LECTURE 3: Secondary Fan + Regular Triangulations, Wall crossing and circuit conditions.

Last time: We have Gale dual cones

$C_\beta \leftarrow$ effective cone, degrees of divisors β_i $\qquad C_v \leftarrow$ rays.

The secondary fan partitions C_β into GKZ chambers Γ_{Σ, I_ϕ}.

Main Theorem today: The chambers of Σ_{GKZ} correspond 1:1 to regular triangulations of $\{v_i\}$

Best case is $\{v\}$ are geometric = nonzero + distinct.

Question to resolve/understand: moving between chamber = crossing a wall, what is the meaning?

~~Players~~ Dramatis Personae

Nef $\subseteq$ Mov $\subseteq$ Effective ~~cu~~ cones

Divisorial, Flipping, Fibering $\leftarrow$ types of maps.

From §14: $G \subseteq (\mathbb{C}^*)^r$, $\bar{\beta} = \{\beta_1, \dots, \beta_r\} \in \hat{G}_{\mathbb{R}}$ and

$\bar{v} = \{v_1, \dots, v_r\} \subseteq N_{\mathbb{R}}$, cones C_β, C_v. $|\Sigma_{GKZ}| = C_\beta$,

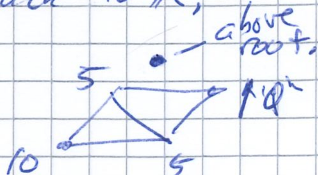
and if $\chi \in \operatorname{relint}(\Gamma_{\Sigma, I_\phi})$ then $\mathbb{C}^r /\!/_{\chi} G \cong X_\Sigma$

(n.b. the last Σ corresponds to the Σ in Γ_{Σ, I_ϕ}).

Definition: For a point configuration $X \subseteq \mathbb{R}^d$, assign a weight to each point and take the convex hull. Then project the "lower faces" of the triangulation back to $\mathbb{R}^d$.

Example: Caved in roof of house (corners all equal)

vs

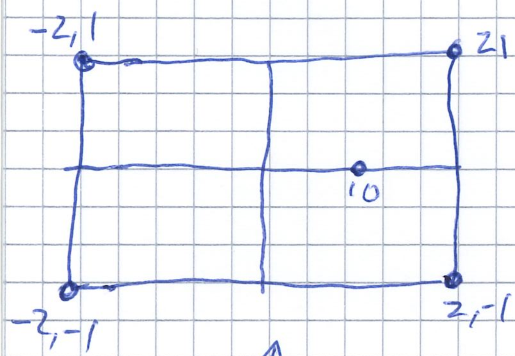


tent w/ crease, tent pole punched thru.

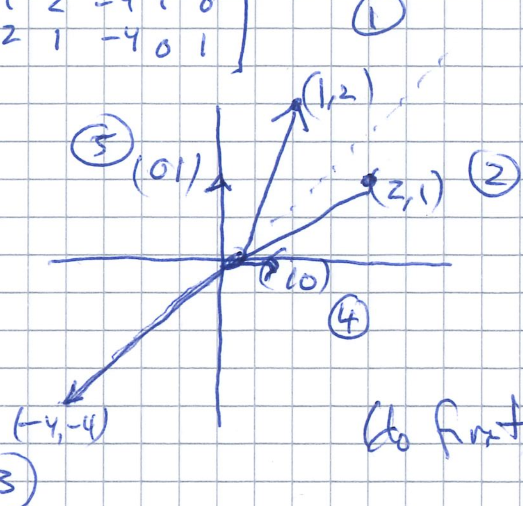
Example 15.2.2 $0 \to \mathbb{Z}^3 \xrightarrow{B} \mathbb{Z}^5 \xrightarrow{A} \mathbb{Z}^2 \to 0$

$$B = \begin{bmatrix} 2 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 0 & 1 \\ -2 & 1 & 1 \\ -2 & -1 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & -4 & 1 & 0 \\ 2 & 1 & -4 & 0 & 1 \end{bmatrix}$$

((can add 0 0 -1 if want))



(do second)



(do first)

Def: $J \subseteq [r]$ is a β-basis if $|J| = \dim G$ and the β_i, $i \in J$ are a basis of $\widehat{G}_\mathbb{R}$. In this case, cone β_J is full dim simplicial cone.

Thm Γ_{Σ, I_ϕ} chamber of Σ_{GKZ}

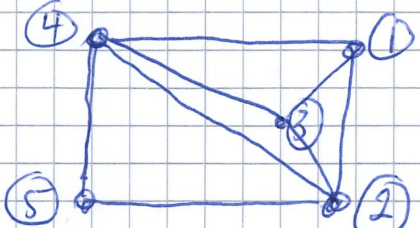
- $\sigma \in \Sigma_{max}$, $J_\sigma = \{ i \mid v_i \notin \sigma \text{ or } i \in I_\phi \}$ is β-basis
- J a β-basis, $\Gamma_{\Sigma, I_\phi} \subseteq \text{Cone } \beta_J \Leftrightarrow J = J_\sigma$, some $\sigma \in \Sigma_{max}$.
- $\Gamma_{\Sigma, I_\phi} = \bigcap_{\sigma \in \Sigma_{max}} \text{Cone } \beta_{J_\sigma}$.

Algorithm to assign reg Δ of v_i to chamber Γ

- Find all the J bases $\Leftrightarrow$ cones containg Γ
- Write down indices of cones.
- Take complements of index sets; these are the Δ's

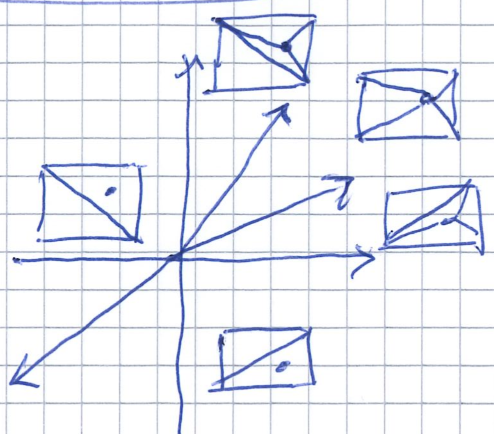
Example, continued: ~~For~~ The cone $\{v_1, v_5\} \subseteq$

	complement
15	234
25	134
45	123
13	245



triangulation ☺

Whole picture:



Recall: D Nef $\Leftrightarrow D \cdot C \geq 0 \;\; \forall$ irr C.

In toric case, we have $0 \to M \to CDiv(X) \to Pic(X) \to 0$

$CDiv(X)$ = Piecewise Linear, Cont. Support Fns SF; $Pic(X) \supseteq Nef(X)$ CSF strictly convex

Besides the Nef cone,

We also have the moving cone

Def: $Mov(X) = \{ D \in Eff(X) \mid$ no fixed component $(D'$ fixed if $D' \leq D_0 \;\forall D_0 \sim D) \}$

Thm If $\vec{v} = \{v_1 \ldots v_k\}$ is geometric, primitive (v 1st lattice pt)

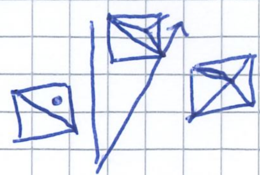
then $Mov(GKZ) = \bigcup_{I_\phi = \phi} \Gamma_{\Sigma, I_\phi}$, i.e.

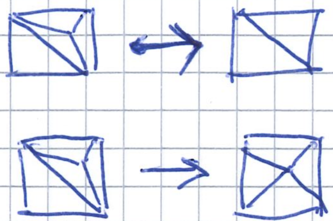
For $Pic(X_\Sigma)_{\mathbb{R}} \simeq \hat{G}_{\mathbb{R}}$ we have

$$Nef(X_\Sigma) \subseteq Mov(X_\Sigma) \subseteq Eff(X_\Sigma)$$

§ Wall Crossing

15.3 In example 15.2.2 we saw 2 kinds of behavior in adjacent chambers:





star subdivision
blow down a divisor. (modify codim 1)

crush 1 curve down
blow up a point (modify codim 2)

Theorem 15.1.3 $\Gamma_{\Sigma, I_\emptyset} \simeq \mathrm{Nef}(X_\Sigma) \times \mathbb{R}^{I_\emptyset}_{\geq 0}$ and the **facets** are one of types

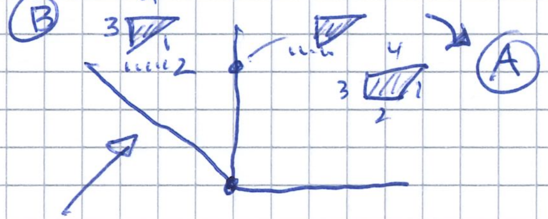
A) $\Gamma_{\Sigma', I_\emptyset} \simeq$ facet of $\mathrm{Nef}(X_\Sigma) \times \mathbb{R}^{I_\emptyset}$, $\underline{\underline{\Sigma \text{ refines } \Sigma'}}$

or

B) $\Gamma_{\Sigma, I_\emptyset \setminus i} \simeq \mathrm{Nef}(X_\Sigma) \times \mathbb{R}^{I_\emptyset \setminus i}$, $i \in I_\emptyset$

Polytope Pic

Example $\mathcal{H}_2$

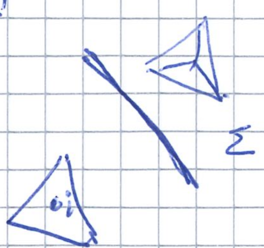


In type B, fans are same, but $I_\emptyset$ changes, going from $\{2\}$ to empty.

In type A, fan changes.

Triangulation pix

Thm: $\bar{\nu}$ geometric and $\Gamma_{\Sigma, I_\emptyset}$ chamber w/ $I_\emptyset \neq \emptyset$, $i \in I_\emptyset$.



- $\Gamma_{\Sigma, I_\emptyset \setminus i}$ is a facet of $\Gamma_{\Sigma, I_\emptyset}$
- Other side of wall is $\Gamma_{\Sigma', I_\emptyset \setminus i}$, and Σ' is a star subdivision at i
- $X_{\Sigma'} \to X_\Sigma$ has exceptional locus codim 1

What happens when I_ϕ all same? Here the fan changes (Case A)

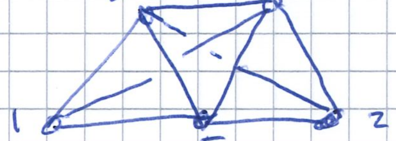
$$\Gamma_{\Sigma, I_\phi} \succ \Gamma_{\Sigma_0, I_\phi} \prec \Gamma_{\Sigma', I_0} \quad \text{then } \Sigma_0 \text{ is}$$

not simplicial, $\Sigma(1) = \Sigma_0(1) = \Sigma'(1)$ and exceptional ~~loci of~~ loci of $X_\Sigma \to X_{\Sigma_0} \leftarrow X_{\Sigma'}$ are codim ≥ 2

Warning: $\star$ is not always as simple as 

Example 15.3.9

On board?



$$\begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & -2 & 1 \end{pmatrix} \quad \text{all in } z_4 = 1.$$

Def Let $v_1 \cdots v_{n+1}$ span $N = \mathbb{Z}^n$, write prim reln $\sum b_i v_i = 0$

$J_- = \{ i \mid b_i < 0 \}$, $J_0 = \{ i \mid b_i = 0 \}$, $J_+ = \{ i \mid b_i > 0 \}$.

$J_- \cup J_+$ is a circuit, $\Sigma_- = \{ \theta \mid \theta \subseteq \text{cone}\{ v_1 \ldots \hat{v_i} \ldots v_{n+1} \}, i \in J_+ \}$

$\Sigma_+ = \quad$ " $\quad , i \in J_-$

So $-3v_1 - 3v_2 + 2v_3 + 2v_4 + 2v_5 = 0 \Rightarrow J_- = \{1, 2\}$, $J_+ = \{3, 4, 5\}$

Thm If $\text{cone}\{v_1 \ldots v_{n+1}\} = \theta_0$ is strongly convex + full dim in $N = \mathbb{Z}^n$

Then
- If $J_- = \{i\}$, the origin in the secondary fan is a divisorial wall, $\Sigma_+ = \theta_0 + \text{faces}$, $\Sigma_- = $ star subdivision at v_i
- If J_+, J_- both have ≥ 2 elements, the origin is a flipping wall, with:

ϕ_+, ϕ_- birational,

[Definition of V] $\left\{ \begin{array}{l} \text{exceptional locus} \\ V(J_+), V(J_-) \text{ respectively} \end{array} \right\}$

$$\begin{tikzcd}
V(\theta_{J_-}) \arrow[r, hook] \arrow[ddr] & X_{\Sigma_-} \arrow[dr, "\phi_-"'] & & X_{\Sigma_+} \arrow[dl, "\phi_+"] & V(\theta_{J_+}) \arrow[l, hook'] \arrow[ddl] \\
& & U_{\theta_0} & & \\
& & V(J_- \cup J_+) \arrow[u, hook] & &
\end{tikzcd}$$

15.3.13 **Pièce de Résistance: "Flipping Thm"** (generalize last result)

If ν is geometric and $\Gamma_{\Sigma_0, I_\phi}$ is a flipping wall between Γ_{Σ, I_ϕ} "$\leq$" Γ_{Σ', I_ϕ}. Then $\exists$ circuit

X_Σ $\quad$ $X_{\Sigma'}$

$$\sum_{i \in J_-} b_i v_i + \sum_{i \in J_+} b_i v_i = 0, \quad J_+ \cap J_- = \phi,$$

$$J_+ \cup J_- \subseteq [r] \setminus I_\phi, \quad b_i < 0 \ J_-, \ b_i > 0 \ J_+$$

- $\theta_{J_-} \subseteq \Sigma$, $\theta_{J_+} \subseteq \Sigma'$, $\theta_{J_- \cup J_+} \subseteq \Sigma_0$
- $X_{\Sigma_0} \setminus V(\theta_{J_- \cup J_+})$ is the simplicial locus of X_{Σ_0} (blow up non simp. stuff)
- $\exists$ comm diagram exactly as on prev page, $\operatorname{codim} V(\theta_{J_-}) = |J_-| \geq 2 \leq |J_+| = \operatorname{codim} V(\theta_{J_+})$ (So, small contractions)

Def: In situation above, $X \xrightarrow{\phi_- \ \phi_+^{-1}} X'$ is [birational, equivariant wrt torus action, iso in codim 1] (*)

AND gets a name: "elementary flip"

Cor 15.3.14 A map satisfying (*) of semiproj torics (= toric var of full dim polyhedron)

Is a composition of elementary flips and a toric isomorphism.

☺