Conformal Bootstrap in Liouville theory

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Workshop ESI "Higher Structures Emerging from Renormalisation"

Context

- Euclidean Quantum Field Theories (QFT) model **statistical physics**.
- Physical content is encoded in expectation values of observables (fields), which are called **correlation functions**.
- At **critical temperature**, such systems feature further conformal symmetries and are thus dubbed Conformal Field Theories (CFT).
- Belavin-Polyakov-Zamolodchikov (**Conformal Bootstrap**, 1984) observed that this extra symmetry constrains the system strongly and used this idea to classify CFTs. They gave explicit expressions for the correlation functions of several CFTs in 2D (minimal models, e.g. critical Ising model).
- In 3D, Conformal Bootstrap has recently led to spectacular numerical predictions (e.g. 3D Ising model) by Rychkov and collaborators.

Two approaches of QFT/CFT

1. **Axiomatic:**
   - Wightman's axioms, Osterwalder-Schrader, Segal's axioms
   - **Conformal Bootstrap** (BPZ): recursive construction of correlation functions recursively via Operator Product Expansion.
     $\Rightarrow$ in maths: **Vertex Operator Algebras** (Borcherds, Frenkel,..) in representation theory.
2. **Constructive:**
   - Find examples satisfying the axioms (Ising, $P(\phi)^n_d$,...)
   - **Path integral** (measure on the set of configurations of the system), renormalization group

     $$F \mapsto \int F(\phi) e^{-S(\phi)} D\phi$$

     Hard to extract quantitative information and to relate to axioms.
   - Perturbative approach, approximative.

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This talk: From (2) to (1) in the context of a non trivial CFT: **Liouville theory**.

Axiomatic of 2D CFTs on the Riemann sphere $\mathbb{C} \cup \{\infty\}$: conformal covariance

- If ψ Möbius ($\psi(z) = \frac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{C}$, $ad - bc = 1$), n point correlation functions are related by

$$\langle \prod_{i=1}^{n} V_{\alpha_i}(\psi(z_i)) \rangle = \Big(\prod_{i=1}^{n} |\psi'(z_i)|^{-2\Delta_{\alpha_i}} \Big) \langle \prod_{i=1}^{n} V_{\alpha_i}(z_i) \rangle$$

  where V_α are **primary fields** and $\Delta_\alpha \in \mathbb{R}$ are **conformal weights**.

- Hence, 3 point correlation functions are determined up to constants $C(\alpha_1, \alpha_2, \alpha_3)$ called **structure constants**:

$$\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\alpha_2}(z_3) \rangle$$
$$= C(\alpha_1, \alpha_2, \alpha_3) |z_1 - z_2|^{2(\Delta_{\alpha_3} - \Delta_{\alpha_1} - \Delta_{\alpha_2})} |z_1 - z_3|^{2(\Delta_{\alpha_2} - \Delta_{\alpha_1} - \Delta_{\alpha_3})} |z_2 - z_3|^{2(\Delta_{\alpha_1} - \Delta_{\alpha_2} - \Delta_{\alpha_3})}$$

- Liouville theory: $C(\alpha_1, \alpha_2, \alpha_3)$ satisfies **DOZZ formula** $C^{\mathrm{DOZZ}}_{\gamma,\mu}(\alpha_1, \alpha_2, \alpha_3)$.

Axiomatic of 2D CFTs: OPE and bootstrap

- **Operator Product Expansion** (axiom):

$$V_{\alpha_1}(z_1)V_{\alpha_2}(z_2) = \sum_{\alpha\in\mathcal{S}} C^{\alpha}_{\alpha_1,\alpha_2}(z_1,z_2,\partial_{z_1},\partial_{\bar{z}_1})V_\alpha(z_1), \qquad |z_1-z_2|\to 0$$

  holds when inserted in correlation functions

$$\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{\alpha_3}(z_3)\dots\rangle = \sum_{\alpha\in\mathcal{S}} C^{\alpha}_{\alpha_1,\alpha_2}(z_1,z_2,\partial_{z_1},\partial_{\bar{z}_1})\langle V_\alpha(z_1)V_{\alpha_3}(z_3)\dots\rangle$$

  where for $z_1=0$
  - $C^{\alpha}_{\alpha_1,\alpha_2}(0,z,\partial_{z_1},\partial_{\bar{z}_1}) = C(\alpha_1,\alpha_2,\bar{\alpha})|z|^{2\Delta_\alpha-2\Delta_{\alpha_1}-2\Delta_{\alpha_2}}(1+\frac{\Delta_\alpha+\Delta_{\alpha_2}-\Delta_{\alpha_1}}{2}(z\frac{\partial}{\partial z_1}+\bar{z}\frac{\partial}{\partial\bar{z}_1})_{|z_1=0}+\cdots)$ are determined by structure constants.
  - the sum runs over some subset $\mathcal{S}$ of indices, called the **spectrum**.

- Iterating the OPE $\Rightarrow$ Solving a CFT boils down to determining the **spectrum** and **structure constants**

- Considering two possible OPEs yields a quadratic relation for structure constants, which may be solved in some cases (cf Ising 3D).

- Liouville theory: $\mathcal{S}=Q+i\mathbb{R}_+$ and $\sum=\int$.

Plan of the talk

Probabilistic construction of Liouville CFT

Main result

Strategy of proof

Liouville CFT

- Consider the metric
$$g(z) = \frac{1}{|z|_+^4}, \quad |z|_+ = \max(1, |z|)$$
- LCFT is formally a measure on some space Σ of maps $\phi : \mathbb{C} \to \mathbb{R}$ defined by
$$F \mapsto \int_\Sigma F(\phi) e^{-S_L(\phi)} D\phi$$
wit $D\phi$ putative Lebesgue measure on Σ and
$$S_L(\phi) = \frac{1}{4\pi} \int_{\mathbb{C}} \left(|\nabla \phi(z)|^2 + 2Q\phi(z)g(z) + 4\pi\mu e^{\gamma\phi(z)} g(z) \right) dz, \quad Q = \frac{\gamma}{2} + \frac{2}{\gamma}$$

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- David-Kupiainen-Rhodes-Vargas' construction (2014) relies on 2 ingredients:
  - the squared gradient term gives rise to a Gaussian measure called **Gaussian free field** (GFF).
  - the term $\exp\left(-\mu \int_{\mathbb{C}} e^{\gamma\phi(z)} g(z)\, dz \right)$ is seen as a Radon-Nykodim derivative w.r.t. the GFF measure. Exponential of the GFF $\Rightarrow$ Gaussian multiplicative Chaos theory (GMC)

Key ingredient 1: Gaussian Free Field

- let X be the GFF with vanishing g-average, namely a Gaussian centered distribution (in the sense of Schwartz)

$$h \text{ test function on } \mathbb{C} \mapsto X_h$$

with covariance

$$\mathbb{E}\left[X_h X_{h'}\right] = \iint_{\mathbb{C}^2} h(z)h'(z')G(z,z')\, g(z)g(z')dzdz'$$

where G the Green function of Laplacian on $\mathbb{C}$ w.r.t. metric g

$$G(z,z') = \ln \frac{1}{|z-z'|} - \frac{1}{4}\ln g(z) - \frac{1}{4}\ln g(z')$$

Formally

$$X_1 = 0, \quad \mathbb{E}[X(z)X(z')] = G(z,z') = \ln \frac{|z|_+|z'|_+}{|z-z'|}$$

- **Definition of the Gaussian measure:** for positive continuous functionals F on $H^{-\epsilon}(\mathbb{C})$

$$\int_\Sigma F(\phi)e^{-\frac{1}{4\pi}\int_{\mathbb{C}}|\nabla\phi|^2\,dz} D\phi := \text{Constant} \times \int_{\mathbb{R}} \mathbb{E}[F(c+X(\cdot))]\,dc$$

Key ingredient 2: Gaussian multiplicative chaos

- Goal: construct the green term in

$$e^{-S_L(\phi)}D\phi = e^{-\frac{1}{4\pi}\int_{\mathbb{C}}(2Q\phi g(z)+4\pi\mu e^{\gamma\phi(z)}g(z))\,dz}e^{-\frac{1}{4\pi}\int_{\mathbb{C}}|\nabla\phi(z)|^2\,dz}D\phi =$$

i.e. make sense of the term (where X GFF)

$$\int_{\mathbb{C}} e^{\gamma X(z)}g(z)\,dz.$$

- **Gaussian Multiplicative Chaos** (GMC, Kahane 85)

$$M_\gamma(dz) := \lim_{\epsilon\to 0} e^{\gamma X_\epsilon(z)-\frac{\gamma^2}{2}\mathbb{E}[X_\epsilon(z)^2]}g(z)\,dz$$

with X_ϵ a regularization of the field X at scale $\epsilon > 0$

$$\mathbb{E}[X_\epsilon(z)X_\epsilon(z')] \underset{z\to z'}{\sim} \ln\frac{1}{|z-z'|+\epsilon} + O(1)$$

Non trivial iff $\gamma \in (0,2)$, in which case total mass $M_\gamma(\mathbb{C})$ is finite almost surely.

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Mathematical definition

Definition of Liouville functional measure for F bounded functional on $H^{-\epsilon}(\mathbb{C})$

$$F \mapsto \langle F \rangle_{\gamma,\mu} := \int_{\mathbb{R}} \mathbb{E}\Big[F(c + X(\cdot))\, e^{-\mu e^{\gamma c} M_\gamma(\mathbb{C})}\Big] e^{-2Qc}\, dc$$

with X the GFF and M_γ the GMC random measure

$$\mathbb{E}[X(z)X(z')] = G(z,z'), \quad \text{and} \quad M_\gamma(\mathbb{C}) = \int_{\mathbb{C}} e^{\gamma X(z) - \frac{\gamma^2}{2}\mathbb{E}[X(z)^2]} g(z)\, dz.$$

Liouville field:

$$\Phi(z) := c + X(z) - 2Q \ln g(z).$$

This is the observable we will be interested in.

Correlation functions

- For $z \in \mathbb{C}$ and $\alpha \in \mathbb{R}$, *vertex operators* are

$$V_\alpha(z) := e^{\alpha\Phi(z)}$$

- Correlation functions in Liouville CFT are "Laplace cumulants" of the measure:

$$\langle \prod_{k=1}^{n} V_{\alpha_k}(z_k) \rangle_{\gamma,\mu}$$

for arbitrary $n \in \mathbb{N}$, $z_1, \ldots, z_n \in \mathbb{C}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$
Remark: a proper definition requires a regularization as Φ is a distribution and cannot be evaluated pointwise.

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Theorem (DKRV, 2014)

For $\gamma \in (0,2)$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, one can define probabilistically the correlation functions $\langle \prod_{k=1}^{n} V_{\alpha_k}(z_k) \rangle_{\gamma,\mu}$. They are non trivial iff:

$$\forall i,\ \alpha_i < Q \quad \text{and} \quad \sum_{i=1}^{n} \alpha_i > 2Q \qquad \text{(Seiberg bounds)}$$

In particular, existence implies $n \geq 3$. Conformal covariance holds with weights $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$.

Correlation functions

- **LCFT correlations are moments of GMC** (DKRV 2014): for $n \geq 3$

$$\langle \prod_{k=1}^{n} V_{\alpha_k}(z_k) \rangle_{\gamma,\mu} = \gamma^{-1} \mu^{-s} \Gamma(s) \Big(\prod_{1 \leq j < k \leq n} \frac{1}{|z_j - z_k|^{\alpha_j \alpha_k}} \Big) \mathbb{E}[Z^{-s}]$$

where $s = \frac{\sum_{i=1}^{n} \alpha_i - 2Q}{\gamma}$ and

$$Z := \int_{\mathbb{C}} \Big(\prod_{i=1}^{n} \frac{|z|_+^{\gamma \alpha_i}}{|z - z_i|^{\gamma \alpha_i}} \Big) M_\gamma(dz)$$

- For $n = 3$, do they satisfy **DOZZ formula**?

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- For $n = 3$, do they satisfy **DOZZ formula**? Yes!

The DOZZ formula

Theorem (Kupiainen, Rhodes, V., 2017)

Assume $\forall i,\ \alpha_i < Q$ *and* $\sum_{i=1}^{3} \alpha_i > 2Q$. *Then*

$$\langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty) \rangle_{\gamma,\mu} = \frac{1}{2} C_{\gamma,\mu}^{DOZZ}(\alpha_1,\alpha_2,\alpha_3)$$

with

$$C_{\gamma,\mu}^{DOZZ}(\alpha_1,\alpha_2,\alpha_3) := \left(\pi\,\mu\,\frac{\Gamma(\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})}\,(\frac{\gamma}{2})^{2-\gamma^2/2}\right)^{\frac{2Q-\bar{\alpha}}{\gamma}} \times \frac{\Upsilon'_{\frac{\gamma}{2}}(0)\Upsilon_{\frac{\gamma}{2}}(\alpha_1)\Upsilon_{\frac{\gamma}{2}}(\alpha_2)\Upsilon_{\frac{\gamma}{2}}(\alpha_3)}{\Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}-2Q}{2})\Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}-2\alpha_1}{2})\Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}-2\alpha_2}{2})\Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}-2\alpha_3}{2})}$$

with $\bar{\alpha} = \alpha_1 + \alpha_2 + \alpha_3$ and the function $\Upsilon_{\frac{\gamma}{2}}$ defined as analytic continuation of the following integral defined for $0 < \Re(z) < \Re(Q)$

$$\ln \Upsilon_{\frac{\gamma}{2}}(z) = \int_0^{\infty} \left(\left(\frac{Q}{2} - z\right)^2 e^{-t} - \frac{(\sinh((\frac{Q}{2}-z)\frac{t}{2}))^2}{\sinh(\frac{t\gamma}{4})\sinh(\frac{t}{\gamma})} \right) \frac{dt}{t}$$

Conformal bootstrap

Theorem (Guillarmou, Kupiainen, Rhodes, V., 2020)

For $\gamma \in (0,2)$ and $\alpha_i < Q$ ($i = 1, \dots, 4$) satisfying

$$\alpha_1 + \alpha_2 > Q \quad \text{and} \quad \alpha_3 + \alpha_4 > Q$$

the following identity holds for $|z| < 1$

$$\langle V_{\alpha_1}(0) V_{\alpha_2}(z) V_{\alpha_3}(1) V_{\alpha_4}(\infty) \rangle_{\gamma,\mu} \\ = \frac{1}{8\pi} \int_{\mathbb{R}_+} C^{\mathrm{DOZZ}}_{\gamma,\mu}(\alpha_1, \alpha_2, Q - iP) C^{\mathrm{DOZZ}}_{\gamma,\mu}(\alpha_3, \alpha_4, Q + iP) |z|^{2(\Delta_{Q+iP} - \Delta_{\alpha_1} - \Delta_{\alpha_2})} |\mathcal{F}_P(z)|^2 dP \tag{1}$$

where $\mathcal{F}_P$ are the holomorphic ***conformal blocks*** *given by $\mathcal{F}_P(z) = \sum_{n=0}^\infty \beta_n z^n$ where the β_n have (complicated) combinatorial expressions in terms of Young diagrams.*

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Remarks:

- This is a reformulation of the OPE at the level of 4 point correlation functions. Similar expressions for higher ($n \geq 5$) order correlation functions hold.
- Equality between $\mathcal{F}_P(z)$ and Nekrasov partition function: Maulik-Okounkov (2012), Schiffmann-Vasserot (2012). In these works, $\mathcal{F}_P(z)$ is a formal power series. **Convergence is an output of our result.**
- On torus, Ghosal-Remy-Sun-Sun (2020) have obtained a beautiful probabilistic representation of the conformal blocks and proven their convergence.

Conformal blocks

The conformal blocks are holomorphic

$$\mathcal{F}_{(\Delta_{\alpha_j})_i,P}(z) = \sum_{n=0}^{\infty} \beta_n z^n$$

where

- $\beta_n = \sum_{|\nu|,|\bar{\nu}|=n} v(\Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{Q+iP}, \nu) Q^{-1}_{\Delta_{Q+iP}}(\nu, \bar{\nu}) v(\Delta_{\alpha_4}, \Delta_{\alpha_3}, \Delta_{Q+iP}, \bar{\nu})$
- $\nu = (k_1 \geq k_2 \geq \dots)$ Young diagram of size $|\nu| = \sum_i k_i$
- $v(\Delta, \Delta', \Delta'', \nu) = \prod_j (k_j \Delta' + \Delta'' - \Delta + \sum_{u<j} k_u)$
- $Q^{-1}_{\Delta_{Q+iP}}(\nu, \bar{\nu})$ Shapovalov matrix with central charge $1 + 6Q^2$, and $Q^{-1}_{\Delta_{Q+iP}}(\nu, \bar{\nu})$ its inverse.

Strategy of proof: the Hilbert space $\mathcal{H}$

- **Introduce Liouville Hilbert space $\mathcal{H}$:** set

$$\Omega := (\mathbb{R}^2)^{\mathbb{N}^*}, \text{with proba} \quad \mathbb{P} := \prod_{n\geq 1} \frac{1}{2\pi} e^{-\frac{x_n^2+y_n^2}{2}} dx_n dy_n.$$

Then

$$\mathcal{H} = L^2(\mathbb{R}\times\Omega; dc \otimes \mathbb{P}).$$

- **Circular GFF:** set

$$\varphi(\theta) = \sum_{n>0} \frac{1}{\sqrt{n}} \Big(x_n \cos(n\theta) - y_n \sin(n\theta) \Big)$$

Under $\mathbb{P}$, φ has the law of a Gaussian Free Field on the unit circle

$$\mathbb{E}[\varphi(\theta)\varphi(\theta')] = \ln \frac{1}{|e^{i\theta} - e^{i\theta'}|}.$$

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Strategy of proof: the GFF decomposition

- Key decomposition
$$X = P\varphi + X_{\mathbb{D}} + X_{\mathbb{D}^c}$$
where $P\varphi$ harmonic extension of φ and $X_{\mathbb{D}}, X_{\mathbb{D}^c}$ two independent Dirichlet GFFs on $\mathbb{D}$ and $\mathbb{D}^c$ (with covariance $G_{\mathbb{D}}$,$G_{\mathbb{D}^c}$).
- Key invariance:
$$X_{\mathbb{D}^c}\Big(\frac{1}{\bar{z}}\Big) \overset{(Law)}{=} X_{\mathbb{D}}(z)$$
- Using above, one gets
$$\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_3}(\infty)\rangle_{\gamma,\mu} = \langle U(V_{\alpha_1}(0)V_{\alpha_2}(z)), U(V_{\alpha_4}(0)V_{\alpha_3}(1))\rangle_{\mathcal{H}}$$
where
$$U(V_\alpha(z_1)V_\beta(z_2))(c,\varphi) = e^{(\alpha+\beta-Q)c}e^{\alpha P\varphi(z_1)+\beta P\varphi(z_2)+\alpha\beta G_{\mathbb{D}}(z_1,z_2)}(1-|z_1|^2)^{\frac{\alpha^2}{2}}(1-|z_2|^2)^{\frac{\beta^2}{2}}$$
$$\times\, \mathbb{E}_\varphi\Big[\exp\Big(-\mu e^{\gamma c}\int_{\mathbb{D}} e^{\gamma\alpha G_{\mathbb{D}}(z,z_1)+\gamma\beta G_{\mathbb{D}}(z,z_2)}M_\gamma(dz)\Big)\Big]$$

Strategy of proof: the general scheme

- Step 1: write 4 point function as a scalar product on Hilbert space $\mathcal{H}$

$$\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_3}(\infty)\rangle_{\gamma,\mu} = \langle U(V_{\alpha_1}(0)V_{\alpha_2}(z)), U(V_{\alpha_4}(0)V_{\alpha_3}(1))\rangle_{\mathcal{H}}$$

- Step 2: Spectral resolution where "$\langle \Psi_{Q+iP,\nu,\tilde{\nu}}, \Psi_{Q+iP',\nu',\tilde{\nu}'}\rangle_{\mathcal{H}} = 2\pi\delta_{P=P'} Q_{\Delta_{Q+iP}}(\nu,\nu')Q_{\Delta_{Q+iP}}(\tilde{\nu},\tilde{\nu}')$"

$$\langle \cdot,\cdot\rangle_{\mathcal{H}} = \frac{1}{2\pi}\sum_{\nu,\tilde{\nu},\nu',\tilde{\nu}'}\int_0^\infty \langle U(V_{\alpha_1}(0)V_{\alpha_2}(z)), \Psi_{Q+iP,\nu,\tilde{\nu}}\rangle_{\mathcal{H}}\langle \Psi_{Q+iP,\nu',\tilde{\nu}'}, U(V_{\alpha_3}(1)V_{\alpha_3}(0))\rangle_{\mathcal{H}}$$
$$\times Q_{\Delta_{Q+iP}}(\nu,\nu')^{-1}Q^{-1}_{\Delta_{Q+iP}}(\tilde{\nu},\tilde{\nu}')\,dP$$

- Step 3: Conformal Ward

$$\langle \Psi_{Q+iP,\nu,\tilde{\nu}}, U(V_{\alpha_1}(0)V_{\alpha_2}(z))\rangle_{\mathcal{H}} = v(\Delta_{\alpha_1},\Delta_{\alpha_2},\Delta_{Q+iP},\tilde{\nu})v(\Delta_{\alpha_1},\Delta_{\alpha_2},\Delta_{Q+iP},\nu)$$
$$\times \frac{1}{2}C^{\mathrm{DOZZ}}_{\gamma,\mu}(\alpha_1,\alpha_2,Q+iP)\bar{z}^{|\nu|}z^{|\tilde{\nu}|}|z|^{2(\Delta_{Q+iP}-\Delta_{\alpha_1}-\Delta_{\alpha_2})}$$

Strategy of proof: Hamiltonian of Liouville CFT

- On the Hilbert space $L^2(\mathbb{R}\times\Omega, dc\otimes\mathbb{P})$, the Hamiltonian **H** of Liouville theory is the Schrödinger type operator

$$\mathbf{H} = -\frac{1}{2}\partial_c^2 + \frac{Q^2}{2} + \mathbf{P} + \mu e^{\gamma c} V$$

with **P** the infinite harmonic oscillator and V a GMC type potential ($\gamma^2 < 2$)

$$\mathbf{P} := \sum_{n=1}^{\infty} n\big(x_n\partial_{x_n} - \partial_{x_n}^2 + y_n\partial_{y_n} - \partial_{y_n}^2\big), \quad V(\varphi) := \int_0^{2\pi} e^{\gamma\varphi(\theta) - \frac{\gamma^2}{2}\mathbb{E}[\varphi^2(\theta)]}\, d\theta$$

Write $\mathbf{H}_0$ for the free Hamiltonian $\mathbf{H}_{\mu=0}$.

- Probabilistic representation for

$$e^{-t\mathbf{H}_0} f(c,\varphi) = e^{-\frac{Q^2}{2}t}\mathbb{E}_\varphi\left[f\left(c + B_t, X\circ e^{-t} - B_t\right)\right]$$

where $B_t = \frac{1}{2\pi}\int_0^{2\pi} X(e^{-t}e^{i\theta})d\theta$ is a Brownian motion.

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$$\mathbf{P} := \sum_{n=1}^{\infty} n\left(x_n\partial_{x_n} - \partial_{x_n}^2 + y_n\partial_{y_n} - \partial_{y_n}^2\right), \quad V(\varphi) := \int_0^{2\pi} e^{\gamma\varphi(\theta) - \frac{\gamma^2}{2}\mathbb{E}[\varphi^2(\theta)]}\, d\theta$$

Write $\mathbf{H}_0$ for the free Hamiltonian $\mathbf{H}_{\mu=0}$.

- Probabilistic representation for $\mathbf{H}_0$

$$e^{-t\mathbf{H}_0} f(c,\varphi) = e^{-\frac{Q^2}{2}t}\mathbb{E}_\varphi\left[f\left(c + B_t, X\circ e^{-t} - B_t\right)\right]$$

where $B_t = \frac{1}{2\pi}\int_0^{2\pi} X(e^{-t}e^{i\theta})d\theta$ is a Brownian motion.

Strategy of proof: Hamiltonian of Liouville CFT

- On the Hilbert space $L^2(\mathbb{R} \times \Omega, dc \otimes \mathbb{P})$, the Hamiltonian $\mathbf{H}$ of Liouville theory is the Schrödinger type operator

$$\mathbf{H} = -\frac{1}{2}\partial_c^2 + \frac{Q^2}{2} + \mathbf{P} + \mu e^{\gamma c} V$$

with $\mathbf{P}$ the infinite harmonic oscillator and V a GMC type potential ($\gamma^2 < 2$)

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- Probabilistic representation for **H**

$$e^{-t\mathbf{H}_0} f(c,\varphi) = e^{-\frac{Q^2}{2}t}\mathbb{E}_\varphi\left[f\left(c + B_t, X \circ e^{-t} - B_t\right) e^{-\mu\int_0^t e^{\gamma(c+B_s)} V(X\circ e^{-s} - B_s)\, ds}\right]$$

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Strategy of proof: Hamiltonian of Liouville CFT

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where $B_t=\frac{1}{2\pi}\int_0^{2\pi}X(e^{-t}e^{i\theta})d\theta$ is a Brownian motion.

Strategy of proof: Diagonalization of the Hamiltonian using scattering theory

Theorem

For $\gamma \in (0,2)$, *the spectrum of* $\mathbf{H} = [\frac{Q^2}{2}, +\infty)$, *is absolutely continuous and there is a complete family of generalized eigenstates* $\Psi_{Q+iP,\nu,\tilde{\nu}} \in L^2_{\text{enlarged}}(\mathbb{R}\times\Omega)$ *labeled by* $P \in \mathbb{R}$ *and Young diagrams* $\nu, \tilde{\nu}$ *(discrete) such that*

$$\mathbf{H}\Psi_{Q+iP,\nu,\tilde{\nu}} = \Big(\frac{Q^2+P^2}{2} + |\nu| + |\tilde{\nu}|\Big)\Psi_{Q+iP,\nu,\tilde{\nu}}$$

and diagonalizing $\mathbf{H}$*: for each* $u, v \in L^2(\mathbb{R}\times\Omega)$

$$\langle u, v\rangle_{L^2} = \frac{1}{2\pi}\sum_{\nu,\tilde{\nu},\nu',\tilde{\nu}'}\int_0^\infty \langle u, \Psi_{Q+iP,\nu,\tilde{\nu}}\rangle_{\mathcal{H}}\langle \Psi_{Q+iP,\nu',\tilde{\nu}'}, v\rangle_{\mathcal{H}} Q_{\Delta_{Q+iP}}(\nu,\nu')^{-1} Q^{-1}_{\Delta_{Q+iP}}(\tilde{\nu},\tilde{\nu}')\, dP$$

Moreover

$$\begin{aligned}\langle \Psi_{Q+iP,\nu,\tilde{\nu}}, U(V_{\alpha_1}(0)V_{\alpha_2}(z))\rangle_2 &= v(\Delta_{\alpha_1},\Delta_{\alpha_2},\Delta_{Q+iP},\tilde{\nu})v(\Delta_{\alpha_1},\Delta_{\alpha_2},\Delta_{Q+iP},\nu)\\ &\times \frac{1}{2}C^{\mathrm{DOZZ}}_{\gamma,\mu}(\alpha_1,\alpha_2,Q+iP)\bar{z}^{|\nu|}z^{|\tilde{\nu}|}|z|^{2(\Delta_{Q+iP}-\Delta_{\alpha_1}-\Delta_{\alpha_2})}\end{aligned}$$

Perspectives

- Modular and higher genus Bootstrap of Liouville CFT on the torus using Segal's axioms (GKRV)
- Probabilistic formulas for $\mathcal{F}_P(z)$ (Ghosal-Remy-Sun-Sun)
- CFT of CLE/SLE: "Imaginary" DOZZ formula, etc... (works by Ang, Holden, Remy, Sun). Construction of imaginary Liouville CFT?

Beyond CFT?

- Rigorous definition of the $2d$ sinh-Gordon model (GKRV). Proof of DOZZ type formulas?
- Rigorous definition of the $2d$ sine-Gordon model?