cone}}} \times \underset{\mathfrak{g}_1}{\underset{\cap}{\mathrm{Cone}}}\right) \times_{SO(1,4)\times SO(5)} PSU(2,2|4)$$

Let us associate to it a differential ideal in PDFs on $PSU(2,2|4)$ in the following way. In a local chart $\Pi T(AdS_5 \times S^5)$ is parameterized by:

$$g\,,\quad J_1 = -(dgg^{-1})_1\,,\quad J_2 = -(dgg^{-1})_2\,,\quad J_3 = -(dgg^{-1})_3$$

with d acting as follows:

$$\begin{aligned} dg &= -(A_0 + J_1 + J_2 + J_3)g \\ dJ_1 &= [J_2, J_3] + [A_0, J_1] \\ dJ_2 &= \{J_1, J_1\} + \{J_3, J_3\} + [A_0, J_2] \end{aligned}$$

where A_0 is some $so(1,4) \oplus so(5)$ connection. The differential ideal is given by:

$$J_2 = 0 \tag{27}$$
$$\{J_1, J_1\} = \{J_3, J_3\} = 0 \tag{28}$$

Let us call this $CAdS$ (“cone bundle of AdS”):

$$CAdS \subset \Pi T(AdS_5 \times S^5) \text{ solving Eqs (27) and (28)}$$

On this constraint, we denote:

$$\lambda_3 = J_3\,,\quad \lambda_1 = J_1$$

In this case the map i was partially constructed in my paper.