Twisted R–Poisson Sigma Models & Higher Geometry

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+ 2206.03683 (ThCh, Noriaki Ikeda, Grga Šimunić) + WiP

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Motivation

- Batalin-Vilkovisky (BV) quantization & Geometry $\rightsquigarrow$ AKSZ construction of TFTs.

  2D: Poisson Sigma Model $\rightsquigarrow$ A- and B-model; 3D: Courant Sigma Model $\rightsquigarrow$ Chern-Simons; QP manifolds

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- Wess-Zumino (WZ) terms require twisted structures & vanilla AKSZ doesn't work.
  2D: WZW-Poisson Sigma Model Klimcik, Strobl '01 3D: 4-form-twisted (pre-)Courant Sigma Model Hansen, Strobl '09
  - ✓ The Q-vs-QP problem: it can happen that $Q \cup P \neq QP$ for the target space, or even $\not{P}$.

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    E.g. Twisted (pre-)Courant sigma models in 3D.
  - ✓ Still 2D but beyond twisted Poisson, e.g. Dirac sigma models.
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  Relation to higher structures? A dictionary?
- Are Poisson and twisted Poisson sigma models just a 2D story? (no)
- Strings on general flux backgrounds & duality ⇝ "R-flux" (3-vector)
  world volume pov: Halmagyi '08; Mylonas, Schupp, Szabo '12; Heller, Ikeda, Watamura '16; ThCh, Jonke, Khoo, Szabo '18
  - ✓ Understand their global formulation.

⇝ Twisted R-Poisson structures & their induced WZ-TFTs

R-Poisson structure & twists

H-twisted R-Poisson manifold $(\mathsf{M}, \Pi, \mathsf{R}, \mathsf{H})$ of order $p+1$: 2- & $(p+1)$-vectors Π & R; $(p+2)$-form H

$$[\Pi, \Pi]_{\mathsf{SN}} = 0 \quad [\Pi, \mathsf{R}]_{\mathsf{SN}} = (-1)^{p+1} \langle \otimes^{p+2} \Pi, \mathsf{H} \rangle \,, \quad \mathrm{d}\mathsf{H} = 0 \,.$$

for $\mathsf{H} = 0 \rightsquigarrow$ (untwisted) R-Poisson; for $\mathsf{H} = 0 = \mathsf{R} \rightsquigarrow$ Poisson.

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Recall: there exists the notion of C-twisted Poisson manifold (M, Π, C) Ševera, Weinstein '01

This is not a C-twisted R-Poisson for $p = 1$ ...

$$\frac{1}{2} [\Pi, \Pi]_{SN} = \langle \otimes^3 \Pi, C \rangle \quad \text{and} \quad \mathrm{d}C = 0 \,.$$

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$$\frac{1}{2} [\Pi, \Pi]_{SN} = \langle \otimes^3 \Pi, C \rangle \quad \text{and} \quad \mathrm{d}C = 0 \,.$$

Bi-twisted R-Poisson manifold (M, Π, R, C, H): 2-vector Π, 3-vector R, 3-form C, 4-form H s.t.

$$\begin{aligned} \frac{1}{2} [\Pi, \Pi]_{SN} &= R + \langle \Pi \otimes \Pi \otimes \Pi, C \rangle \,, \\ \mathrm{d}C &= H \,. \end{aligned}$$

For $R = 0 = H$ reduces to C-twisted Poisson manifold with a closed 3-form C.

C–Twisted Poisson & Q–manifold

Lie algebroids $(E \xrightarrow{\pi} M, [\cdot,\cdot]_E, \rho : E \to TM) \quad \Leftrightarrow \quad$ Q-manifolds $(E[1], Q_E)$. Vaintrob '97

Twisted Poisson structure $(M, \Pi, C) \rightsquigarrow$ Lie algebroid on $T^*M \rightsquigarrow (T^*[1]M, Q_{T^*M})$:

Coordinates $\quad (x^\mu, a_\mu) \quad$ of degree $\quad (0, 1)$,

$$Q_{T^*M} = \Pi^{\mu\nu}(x) a_\mu \frac{\partial}{\partial x^\nu} - \frac{1}{2}(\partial_\rho \Pi^{\mu\nu} + \Pi^{\mu\kappa}\Pi^{\nu\lambda} C_{\kappa\lambda\rho}) a_\mu a_\nu \frac{\partial}{\partial a_\rho} ,$$

$$Q^2_{T^*M} = 0 \quad \Leftrightarrow \quad \frac{1}{2}[\Pi, \Pi]_{SN} = \langle \otimes^3 \Pi, C \rangle .$$

Covariantization Warm-Up

Rewrite the Q-vector in covariant form in terms of an affine connection ∇ on M

$$Q_{T^*M} = \Pi^{\mu\nu} a_\mu \mathrm{D}_\nu^{(0)} - \frac{1}{2} \mathring{\nabla}_\rho \Pi^{\mu\nu} a_\mu a_\nu \frac{\partial}{\partial a_\rho}\,, \quad (\mathrm{D}_\nu^{(0)} = \frac{\partial}{\partial x^\nu} + \Gamma^\sigma_{\nu\rho} a_\sigma \frac{\partial}{\partial a_\rho})$$

where Γ are coefficients of ∇ and $\mathring{\nabla}$ is the torsionless piece. They differ by the torsion

$$T = \langle \Pi, C \rangle \quad \left(\text{that is} \quad \Gamma^\rho_{\mu\nu} = \mathring{\Gamma}^\rho_{\mu\nu} - \frac{1}{2} \Pi^{\rho\sigma} C_{\mu\nu\sigma}\right)\,.$$

One may now ask what is the other object, namely $-\mathring{\nabla}\Pi$, in geometrical terms.

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Rewrite the Q-vector in covariant form in terms of an affine connection ∇ on M

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It's the E-torsion of an E-connection on the Lie algebroid $(T^*M, [\cdot,\cdot]_K, \Pi^\sharp : T^*M \to TM)$

$$^E\nabla : \Gamma(E \otimes V) \to \Gamma(V)\,, \quad {}^E\nabla_e(fv) = f\,{}^E\nabla_e v + \rho(e) f\, v\,, \quad e \in \Gamma(E)\,,\ v \in \Gamma(V)\,.$$

$$(V = E)\,, \quad {}^E\nabla_e e' := \nabla_{\Pi^\sharp(e)} e'\,, \quad {}^E T(e,e') = {}^E\nabla_e e' - {}^E\nabla_{e'} e - [e,e']_K\,.$$

N.B. One can unite $^E T$ and $\langle \Pi, T \rangle$ as components of the Gualtieri torsion tensor on a Courant algebroid ...

H-twisted R-Poisson & Q-manifold

Similarly, for a H-twisted R-Poisson structure $(\mathrm{M}, \Pi, \mathrm{R}, \mathrm{H}) \rightsquigarrow (\mathrm{T}^*[p]\mathrm{T}^*[1]\mathrm{M}, Q)$:

Coordinates $\quad (x^\mu, a_\mu, y^\mu, z_\mu) \quad$ of degreee $\quad (0, 1, p-1, p)\,.$

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Coordinates $(x^\mu, a_\mu, y^\mu, z_\mu)$ of degreee $(0, 1, p-1, p)$.

(There exists a cohomological Q-vector

$$Q = \Pi^{\nu\mu} a_\nu \frac{\partial}{\partial x^\mu} - \frac{1}{2}\partial_\rho \Pi^{\mu\nu} a_\mu a_\nu \frac{\partial}{\partial a_\rho} +$$

$$+ \left((-1)^p \Pi^{\nu\mu} z_\nu - \partial_\nu \Pi^{\mu\rho} a_\rho y^\nu + \frac{1}{p!} \mathrm{R}^{\mu\nu_1\dots\nu_p} a_{\nu_1} \dots a_{\nu_p}\right) \frac{\partial}{\partial y^\mu} +$$

$$+ \left(\partial_\rho \Pi^{\mu\nu} a_\nu z_\mu - \frac{(-1)^p}{2} \partial_\rho \partial_\sigma \Pi^{\mu\nu} y^\sigma a_\mu a_\nu + \frac{(-1)^p}{(p+1)!} f_\rho^{\mu_1\dots\mu_{p+1}} a_{\mu_1} \dots a_{\mu_{p+1}}\right) \frac{\partial}{\partial z_\rho},$$

where $f_\rho^{\mu_1\dots\mu_{p+1}} = \partial_\rho \mathrm{R}^{\mu_1\dots\mu_{p+1}} + \prod_{r=1}^{p+1} \Pi^{\mu_r\nu_r} \mathrm{H}_{\rho\nu_1\dots\nu_{p+1}}$.)

$Q^2 = 0 \quad \Leftrightarrow \quad [\Pi, \Pi]_{\mathrm{SN}} = 0$ and $[\Pi, \mathrm{R}]_{\mathrm{SN}} = (-1)^{p+1} \langle \otimes^{p+2} \Pi, \mathrm{H}_{p+2} \rangle$.

Covariantization

Introduce an affine connection without torsion $\mathring{\nabla}$ on M and rewrite the Q-vector as

subject to the redefinition $z^{\mathring{\nabla}}_\mu = z_\mu + \mathring{\Gamma}^\rho_{\mu\nu} y^\nu a_\rho$ and in terms of suitable "Ds" of corresponding degree; $\mathring{\mathcal{R}}$ is the curvature of ∇

$$
\begin{aligned}
Q \;=\;& \Pi^{\mu\nu} a_\mu \mathrm{D}^{(0)}_\nu - \frac{1}{2}\mathring{\nabla}_\rho \Pi^{\mu\nu} a_\mu a_\nu \mathrm{D}^\rho_{(-1)} \\
&+ \left((-1)^p \Pi^{\mu\nu} z^{\mathring{\nabla}}_\mu - \mathring{\nabla}_\mu \Pi^{\nu\rho} a_\rho y^\mu + \frac{1}{p!} \mathrm{R}^{\nu\mu_1\dots\mu_p} a_{\mu_1}\dots a_{\mu_p} \right) \mathrm{D}^{(1-p)}_\nu \\
&+ \left(\mathring{\nabla}_\nu \Pi^{\mu\rho} a_\rho z^{\mathring{\nabla}}_\mu - \frac{(-1)^p}{2} \left(\mathring{\nabla}_\nu \mathring{\nabla}_\mu \Pi^{\rho\sigma} - 2\Pi^{\kappa[\rho} \mathring{\mathcal{R}}^{\sigma]}{}_{\mu\kappa\nu} \right) y^\mu a_\rho a_\sigma \right) \mathrm{D}^\nu_{(-p)} \\
&- \frac{(-1)^p}{(p+1)!} \left(\mathring{\nabla}_\nu \mathrm{R}^{\mu_1\dots\mu_{p+1}} + \prod_{r=1}^{p+1} \Pi^{\mu_r\nu_r} \mathrm{H}_{\nu\nu_1\dots\nu_{p+1}} \right) a_{\mu_1}\dots a_{\mu_{p+1}} D^\nu_{(-p)}\,.
\end{aligned}
$$

One may now ask what is the geometric interpretation of all these new objects?

$$\Pi, \mathring{\nabla}\Pi$$

$$\mathrm{R}, \mathring{\nabla}\mathrm{R} + \langle \otimes^{p+1}\Pi, \mathrm{H}\rangle \quad \text{and} \quad \mathring{\nabla}\mathring{\nabla}\Pi - 2\mathrm{Alt}\langle \Pi, \mathring{\mathcal{R}}\rangle$$

“Basic” E–curvature

Let us focus on the last of these objects, for which we already have the ingredients.

In general, equipped with an affine connection ∇ on M & a Lie algebroid on E, define

$$^{\mathrm{E}}\overline{\nabla}_e X := \rho(\nabla_X e) + [\rho(e), X] \,.$$

Then the basic E-curvature is a map Blaom '06

$$^{\mathrm{E}}S : \Gamma(\mathrm{E} \otimes \mathrm{E} \otimes \mathrm{TM}) \to \Gamma(\mathrm{E})$$

$$^{\mathrm{E}}S(e, e')X = \nabla_X [e, e']_{\mathrm{E}} - [\nabla_X e, e']_{\mathrm{E}} - [e, \nabla_X e']_{\mathrm{E}} - \nabla_{^{\mathrm{E}}\overline{\nabla}_{e'} X} e + \nabla_{^{\mathrm{E}}\overline{\nabla}_e X} e' \,.$$

For $\mathrm{E} = \mathrm{T}^*\mathrm{M}$ it turns out that $^{\mathrm{E}}S = -\nabla(^{\mathrm{E}}T) - 2\mathrm{Alt}\langle \Pi, \mathcal{R} \rangle$ $\rightsquigarrow$ precisely the desired term.
see Kotov, Strobl '16

Lessons:

- E-geometry (E-connections, E-torsion, basic E-curvature) controls the Q-structure.
- Basic E-curvature $^{\mathrm{E}}S$ has more content than E-curvature $^{\mathrm{E}}\mathcal{R}$ of an E-connection.

$\mathbf{Q} \cup \mathbf{P} \neq \mathbf{QP}$

We work on a cotangent bundle $\rightsquigarrow$ equipped with a (graded) symplectic (P) structure.

QP-manifolds: dg symplectic with compatibility of the vector Q & the symplectic form ω

$$\mathcal{L}_Q \omega = 0 \,.$$

In general, for twisted Poisson & twisted R-Poisson: $\mathcal{L}_Q \omega \neq 0$ when $\mathrm{H}_{p+2} \neq 0 \,.$

Lesson: Twists (WZ terms in the TFT) obstruct QP-ness.

Enter Field Theory

Goals and main pointers

- Construct WZ-TFTs induced by twisted R-Poisson structure in any dimension ≥ 2.

***

- General class of TFTs in $(p+1)$D with “nonlinear openness” & high reducibility
  Gauge algebra closes on **products** of field equations / forms of degree > 1
- BV operator and BV action very demanding no QP, no AKSZ
  - ✓ Fully solved for D $= 3 \rightsquigarrow$ 1st example of BV for a pre-Courant sigma model
  - ✓ Closed formulas for any D in the untwisted case, alternative to AKSZ with advantages ...
- Target space covariance Role of E-geometry and **Ep-geometry** (connections, torsion and basic curvature)
- “Islands” of TFTs in special D by deformation $\rightsquigarrow$ ex: bi-twisted R-Poisson 3D TFT.
  in the present context, only in low dimensions, 2, 3 and 4

Warm-Up: The C-twisted Poisson Sigma Model

Klimcik, Strobl '01; Ikeda, Strobl '19

2D TFT with scalars & 1-forms (X^μ, A_μ) & C-twisted Poisson manifold as target space

$X : \Sigma_2 \to \mathrm{M}$ & $A \in \Omega^1(\Sigma_2, X^*\mathrm{T}^*\mathrm{M})$.

$$S_{\text{C-PSM}} = \int_{\Sigma_2} \left(A_\mu \wedge \mathrm{d}X^\mu + \frac{1}{2}\Pi^{\mu\nu}(X)\, A_\mu \wedge A_\nu \right) + \int_{\Sigma_3} X^* \mathrm{C}\,.$$

Symmetries/EOMs scalar ϵ_μ ; the gauge algebra is "soft" and "open" even for $C = 0$, $[\delta_1, \delta_2]A = \delta_{12}A + (\ldots)F$

$$\delta X^\mu = \Pi^{\nu\mu}\epsilon_\nu\,, \qquad \delta A_\mu = \mathrm{d}\epsilon_\mu + \partial_\mu \Pi^{\nu\rho} A_\nu \epsilon_\rho + \frac{1}{2}\Pi^{\nu\rho}\mathrm{C}_{\mu\nu\sigma}(\mathrm{d}X^\sigma - \Pi^{\sigma\lambda}A_\lambda)\epsilon_\rho\,.$$

$$F^\mu := \mathrm{d}X^\mu + \Pi^{\mu\nu}A_\nu = 0\,, \qquad G_\mu := \mathrm{d}A_\mu + \frac{1}{2}\partial_\mu\Pi^{\nu\rho}A_\nu \wedge A_\rho + \frac{1}{2}\mathrm{C}_{\mu\nu\rho}\mathrm{d}X^\nu \wedge \mathrm{d}X^\rho = 0\,.$$

The covariant transformation of A and its manifestly covariant field strength are

$$\delta^\nabla A = \mathrm{D}\epsilon - {}^{\mathrm{E}}T(A, \epsilon) \quad \text{and} \quad G^\nabla = \mathrm{D}A - \frac{1}{2}\,{}^{\mathrm{E}}T(A, A)\,,$$

Recall that ${}^{E}T$ does not see C; all C-dependence is through D, the fully covariant exterior derivative

H-twisted R-Poisson Sigma Models

TFTs on Σ_{p+1} with $X : \Sigma_{p+1} \to \mathrm{M}$ and a WZ term from a closed $(p+2)$-form H on M.

Field content $(X^\mu, A_\mu, Y^\mu, Z_\mu)$ (chosen as to accommodate a 2-vector background)

$$A \in \Omega^1(\Sigma_{p+1}, X^*\mathrm{T}^*\mathrm{M}) \qquad Y \in \Omega^{p-1}(\Sigma_{p+1}, X^*\mathrm{TM}) \qquad Z \in \Omega^p(\Sigma_{p+1}, X^*\mathrm{T}^*\mathrm{M})\,.$$

The general classical action functional for $p > 0$ with target $(\mathrm{M}, \Pi, \mathrm{R}, \mathrm{H})$ of order $p+1$

N.B. for $\mathrm{R} = 0 = \mathrm{H}$, this is a Poisson sigma model in any dimension ...

$$S^{(p+1)} = \int_{\Sigma_{p+1}} \Big(Z_\mu \wedge \mathrm{d}X^\mu - A_\mu \wedge \mathrm{d}Y^\mu + \Pi^{\mu\nu}(X)\, Z_\mu \wedge A_\nu - \frac{1}{2}\, \partial_\rho \Pi^{\mu\nu}(X)\, Y^\rho \wedge A_\mu \wedge A_\nu \, + \\ + \frac{1}{(p+1)!} \mathrm{R}^{\mu_1 \cdots \mu_{p+1}}(X)\, A_{\mu_1} \wedge \cdots \wedge A_{\mu_{p+1}} \Big) + \int_{\Sigma_{p+2}} X^*\mathrm{H}\,.$$

Even the action functional does not look very covariant at first sight in this case.

Gauge symmetries & nonlinear openness

Three gauge parameters $(\epsilon_\mu, \chi^\mu, \psi_\mu)$ of form degrees $(0, p-2, p-1)$,

$$\delta X^\mu = \Pi^{\nu\mu}\epsilon_\nu \,,$$

$$\delta A_\mu = \mathrm{d}\epsilon_\mu + \partial_\mu \Pi^{\nu\rho} A_\nu \epsilon_\rho \,,$$

$$\delta Y^\mu = \mathrm{d}\chi^\mu + \text{terms}(\Pi, \partial\Pi, \mathrm{R})$$

$$\delta Z_\mu = \mathrm{d}\psi_\mu + \text{terms}(\Pi, \partial\Pi, \partial\partial\Pi, \partial\mathrm{R}) - \frac{1}{(p+1)!}\Pi^{\rho\nu}\mathrm{H}_{\mu\nu\lambda_1\dots\lambda_p}\epsilon_\rho \sum_{r=1}^{p+1}(-1)^r \prod_{s=1}^{r-1}\mathrm{d}X^{\lambda_s}\prod_{t=r}^{p}\Pi^{\lambda_t\kappa_t}A_{\kappa_t} \,.$$

4 EOMs, $F^\mu \supset \mathrm{d}X^\mu \,,\ G_\mu \supset \mathrm{d}A_\mu \,,\ \mathcal{F}^\mu \supset \mathrm{d}Y^\mu \,,\ \mathcal{G}_\mu \supset \mathrm{d}Z_\mu \ \dots = 0$.

A "soft", "open" and highly reducible constrained Hamiltonian system. Notably:

$$[\delta_1, \delta_2]Z_\mu \approx \delta_{12}Z_\mu + (\dots)_\mu^\nu G_\nu + (\dots)_{\mu\nu}\mathcal{F}^\nu + (\dots)_{\mu\nu}F^\nu +$$

$$+(\dots)_{\mu\nu\rho}F^\nu F^\rho + \dots + (\dots)_{\mu\nu_1\dots\nu_p}F^{\nu_1}\dots F^{\nu_p} \,.$$

Unveiling target space covariance

Introduce an ordinary connection $\mathring{\nabla}$ (without torsion) on TM. Then, e.g.

Recall: $Z_\mu^{\mathring{\nabla}} = Z_\mu + \mathring{\Gamma}^\rho_{\mu\nu} Y^\nu \wedge A_\rho$

$$\mathcal{F}^{\mathring{\nabla}} = \mathring{\mathrm{D}} Y - {}^{\mathrm{E}}T(A, Y) + (-1)^p \Pi(Z^{\mathring{\nabla}}) - \frac{1}{p!} \mathrm{R}(A, \dots, A)\,,$$

$$\mathcal{G}^{\mathring{\nabla}} = (-1)^{p+1} \mathring{\mathrm{D}} Z^{\mathring{\nabla}} - {}^{\mathrm{E}}T(Z^{\mathring{\nabla}}, A) + \frac{1}{2}\, {}^{\mathrm{E}}S(Y, A, A) + \frac{1}{(p+1)!} (\mathring{\nabla}\mathrm{R} + \mathcal{T})(A, \dots, A)\,.$$

$\rightsquigarrow$ $\mathrm{T}^*\mathrm{M}$-torsion, basic $\mathrm{T}^*\mathrm{M}$-curvature and $\mathcal{T} := \langle \otimes^{p+1} \Pi, \mathrm{H}_{p+2} \rangle \in \Gamma(\mathrm{T}^*\mathrm{M} \otimes \bigwedge^{p+1} \mathrm{TM})$.

The action may be expressed in covariant form as pull-backs understood ...

$$S^{(p+1)} = \int_{\Sigma_{p+1}} \left(\langle Z^{\mathring{\nabla}}, F \rangle - \langle Y, G^{\mathring{\nabla}} \rangle + \frac{1}{(p+1)!} \mathrm{R}(A, \dots, A) \right) + \int_{\Sigma_{p+2}} X^* \mathrm{H}\,.$$

In 3D/4D, the geometric completion of local patch results for string/M-theory fluxes.

Mylonas, Schupp, Szabo '12; Th. Ch, Jonke, Lechtenfeld '15; Heller, Ikeda, Watamura '16; Th. Ch., Jonke, Lüst, Szabo '19

Enter BV

Classical BV in a nutshell

Given the classical action S_0 and its gauge symmetries,

- Enlarge the configuration space by ghosts, ghosts for ghosts &c. and antifields.
- Define an odd symplectic structure on this space, the BV (anti)bracket $(\cdot, \cdot)_{\mathrm{BV}}$.
- Extend S_0 with all possible terms with ghosts/antifields to an action S.
- Solve the Classical Master Equation (CME) $(S, S)_{\mathrm{BV}} = 0$.

NB: the BRST operator s_0 is not nilpotent off-shell, but the BV operator $s = (S, \cdot)_{\mathrm{BV}}$ is.

Once WZ terms are turned on, AKSZ does not apply. Example: Twisted PSM

TPSM: no higher form gauge parameters, or ghosts for ghosts, or nonlinear openness

The twisted R-Poisson class in arbitrary dimensions features all the above. $4(p+2)$ fields

Field/Ghost	X^μ	A_μ	Y^μ	Z_μ	ϵ_μ	$\chi^\mu_{(r)}$	$\psi_\mu^{(r)}$
Ghost degree	0	0	0	0	1	$r+1$	$r+1$
Form degree	0	1	$p-1$	p	0	$p-2-r$	$p-1-r$

Antifield	X^+_μ	A^μ_+	Y^+_μ	Z^μ_+	ϵ^μ_+	$\chi^{+(r)}_\mu$	$\psi^\mu_{+(r)}$
Ghost degree	-1	-1	-1	-1	-2	$-r-2$	$-r-2$
Form degree	$p+1$	p	2	1	$p+1$	$r+3$	$r+2$

BV operator and action

- Now the action would be $S_{\text{BV}} = S^{(0)} + S^{(1)} + \cdots + S^{(p+1)} \rightsquigarrow$ tough ...
- Instead use a “refinement strategy” to determine the BV operator on all fields

  starting from the known BRST operator on the classical fields

$$s\varphi \quad \text{such that} \quad s^2\varphi = 0\,.$$

  Turns out to be much more tractable due to repeating patterns.
- Essentially $s\varphi = s_{\text{AKSZ}}\varphi + (\Delta s\,\varphi)(H, F)$ unlike TPSM, all but one ψ-ghosts receive EOM-corrections

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  Turns out to be much more tractable due to repeating patterns.

- Essentially $s\varphi = s_{\text{AKSZ}}\varphi + (\Delta s\,\varphi)(H, F)$ unlike TPSM, all but one ψ-ghosts receive EOM-corrections

- Worth noting: in the twisted Poisson $s_0^2 A_\mu \propto {}^{\text{E}}S^{\rho\sigma}_{\mu\nu}\epsilon_\rho\epsilon_\sigma F^\nu$ (basic curvature).
  Here the square of s_0 on the highest-form contains, covariantly & schematically:

$$s_0^2 Z^{\mathring{\nabla}} \supset \left[\mathring{\nabla}(\mathring{\nabla}\text{R} + \langle\otimes^{p+1}\Pi, \text{H}\rangle) - (p+1)\text{Alt}\langle \text{R}, \mathring{\mathcal{R}}\rangle\right](\epsilon, \epsilon, A, \dots, A, F)\,.$$

- Reflect the openness, as usual in BV they appear along with "4-fermion" terms.
  Recall the BV action of the topological A- and B-models

3D Twisted R-Poisson-Courant Sigma Model

Out of the 8 fields and ghosts, 4 unmodified w.r.t. AKSZ: $X^\mu, A_\mu, \epsilon_\mu, \chi^\mu$.

2 modified only with H-components but not with field equations: indices of H raised with Π

$$\Delta s\, Y^\mu = \frac{1}{4} \mathsf{H}_\sigma{}^{\mu\nu\rho}\, Z_+^\sigma \epsilon_\nu \epsilon_\rho \,, \quad \Delta s\, \psi_\mu^{(1)} = \frac{1}{3!} \mathsf{H}_\mu{}^{\nu\rho\sigma}\, \epsilon_\nu \epsilon_\rho \epsilon_\sigma \,.$$

2 modified also by HF-dependent terms:

$$\Delta s\, \psi_\mu = \left(\frac{1}{4} \mathsf{H}_{\mu\nu}{}^{\rho\sigma}\, F^\nu + \frac{1}{2} \mathsf{H}_\mu{}^{\rho\sigma\nu}\, A_\nu - \frac{1}{3!} \partial_{(\lambda} \mathsf{H}_{\mu)}{}^{\rho\sigma\nu}\, Z_+^\lambda \epsilon_\nu \right) \epsilon_\rho \epsilon_\sigma \,,$$

$$\begin{aligned} \Delta s\, Z_\mu = & \left\{ \frac{1}{3!} \mathsf{H}_{\mu\kappa\lambda}{}^{\nu}\, F^\kappa F^\lambda + \frac{1}{2} \mathsf{H}_{\mu\lambda}{}^{\nu\kappa}\, A_\kappa F^\lambda + \frac{1}{2} \mathsf{H}_\mu{}^{\nu\kappa\lambda} \left(A_\kappa A_\lambda - \frac{1}{2} \epsilon_\kappa Y_\lambda^+ \right) \right. \\ & + \frac{1}{3!} \partial_{(\mu} \mathsf{H}_{\rho)\lambda}{}^{\nu\kappa}\, F^\lambda Z_+^\rho \epsilon_\kappa + \frac{1}{3!} \partial_{(\rho} \mathsf{H}_{\mu)}{}^{\nu\kappa\lambda} \left(\epsilon_\lambda \psi_+^\rho + 3 A_\lambda Z_+^\rho \right) \epsilon_\kappa \\ & \left. - \left(\frac{1}{2 \cdot 3!} \partial_{(\rho} \partial_\sigma \mathsf{H}_{\mu)}{}^{\nu\kappa\lambda} + \frac{1}{8} \partial_{(\rho} \partial_\sigma \Pi^{\nu\tau} \mathsf{H}_{\mu)\tau}{}^{\kappa\lambda} \right) \epsilon_\kappa \epsilon_\lambda Z_+^\rho Z_+^\sigma \right\} \epsilon_\nu \,. \end{aligned}$$

Remarkably, the BV action is then fully and uniquely determined. details in ThCh, Ikeda, Šimunić

Deformations
or
Can we twist the twisted R–Poisson?

dim Σ_{p+1}	Admissible deformations
2	$f^{\mu\nu}(X,Y)\,Z_\mu \wedge Z_\nu,\ \ f^{\mu\nu}(Y)\,A_\mu \wedge A_\nu$
3	$f_{\mu\nu\rho}(X) Y^\mu \wedge Y^\nu \wedge Y^\rho,\ \ f^\mu_\nu(X)\,Z_\mu \wedge Y^\nu,\ f^\rho_{\mu\nu}(X) Y^\mu \wedge Y^\nu \wedge A_\rho$
4	$f_{\mu\nu}(X) Y^\mu \wedge Y^\nu$

Island TFTs & Bi-twisted R-Poisson

- 2D $\rightsquigarrow$ different choices of the deformation action yield: Generically doubled sigma models
  - Ex1: H-twisted R-Poisson of order 2, Ex2: H-twisted Poisson + BF theory
- 3D $\rightsquigarrow$ R is a 3-vector & the theory extends Courant sigma models by a WZ term.
  cf. Hansen, Strobl '09
  - Bi-twisted R-Poisson $(\mathrm{M}, \Pi, \mathrm{R}, C, \mathrm{H})$
  - The covariant formulation requires a connection with torsion. Q-structure is modified.
- 4D $\rightsquigarrow$ aside the TRPSM, a theory with a symmetric term
  cf. Ikeda, Uchino '10

$$S^{(4)}_{\text{def}} = \int_{\Sigma_4} \frac{1}{2}\, g_{\mu\nu}(X)\, Y^{\mu} \wedge Y^{\nu}\ .$$

A strong necessary condition: $\Pi^{\mu\nu} g_{\nu\rho} = 0$. (Would be interesting to relax it.)

Enter Ep-Geometry

work in progress

The geometric remains as Ep-tensors

E-geometrically, $\Pi, \mathring{\nabla}\Pi, \nabla\mathring{\nabla}\Pi + 2\mathrm{Alt}\langle \Pi, \mathcal{R}\rangle$: anchor, E-torsion, basic E-curvature.

Still require explanation: $\mathrm{R}, \mathring{\nabla}\mathrm{R} + \langle \otimes^{p+1}\Pi, \mathrm{H}\rangle, \mathring{\nabla}(\mathring{\nabla}\mathrm{R} + \langle \otimes^{p+1}\Pi, \mathrm{H}\rangle) - (p+1)\mathrm{Alt}\langle \mathrm{R}, \mathring{\mathcal{R}}\rangle$.

The geometric remains as Ep-tensors

E-geometrically, $\Pi, \mathring{\nabla}\Pi, \nabla\mathring{\nabla}\Pi + 2\mathrm{Alt}\langle \Pi, \mathcal{R}\rangle$: anchor, E-torsion, basic E-curvature.

Still require explanation: $\mathrm{R}, \mathring{\nabla}\mathrm{R} + \langle \otimes^{p+1}\Pi, \mathrm{H}\rangle, \mathring{\nabla}(\mathring{\nabla}\mathrm{R} + \langle \otimes^{p+1}\Pi, \mathrm{H}\rangle) - (p+1)\mathrm{Alt}\langle \mathrm{R}, \mathring{\mathcal{R}}\rangle$.

The (p+1)-vector $\mathrm{R} \in \Gamma(\wedge^{p+1}\mathrm{TM})$ induces a map $\mathrm{R}^\sharp : \wedge^p \mathrm{T}^*\mathrm{M} \to \mathrm{TM}$:

$$\mathrm{R}^\sharp(\widehat{e}) = \frac{1}{p!}\mathrm{R}^{\mu_1\cdots\mu_{p+1}}\widehat{e}_{\mu_1\ldots\mu_p}\partial_{\mu_{p+1}}\,, \quad \widehat{e} \in \Gamma(\wedge^p \mathrm{T}^*\mathrm{M})\,.$$

For our twisted R-Poisson purposes it suffices to focus on decomposable p-forms:

$$\widehat{e} = e^{(1)} \wedge \ldots \wedge e^{(p+1)}\,, \quad e^{(r)} \in \Gamma(\mathrm{T}^*\mathrm{M})\,.$$

In any case, one can define an “Ep-connection” by means of the identification

$${}^{\mathrm{Ep}}\nabla_{\widehat{e}}\, e = \nabla_{\mathrm{R}^\sharp(\hat{e})}\, e\,,$$

with good Leibniz rule provided with the assistance of map $\mathrm{R}^\sharp$.

Ep-torsion

The Ep-torsion of the higher Ep-connection is a map ${}^{\mathrm{Ep}}T : \Gamma(\otimes^{p+1}\mathrm{T}^*\mathrm{M}) \to \Gamma(\mathrm{T}^*\mathrm{M})$,

$$ {}^{\mathrm{Ep}}T(e^{(1)},\dots,e^{(p+1)}) = \sum_{r=1}^{p+1}(-1)^{p-r+1}\,{}^{\mathrm{Ep}}\nabla_{\widehat{e}[r]}e^{(r)} - [e^{(1)},\dots,e^{(p+1)}]_{\mathrm{Kp}}\,, $$

where $e^{(r)} \in \Gamma(\mathrm{T}^*\mathrm{M})$ and $\widehat{e}[r]$ is the decomposable p-form

$$ \widehat{e}[r] = e^{(1)} \wedge \dots e^{(r-1)} \wedge e^{(r+1)} \wedge \dots \wedge e^{(p+1)}\,, $$

and the generalized $(p+1)$-ary twisted Koszul bracket is given as

$$ [e^{(1)},\dots,e^{(p+1)}]_{\mathrm{Kp}} = \sum_{r=1}^{p+1}(-1)^{p-r+1}\mathcal{L}_{\mathrm{R}^\sharp(\widehat{e}[r])}e^{(r)} - \frac{1}{(p-1)!}\,\mathrm{d}(\mathrm{R}(e^{(1)},\dots,e^{(p+1)})) + \mathrm{Tw}(e^{(r)})\,, $$

$$ \mathrm{Tw}(e^{(1)},\dots,e^{(p+1)}) = \frac{1}{2}\sum_{r=1}^{p+1}(-1)^{p-r}\langle e^{(r)}, T\rangle(\mathrm{R}^\sharp(\widehat{e}[r]),\cdot)\,. $$

For $p=1 \rightsquigarrow {}^{\mathrm{E1}}T = {}^{\mathrm{E}}T$ & $[\cdot,\cdot]_{\mathrm{K1}} = [\cdot,\cdot]_{\mathrm{K}}$. For $T = \langle \Pi, C\rangle$, $\mathrm{Tw} = C(\Pi^\sharp(e^{(1)}),\Pi^\sharp(e^{(2)}))$.

Therefore it reproduces the known results of Poisson for $p=1$.

Ep-torsion and Q-structure / BV

For arbitrary p, the Ep-torsion is $\quad {}^{\mathrm{Ep}}T = -\mathring{\nabla}\mathrm{R}$.

A near-miss? No; for twisted R-Poisson, use the Π too in the $(p+1)$-ary bracket:

$$[\cdot,\ldots,\cdot]_{\mathrm{Kp}} \longrightarrow [\cdot,\ldots,\cdot]_{\mathrm{Kp}} + \mathrm{H}(\Pi^\sharp(e^{(1)}),\ldots,\Pi^\sharp(e^{(p+1)}))\,.$$

Then the Ep-torsion is precisely the appearing tensor in the Q-structure and in the BV

$${}^{\mathrm{Ep}}T = -\mathring{\nabla}\mathrm{R} - \langle \otimes^{p+1}\Pi, \mathrm{H}\rangle\,.$$

Basic Ep-curvature

First introduce the induced Ep-connection on TM given by

$$^{\mathrm{Ep}}\overline{\nabla}_{\widehat{e}}X = \mathrm{R}^{\sharp}(\nabla_X \widehat{e}) + [\mathrm{R}^{\sharp}(\widehat{e}), X]\,.$$

The basic Ep-curvature is a map $^{\mathrm{Ep}}S : \Gamma(\otimes^{p+1}\mathrm{T}^*\mathrm{M} \otimes \mathrm{TM}) \to \Gamma(\mathrm{T}^*\mathrm{M})$ given by
for $p=1$, $^{\mathrm{E1}}S = {}^{\mathrm{E}}S$

$$\begin{aligned} ^{\mathrm{Ep}}S(e^{(1)},\dots,e^{(p+1)})X \quad &= \quad \nabla_X[e^{(1)},\dots,e^{(p)}]_{\mathrm{Kp}} - \sum_{r=1}^{p+1}(-1)^{p-r}\nabla_{{}^{\mathrm{Ep}}\overline{\nabla}_{\widehat{e}[r]}X}e^{(r)} - \\ &\quad - \sum_{r=1}^{p+1}[e^{(1)},\dots,e^{(r-1)},\nabla_X e^{(r)},e^{(r+1)},\dots,e^{(p+1)}]_{\mathrm{Kp}}\,. \end{aligned}$$

A direct computation leads to the result

$$^{\mathrm{Ep}}S = -\nabla(^{\mathrm{Ep}}T) - (p+1)\,\mathrm{Alt}\langle \mathrm{R}, \mathcal{R}\rangle\,,$$

which is the final object encountered in the Q-structure & the BV of twisted R-Poisson.

Take-home messages

- ✓ New examples of solutions to the Classical Master Equation for ~~QP~~ TFTs.
  - ✓ First example in more than 2D.
- ✤ Twisted R-Poisson TFTs encompass all generic features of gauge theory.
  - ✤ Even the unorthodox nonlinear openness.
- ✿ E- and Ep-geometry are the backbones of twisted R-Poisson TFTs and their BV.
  - ✿ Notably, the corresponding notions of torsion and basic curvature.

Some open questions

- Systematics of WZ-AKSZ-BV? Systematics of Ep-geometry?
- Quantum BV action of twisted R-Poisson / relation to deformation quantization?
- TFTs for general homotopy Poisson (P_∞) structures? Th. Voronov '05; Cattaneo, Felder '07
- Beyond $T[1]\Sigma$ world volumes? cf. ThCh, Karagiannis, Schupp '20

Enter Back-Up Slides

mostly long formulas

The BV action of 3D H-twisted R-Poisson

$$S_{\text{BV}}^{(3)} = S^{(3)} - \sum_\alpha \int (-1)^{\text{gh}(\varphi)} \varphi_\alpha^+ s_0 \varphi^\alpha + \int \left(L_\mu Z_+^\mu + M_{\kappa\lambda} Z_+^\kappa Z_+^\lambda + N_{\kappa\lambda\mu} Z_+^\kappa Z_+^\lambda Z_+^\mu \right) ,$$

$$\begin{aligned}
L_\kappa &= -\partial_\kappa \Pi^{\mu\nu} \widetilde{\psi}_\nu Y_\mu^+ + \partial_\kappa \partial_\lambda \Pi^{\mu\nu} \left(\frac{1}{2} \epsilon_\mu \epsilon_\nu A_+^\lambda - \epsilon_\nu \chi^\lambda Y_\mu^+ + \epsilon_\mu \widetilde{\psi}_\nu \psi_+^\lambda\right) + \\
&+ \frac{1}{2} \partial_\kappa \partial_\lambda \partial_\mu \Pi^{\rho\sigma} \epsilon_\rho \epsilon_\sigma \chi^\lambda \psi_+^\mu - \frac{1}{2} (\partial_\kappa R^{\rho\sigma\lambda} + \frac{1}{2} H_\kappa{}^{\rho\sigma\lambda}) \epsilon_\sigma \epsilon_\lambda Y_\rho^+ + \\
&+ \frac{1}{6} \partial_{(\kappa} f_{\mu)}{}^{\rho\sigma\lambda} \epsilon_\rho \epsilon_\sigma \epsilon_\lambda \psi_+^\mu ,\\
M_{\kappa\lambda} &= \frac{1}{2} \partial_\kappa \partial_\lambda \Pi^{\rho\sigma} (\epsilon_\rho \psi_\sigma - A_\rho \widetilde{\psi}_\sigma) + \frac{1}{2} \partial_\kappa \partial_\lambda \partial_\mu \Pi^{\rho\sigma} (\epsilon_\rho A_\sigma \chi^\mu + \frac{1}{2} \epsilon_\rho \epsilon_\sigma Y^\mu) - \\
&- \frac{1}{4} \partial_{(\kappa} f_{\lambda)}{}^{\rho\sigma\mu} \epsilon_\rho \epsilon_\sigma A_\mu - \frac{1}{12} \partial_{(\kappa} H_{\lambda)\mu}{}^{\rho\sigma} \epsilon_\rho \epsilon_\sigma F^\mu ,\\
N_{\kappa\lambda\mu} &= -\frac{1}{6} \partial_\kappa \partial_\lambda \partial_\mu \Pi^{\rho\sigma} \epsilon_\rho \widetilde{\psi}_\sigma - \frac{1}{12} \partial_\kappa \partial_\lambda \partial_\mu \partial_\nu \Pi^{\rho\sigma} \epsilon_\rho \epsilon_\sigma \chi^\nu - \\
&- \left(\frac{1}{36} \partial_{(\kappa} \partial_\lambda f_{\mu)}{}^{\rho\sigma\nu} + \frac{1}{24} \partial_{(\kappa} \partial_\lambda \Pi^{\rho\tau} H_{\mu)\tau}{}^{\sigma\nu} \right) \epsilon_\rho \epsilon_\sigma \epsilon_\nu .
\end{aligned}$$

BV operator for all χ-ghosts in any dimension

up to factors and limits, see ThCh, Ikeda, Šimunić

Full untwisted BV operator including $Y \sim \chi_{(-1)}$:

$$
\begin{aligned}
s\chi^{\mu}_{(r)} &= \mathrm{d}\chi^{\mu}_{(r+1)} + \sum_{s} \#\, \partial_\kappa \partial_{\lambda_1} \ldots \partial_{\lambda_s} \Pi^{\mu\nu} \sum_{s'} (-1)^{s'} \mathcal{O}^{\lambda_1\cdots\lambda_s}(s,s') \mathfrak{X}^{\kappa}_{\nu}(s,s') + \\
&+ \sum_{s} \#\, \partial_{\lambda_1} \ldots \partial_{\lambda_s} \Pi^{\mu\nu} \sum_{s'} (-1)^{s'} \mathcal{O}^{\lambda_1\cdots\lambda_s}(s,s') \psi_{\nu}^{(r+s+s'+1)} + \\
&- \sum_{t,s,s',t} \#\, \partial_{\lambda_1} \ldots \partial_{\lambda_s} \mathrm{R}^{\mu\nu_1\cdots\nu_a\kappa_1\cdots\kappa_{p-a}} \mathcal{O}^{\lambda_1\cdots\lambda_s}(s,s') \widetilde{\mathcal{O}}_{\kappa_1\ldots\kappa_t}(t,t') \epsilon_{\nu_1} \ldots \epsilon_{\nu_a} A_{\kappa_{t+1}} \ldots A_{\kappa_{p-a}} ,
\end{aligned}
$$

where we denote $a := r + s + s' + t + t' + 2$ and we define the following operators,

$$
\mathcal{O}^{\lambda_1\cdots\lambda_s}(s,s') = \sum_{\substack{m_i=-1 \\ 1\le i\le s-1}}^{s'-1} \left(\prod_{u=1}^{s-1} \psi_{+}^{\lambda_u}{}_{(m_u)} \right) \psi_{+}^{\lambda_s}{}_{(s'-s-\sum_{i=1}^{s-1} m_i)} , \quad \left(\mathcal{O}(0,s') = \delta_{0,s'} \right) ,
$$

$$
\widetilde{\mathcal{O}}_{\kappa_1\ldots\kappa_t}(t,t') = \sum_{\substack{m_i=-1 \\ 1\le i\le t-1}}^{t'-1} (-1)^{\sum_{q=0}^{\lfloor t/2\rfloor -1}(1+m_{t-1-2q})} \left(\prod_{u=1}^{t-1} \chi^{+}_{\kappa_u}{}^{(m_u)} \right) \chi^{+}_{\kappa_t}{}_{(t'-t-\sum_{i=1}^{t-1} m_i)} , \quad \left(\widetilde{\mathcal{O}}(0,t') = \delta_{0,t'} \right)
$$

$$
\mathfrak{X}^{k}_{j}(s,s') = \sum_{u=0}^{p-r-s-s'-2} \widetilde{\mathcal{O}}_{j}(1,u-2) \chi^{k}_{(r+s+s'+u)} ,
$$

BV operator for all ψ-ghosts in any dimension

up to factors and limits, see ThCh, Ikeda, Šimunić

Full untwisted BV operator including $Z \sim \psi_{(-1)}$:

$$
\begin{aligned}
s\psi_\mu^{(r)} \;=\;& \mathrm{d}\psi_\mu^{(r+1)} + \sum_s \#\,\partial_\mu \partial_{\lambda_1} \dots \partial_{l_s} \Pi^{\nu\rho} \sum_{s'} (-1)^{s'} \mathcal{O}^{\lambda_1 \dots \lambda_s}(s,s') \widetilde{\mathfrak{X}}_{\nu\rho}(s,s') + \\
&+ \sum_{s,s',t,t'} \#\,\partial_\mu \partial_\lambda \partial_{\lambda_1} \dots \partial_{\lambda_s} \Pi^{\nu\rho} \mathcal{O}^{\lambda_1 \dots \lambda_s}(s,s') \widetilde{\mathcal{O}}_\nu(1,t'-2) \widetilde{\mathcal{O}}_k(1,t-t'-2) \chi^l_{(t+r+s+s'-1)} + \\
&+ \sum_{t,s,s',t'} \#\,\partial_{\lambda_1} \dots \partial_{\lambda_s} \partial_\mu R^{\nu_1 \dots \nu_a \kappa_1 \dots \kappa_{p-a+1}} \mathcal{O}^{\lambda_1 \dots \lambda_s}(s,s') \widetilde{\mathcal{O}}_{\kappa_1 \dots \kappa_t}(t,t') \epsilon_{\nu_1} \dots \epsilon_{\nu_a} A_{\kappa_{t+1}} \dots A_{\kappa_{p-a+1}} \,,
\end{aligned}
$$

where the only new operator that appears is defined as

$$
\widetilde{\mathfrak{X}}_{\nu\rho} = \sum_{u=0}^{p-r-s-s'-1} \widetilde{\mathcal{O}}_\nu(1,u-2) \psi_\rho^{(r+s+s'+u)} \,.
$$