Tutorial on
The Universal Teichmüller space

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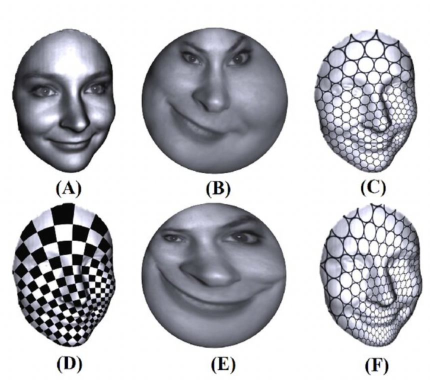
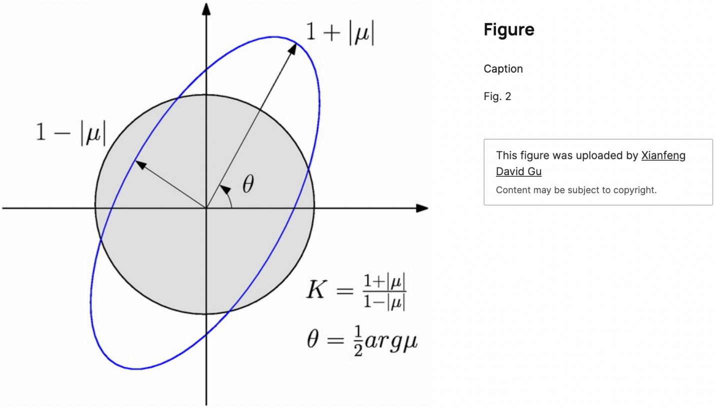


Figure

Caption

Fig. 1 CONFORMAL AND QUASI-CONFORMAL MAPS FOR A TOPOLOGICAL DISK. (A) IS THE ORIGINAL FACE. (B) IS THE CONFORMAL MAPPING OF (A). (C) IS THE CIRCLE PACKING INDUCED BY (B). (D) IS THE CHECKERBOARD TEXTURE INDUCED BY (B). (E) IS THE QUASI-CONFORMAL MAPPING. (F) IS THE CIRCLE PACKING INDUCED BY (E).

Figure: W. Zeng, L.M. Lui, F. Luo, J.S. Liu, T.F. Chan, S.T. Yau, X.F. Gu, Computing Quasiconformal Maps on Riemann surfaces using Discrete Curvature Flow, https://doi.org/10.48550/arXiv.1005.4648



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Quasiconformal and quasisymmetric mappings

Definition (quasiconformal mappings)

An orientation preserving homeomorphim f of an open subset A in $\mathbb{C}$ is called quasiconformal if

- f admits distributional derivatives $\partial_z f$, $\partial_{\bar{z}} f \in L^1_{loc}(A, \mathbb{C})$;
- there exists $0 \leq k < 1$ such that $|\partial_{\bar{z}} f(z)| \leq k|\partial_z f(z)|$ for every $z \in A$.

Such an homeomorphism is said to be K-conformal, where $K = \frac{1+k}{1-k}$.

For example, $f(z) = \alpha z + \beta \bar{z}$ with $|\beta| < |\alpha|$ is $\frac{|\alpha|+|\beta|}{|\alpha|-|\beta|}$-quasiconformal.

Quasiconformal and quasisymmetric mappings

Theorem (Lehto 1987)

An orientation preserving homeomorphism f defined on an open set $A \subset \mathbb{C}$ is ***quasiconformal*** *if and only if it admits distributional derivatives $\partial_z f$, $\partial_{\bar{z}} f \in L^1_{loc}(A, \mathbb{C})$ which satisfy the* ***Beltrami equation***

$$\partial_{\bar{z}} f(z) = \mu(z) \partial_z f(z), \quad z \in A$$

for some $\mu \in L^\infty(A, \mathbb{C})$ with $\|\mu\|_\infty < 1$, called the ***Beltrami coefficient*** *or the complex dilatation of f.*

Let $\mathbb{D}$ denote the open unit disc in $\mathbb{C}$.

Theorem (Ahlfors-Bers)

Given $\mu \in L^\infty(\mathbb{D}, \mathbb{C})$ with $\|\mu\|_\infty < 1$, there exists a unique quasiconformal mapping $\omega_\mu : \mathbb{D} \to \mathbb{D}$ with Beltrami coefficient μ, extending continuously to $\overline{\mathbb{D}}$, and fixing $1, -1, i$.

Quasiconformal and quasisymmetric mappings

Definition (quaisymmetric mappings)

An orientation preserving homeomorphism η of the circle $\mathbb{S}^1$ is called **quasisymmetric** if there is a constant $M > 0$ such that for every $x \in \mathbb{R}$ and every $|t| \leq \frac{\pi}{2}$

$$\frac{1}{M} \leq \frac{\tilde{\eta}(x+t) - \tilde{\eta}(x)}{\tilde{\eta}(x) - \tilde{\eta}(x-t)} \leq M,$$

where $\tilde{\eta}$ is the increasing homeomorphism on $\mathbb{R}$ uniquely determined by $0 \leq \tilde{\eta}(0) < 1$, $\tilde{\eta}(x+1) = \tilde{\eta}(x) + 1$, and the condition that its projects onto η.

Theorem (Beurling-Ahlfors extension Theorem)

Let η be an orientation preserving homeomorphism of $\mathbb{S}^1$. Then η is quasisymmetric if and only if it extends to a quasiconformal homeomorphism of the open unit disc $\mathbb{D}$ into itself.

$T(1)$ as a Banach manifold.

Denote by $L^\infty(\mathbb{D})_1$ the unit ball in $L^\infty(\mathbb{D}, \mathbb{C})$.

Definition (The Universal Teichmüller space via Beltrami coefficients)

By Ahlfors-Bers theorem, for any $\mu \in L^\infty(\mathbb{D})_1$, one can consider the unique quasiconformal mapping $w_\mu \; : \mathbb{D} \to \mathbb{D}$ which fixes $-1, -i$ and 1 and satisfies the Beltrami equation on $\mathbb{D}$

$$\frac{\partial}{\partial \bar{z}} \omega_\mu = \mu \frac{\partial}{\partial z} \omega_\mu.$$

Therefore one can define the following equivalence relation on $L^\infty(\mathbb{D})_1$. For μ, $\nu \in L^\infty(\mathbb{D})_1$, set $\mu \sim \nu$ if $w_\mu|\mathbb{S}^1 = w_\nu|\mathbb{S}^1$. The universal Teichmüller space is defined by the quotient space

$$T(1) = L^\infty(\mathbb{D})_1 / \sim .$$

$T(1)$ as a complex Banach manifold.

Theorem (Lehto 1987)

The space $T(1)$ has a unique structure of complex Banach manifold such that the projection map $\Phi : L^\infty(\mathbb{D})_1 \to T(1)$ is a holomorphic submersion.

The differential of Φ at the origin $D_0\Phi : L^\infty(\mathbb{D}) \to T_{[0]}T(1)$ is a complex linear surjection and induces a splitting of $L^\infty(\mathbb{D}, \mathbb{C})$ into [TaTe2004]:

$$L^\infty(\mathbb{D}, \mathbb{C}) = \text{Ker}\, D_0\Phi \oplus \Omega_\infty(\mathbb{D}, \mathbb{C}),$$

where $\Omega^\infty(\mathbb{D}, \mathbb{C})$ is the Banach space of bounded harmonic Beltrami differentials on $\mathbb{D}$ defined by

$$\Omega_\infty(\mathbb{D}, \mathbb{C}) := \left\{\mu \in L^\infty(\mathbb{D}, \mathbb{C}) \mid \mu(z) = (1 - |z|^2)^2\overline{\phi(z)}, \phi \text{ holomorphic on } \mathbb{D}\right\}.$$

$T(1)$ as a group via quasisymmetric homeomorphisms

T(1) as a group

By the Beurling-Ahlfors extension theorem, a quasiconformal mapping on $\mathbb{D}$ extends to a quasisymmetric homeomorphism on the unit circle leading to the bijection

$$\begin{array}{ccc} T(1) & \stackrel{\sim}{\to} & \mathrm{QS}(\mathbb{S}^1)/PSU(1,1) \\ [\mu] & \mapsto & \left[w_\mu|\mathbb{S}^1\right]. \end{array}$$

- The coset $\mathrm{QS}(\mathbb{S}^1)/PSU(1,1)$ herits from its identification with $T(1)$ a complex Banach manifold structure.
- the coset $\mathrm{QS}(\mathbb{S}^1)/PSU(1,1)$ can be identified with the subgroup of quasisymmetric homeomorphisms fixing $-1, i$ and 1. This identification allows to endow the universal Teichmüller space with a group structure.

Relative to this differential structure, the right translations in $T(1)$ are biholomorphic mappings, whereas the left translations are not even continuous in general. Consequently $T(1)$ is not a topological group.

The WP-metric and the Hilbert manifold structure on $T(1)$.

The Banach manifold $T(1)$ carries a Weil-Petersson metric, which is defined only on a distribution of the tangent bundle [NagVerjovsky1990]. In order to resolve this problem the idea in [TaTe2004] is to change the differentiable structure of $T(1)$.

Theorem (TaTe2004)

The universal Teichmüller space $T(1)$ admits a structure of Hilbert manifold on which the Weil-Petersson metric is a right-invariant strong hermitian metric.

For this Hilbert manifold structure, the tangent space at [0] in $T(1)$ is isomorphic to

$$\Omega_2(\mathbb{D}) := \left\{ \mu(z) = (1 - |z|^2)^2 \overline{\phi(z)}, \quad \phi \text{ holomorphic on } \mathbb{D}, \quad \|\mu\|_2 < \infty \right\},$$

where $\|\mu\|_2^2 = \int\int_{\mathbb{D}} |\mu|^2 \rho(z) d^2 z$ is the L^2-norm of μ with respect to the hyperbolic metric of the Poincaré disc $\rho(z)d^2z = 4(1 - |z|^2)^{-2} d^2 z$.

The WP-metric and the Hilbert manifold structure on $T(1)$.

Definition (Weil-Petersson metric on $T(1)$)

The Weil-Petersson metric on $T(1)$ is given at the tangent space at $[0] \in T(1)$ by

$$\langle \mu, \nu \rangle_{WP} := \iint_{\mathbb{D}} \mu \, \overline{\nu} \, \rho(z) d^2 z$$

With respect to its Hilbert manifold structure, $T(1)$ admits uncountably many connected components. For this Hilbert manifold structure, the identity component $T_0(1)$ of $T(1)$ is a topological group.

The WP-metric and Virasoro coadjoint orbit.

T(1) and Virasoro coadjoint orbit

The pull-back of the Weil-Petersson metric on the quotient space $\mathrm{Diff}_+(\mathbb{S}^1)/\,\mathrm{PSU}(1,1) \subset QS(\mathbb{S}^1)/\,\mathrm{PSU}(1,1)$ is given at [Id] by

$$h_{WP}([\mathrm{Id}])([u],[v]) = 2\pi \sum_{n=2}^{\infty} n(n^2-1)u_n\overline{v_n}.$$

Hence $T_0(1)$ of $T(1)$ can be seen as the completion of $\mathrm{Diff}_+(\mathbb{S}^1)/\,\mathrm{PSU}(1,1)$ for the $H^{3/2}$-norm.

This metric make $T(1)$ into a strong Kähler-Einstein Hilbert manifold, with respect to the complex structure given at [Id] by the Hilbert transform. The tangent space at [Id] consists of Sobolev class $H^{3/2}$ vector fields modulo $\mathfrak{psu}(1,1)$.

The WP-metric and Virasoro coadjoint orbit.

T(1) and Virasoro coadjoint orbit

The associated Riemannian metric is given by

$$g_{WP}([\mathrm{Id}])([u],[v]) = \pi \sum_{n\neq -1,0,1} |n|(n^2-1)u_n\overline{v_n},$$

and the imaginary part of the Hermitian metric is the two-form

$$\omega_{WP}([\mathrm{Id}])([u],[v]) = -i\pi \sum_{n\neq -1,0,1} n(n^2-1)u_n\overline{v_n}.$$

Note that ω_{WP} coincides with the Kirillov-Kostant-Souriau symplectic form obtained on $\mathrm{Diff}_+(\mathbb{S}^1)/\,\mathrm{PSU}(1,1)$ when considered as a coadjoint orbit of the Bott-Virasoro group.

The Universal Teichmüller space and Shapes in the plane

The fingerprint map

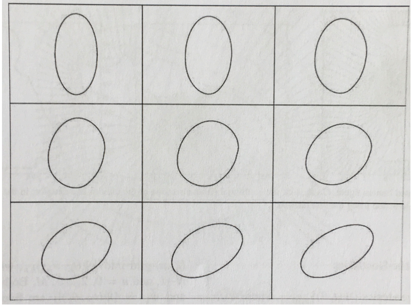
Consider a Jordan curve γ in the plane. Denote by $\mathscr{O}$ and $\mathscr{O}^*$ the two connected components of $\mathbb{C} \setminus \gamma$. By the Riemann mapping theorem, there exists two conformal maps $f : \mathbb{D} \to \mathscr{O}$ and $g : \mathbb{D}^* \to \mathscr{O}^*$ from the unit disc $\mathbb{D}$ and $\mathbb{D}^* := \{z \in \mathbb{C}, |z| > 1\}$ into $\mathscr{O}$ and $\mathscr{O}^*$ respectively. Both f and g extends to homeomorphisms between the closure of the domains and one can form the **conformal welding**

$$h := g^{-1} \circ f : \mathbb{S}^1 \to \mathbb{S}^1.$$

There exists homeomorphisms of $\mathbb{S}^1$ that are not conformal weldings, but any quasi-symmetric homeomorphism h is a conformal welding and the decomposition $h = g^{-1} \circ f$, where $f : \mathbb{D} \to \mathscr{O}$ and $g : \mathbb{D}^* \to \mathscr{O}^*$ are conformal, is unique. The Jordan curve $\gamma := f(\mathbb{S}^1) = g(\mathbb{S}^1)$ is called the **quasi-circle** associated to h. Reciprocally, the map that to a quasi-circle associates a quasi-symmetric homeomorphism of $\mathbb{S}^1$ is called the **fingerprint map** of γ.

The Universal Teichmüller space and Shapes in the plane

Using the fingerprint map, we can pull-back the Weil-Petersson metric of $\mathrm{QS}(\mathbb{S}^1)/PSU(1,1) = T(1)$ to the set of quasi-circles modulo translations and scaling. The geodesics between quasi-circles for the Weil-Petersson metric fournish interpolations between 2D-contours in the plane [SharonMumford2006].



The Siegel disc.

Let $\mathscr{V} = H^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R})/\mathbb{R}$ be the Hilbert space of real valued $H^{\frac{1}{2}}$ functions with mean-value zero. The real Hilbert inner product on $\mathscr{V}$ is given by

$$\langle u, v \rangle_{\mathscr{V}} = \sum_{n \in \mathbb{Z}} |n| u_n \overline{v_n}.$$

Endow the real Hilbert space $\mathscr{V}$ with the following complex structure (called the Hilbert transform)

$$\mathrm{J}\left(\sum_{n \neq 0} u_n e^{inx} \right) = i \sum_{n \neq 0} \operatorname{sgn}(n) u_n e^{inx}.$$

Now $\langle \cdot, \cdot \rangle_{\mathscr{V}}$ and J are compatible in the sense that J is orthogonal with respect to $\langle \cdot, \cdot \rangle_{\mathscr{V}}$. The associated (real) symplectic form is defined by

$$\Omega(u, v) = \langle u, J(v) \rangle_{\mathscr{V}} = \frac{1}{2\pi} \int_{\mathbb{S}^1} u(x) \partial_x v(x) dx = -i \sum_{n \in \mathbb{Z}} n u_n \overline{v_n}.$$

The Siegel disc.

Let us consider the **complexified Hilbert space** $\mathscr{H} := H^{1/2}(\mathbb{S}^1, \mathbb{C})/\mathbb{C}$ and the complex linear extensions of J and Ω still denoted by the same letters. Each element $u \in \mathscr{H}$ can be written as

$$u(x) = \sum_{n\in\mathbb{Z}} u_n e^{inx} \quad \text{with} \quad u_0 = 0 \quad \text{and} \quad \sum_{n\in\mathbb{Z}} |n||u_n|^2 < \infty.$$

The eigenspaces $\mathscr{H}_+$ and $\mathscr{H}_-$ of the operator J are

$$\mathscr{H}_+ = \left\{ u \in \mathscr{H} \,\middle|\, u(x) = \sum_{n=1}^{\infty} u_n e^{inx} \right\}$$
$$\mathscr{H}_- = \left\{ u \in \mathscr{H} \,\middle|\, u(x) = \sum_{n=-\infty}^{-1} u_n e^{inx} \right\},$$

and one has the Hilbert decomposition $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$ into the sum of closed orthogonal subspaces.

The Siegel disc

Definition (The Siegel disc)

The Siegel disc associated with $\mathscr{H}$ is defined by

$$\mathfrak{D}(\mathscr{H}) := \{Z \in L(\mathscr{H}_-, \mathscr{H}_+) \mid Z^T = Z, \forall\, u, v \in \mathscr{H}_- \quad \text{and} \quad I - Z\bar{Z} > 0\}.$$

The restricted Siegel disc associated with $\mathscr{H}$ is by definition

$$\mathfrak{D}_{\text{res}}(\mathscr{H}) := \{Z \in \mathfrak{D}(\mathscr{H}) \mid Z \in L^2(\mathscr{H}_-, \mathscr{H}_+)\}.$$

The restricted Siegel disc as an homogeneous space.

Symplectic group and its restricted version

Consider the symplectic group $\mathrm{Sp}(\mathscr{V}, \Omega)$ of bounded linear maps on $\mathscr{V}$ which preserve the symplectic form Ω

$$\mathrm{Sp}(\mathscr{V}, \Omega) = \{a \in \mathrm{GL}(\mathscr{V}) \mid \Omega(au, av) = \Omega(u, v), \text{ for all } u, v \in \mathscr{V}\}.$$

The restricted symplectic group $\mathrm{Sp}_{\mathrm{res}}(\mathscr{V}, \Omega)$ is

$$\mathrm{Sp}_{\mathrm{res}}(\mathscr{V}, \Omega) := \left\{ \left(\begin{array}{cc} g & h \\ \bar{h} & \bar{g} \end{array} \right) \in \mathrm{Sp}(\mathscr{V}, \Omega) \middle| h \in L^2(\mathscr{H}_-, \mathscr{H}_+) \right\}.$$

The restricted Siegel disc as an homogeneous space.

Theorem

The restricted symplectic group acts transitively on the restricted Siegel disc by

$$\mathrm{Sp}_{\mathrm{res}}(\mathscr{V}, \Omega) \times \mathfrak{D}_{\mathrm{res}}(\mathscr{H}) \longrightarrow \mathfrak{D}_{\mathrm{res}}(\mathscr{H}),$$

$$\left(\begin{pmatrix} g & h \\ \bar{h} & \bar{g} \end{pmatrix}, Z \right) \longmapsto (gZ + h)(\bar{h}Z + \bar{g})^{-1}.$$

The isotropy group of $0 \in \mathfrak{D}_{\mathrm{res}}(\mathscr{H})$ *is the unitary group* $\mathrm{U}(\mathscr{H}_+)$ *of* $\mathscr{H}_+$*, and the restricted Siegel disc is diffeomorphic as Hilbert manifold to the homogeneous space*

$$\mathrm{Sp}_{\mathrm{res}}(\mathscr{V}, \Omega)/\,\mathrm{U}(\mathscr{H}_+).$$

The Siegel disc as a generalization of the Poincaré disc

In the previous construction, replace $\mathscr{V}$ by $\mathbb{R}^2$ endowed with its natural symplectic structure. The corresponding Siegel disc is nothing but the open unit disc $\mathbb{D}$. The action of $\mathrm{Sp}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})$ is the standard action of $\mathrm{SU}(1,1)$ on $\mathbb{D}$ given by

$$z \in \mathbb{D} \longmapsto \frac{az+b}{\bar{b}z+\bar{a}} \in \mathbb{D}, \quad |a|^2 - |b|^2 = 1,$$

and the Hermitian metric obtained on $\mathbb{D}$ is given by the hyperbolic metric

$$h_{\mathfrak{D}}(z)(u,v) = \frac{1}{(1-|z|^2)^2} u\bar{v}.$$

Therefore, $\mathfrak{D}_{\mathrm{res}}(\mathscr{H})$ can be seen as an infinite-dimensional generalization of the Poincaré disc.

Period mapping

Theorem (Theorem 3.1 in NagSullivan1995)

For φ a orientation preserving homeomorphism and any $f \in \mathscr{V} = H^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R})/\mathbb{R}$, define

$$V_\varphi f = f \circ \varphi - \frac{1}{2\pi} \int_{\mathbb{S}^1} f \circ \varphi.$$

Then V_φ maps $\mathscr{V}$ into itself iff φ is quasisymmetric.

Theorem (Proposition 4.1 in NagSullivan1995)

The group $\mathrm{QS}(\mathbb{S}^1)$ of quasisymmetric homeomorphisms of the circle acts on the right by symplectomorphisms on $\mathscr{V}$ by

$$V_\varphi f = f \circ \varphi - \frac{1}{2\pi} \int_{\mathbb{S}^1} f \circ \varphi,$$

$\varphi \in \mathrm{QS}(\mathbb{S}^1)$, $f \in \mathscr{V}$.

Period mapping

Theorem (Theorem 7.1 in NagSullivan1995)

This action defines a map $\Pi : \mathrm{QS}(\mathbb{S}^1) \to \mathrm{Sp}(\mathscr{V}, \Omega)$. *The operator* $\Pi(\varphi)$ *preserves the subspaces* $\mathscr{H}_+$ *and* $\mathscr{H}_-$ *iff* φ *belongs to* $\mathrm{PSU}(1,1)$. *The resulting map is an injective equivariant holomorphic immersion*

$$\Pi \; : T(1) = \mathrm{QS}(\mathbb{S}^1)/\,\mathrm{PSU}(1,1) \to \mathrm{Sp}(\mathscr{V}, \Omega)/\,\mathrm{U}(H_+) \simeq \mathfrak{D}(\mathscr{H})$$

called the ***period mapping*** *of* $T(1)$.

The Hilbert version of the period mapping is given by the following

Theorem (TaTe2004)

For $[\mu] \in T(1)$, $\Pi([\mu])$ *belongs to the restricted Siegel disc if and only if* $[\mu] \in T_0(1)$. *Moreover the pull-back of the natural Kähler metric on* $\mathfrak{D}_{\mathrm{res}}(\mathscr{H})$ *coincides, up to a constant factor, with the Weil-Petersson metric on* $T_0(1)$.

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