On topological aspects of smooth-automorphic forms

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The idea of smooth-automorphic forms is not new

In his 1989 paper *Introduction to the Schwartz space of* $\Gamma\backslash G$, W. Casselman wrote:

> *"... one plausible, and perhaps useful, extension of the notion of automorphic form would be to include functions in* $A_{umg}(\Gamma\backslash G)$ *which are* $Z(\mathfrak{g})$*-finite but not necessarily* K*-finite."*

Recently. . .

Smooth-automorphic forms played a role in:

- the formulation of global GGP conjectures (2012)
- a proof of the global GGP conjectures for unitary groups $U_n \times U_{n+1}$ in the endoscopic cases (Beuzart-Plessis–Chaudouard–Zydor 2020).

A detailed study of smooth-automorphic forms has begun:

- explicit construction of cuspidal smooth-automorphic forms using Poincaré series (Muić 2016).

A basic question about smooth-automorphic forms

- How to topologize spaces of smooth-automorphic forms, turning them into smooth group representations?

Throughout this talk, let:

- G be a connected reductive group def. over a number field F
- $\mathbb{A}$ = the adele ring of F
- $\mathbb{A}_f$ = the subring of finite adeles of F
- $S = S_\infty \cup S_f$ be the set of (non-trivial) places of F
- $G_\infty = \prod_{v \in S_\infty} G(F_v)$ and $\mathfrak{g}_\infty = \mathrm{Lie}(G_\infty)$ $\rightsquigarrow$ $G(\mathbb{A}) = G_\infty \times G(\mathbb{A}_f)$
- $\mathcal{U}(\mathfrak{g})$ = the universal enveloping algebra of $\mathfrak{g}_\infty \otimes_{\mathbb{R}} \mathbb{C}$
- $\mathcal{Z}(\mathfrak{g})$ = the center of $\mathcal{U}(\mathfrak{g})$
- $P_0 = L_0 \ltimes N_0$ be a fixed minimal parabolic F-subgroup of G
- $K_{\mathbb{A}} = K_\infty \times K_{\mathbb{A}_f}$ be a maximal compact subgroup of $G(\mathbb{A})$ that is in good position with respect to $P_0 = L_0 \ltimes N_0$
- $(K_n)_{n \in \mathbb{Z}_{>0}}$ be a decreasing cofinal sequence of compact open subgroups K_n of $G(\mathbb{A}_f)$
- $G(\mathbb{A})^1 = \bigcap_{\chi \in X^*(G)} \ker |\chi|$ and $A_G^{\mathbb{R}} \cong \mathbb{R}^R_{>0}$ s.t. $G(\mathbb{A}) = A_G^{\mathbb{R}} \times G(\mathbb{A})^1$
- $[G] = G(F) A_G^{\mathbb{R}} \backslash G(\mathbb{A})$ $\rightsquigarrow$ $\mathrm{vol}([G]) < \infty$.

Functions of uniform moderate growth

We fix an embedding $\iota : G \to \mathrm{GL}_N$ defined over F and define the adelic group norm $\|\cdot\| : G(\mathbb{A}) \to \mathbb{R}_{>0}$,

$$\|g\| = \prod_{\mathrm{v}\in S} \max_{1\le i,j\le N} \left\{ \left|\iota(g_{\mathrm{v}})_{i,j}\right|_{\mathrm{v}}, \left|\iota(g_{\mathrm{v}}^{-1})_{i,j}\right|_{\mathrm{v}} \right\}.$$

A function $f \in C^\infty(G(F)\backslash G(\mathbb{A}))$ is of **uniform moderate growth of exponent** $d \in \mathbb{Z}_{\ge 0}$ if

$$p_{d,X}(f) := \sup_{g\in G(\mathbb{A})} |(Xf)(g)| \, \|g\|^{-d} < \infty, \qquad X \in \mathcal{U}(\mathfrak{g}).$$

- ☹ The space $C^\infty_{umg,d}(G)$ of such functions, equipped with the locally convex topology defined by seminorms $p_{d,X}$, is not complete!
- ☺ For every $n \in \mathbb{Z}_{>0}$, the seminorms $p_{d,X}$ define a Fréchet topology on the subspace $C^\infty_{umg,d}(G)^{K_n}$.

Strict inductive limits of sequences of locally convex spaces

Let $(V_n)_{n\in\mathbb{Z}_{>0}}$ be a sequence of complex, Hausdorff, locally convex spaces such that for every n,

$$V_n \text{ is a closed topological vector subspace of } V_{n+1}.$$

The **strict inductive limit**

$$\varinjlim_n V_n$$

is the space $V = \bigcup_{n=1}^{\infty} V_n$ equipped with the finest locally convex topology with respect to which the inclusion maps $V_n \hookrightarrow V$ are continuous.

Properties:

- For every n, V_n is a closed topological subspace of $\varinjlim_n V_n$.
- If each V_n is complete (resp., barrelled), then so is $\varinjlim_n V_n$.

If each V_n is Fréchet, we say that $\varinjlim_n V_n$ is an **LF-space**.

Smooth functions of uniform moderate growth

Recall:

- ☺ For every $n \in \mathbb{Z}_{>0}$, the seminorms

$$p_{d,X}(f) := \sup_{g \in G(\mathbb{A})} |(Xf)(g)| \, \|g\|^{-d} < \infty, \qquad X \in \mathcal{U}(\mathfrak{g}),$$

  define a Fréchet topology on $C^\infty_{umg,d}(G)^{K_n}$.

$\rightsquigarrow$ We define the LF-space

$$C^\infty_{umg,d}(G) = \varinjlim_n C^\infty_{umg,d}(G)^{K_n}.$$

Smooth-automorphic forms

The space $\mathcal{A}^\infty(G)$ of **smooth-automorphic forms** on $G(\mathbb{A})$ consists of the functions $f \in C^\infty_{umg}(G) := \bigcup_d C^\infty_{umg,d}(G)$ that are $\mathcal{Z}(\mathfrak{g})$-finite, i.e. satisfy

$$\dim_{\mathbb{C}} \mathcal{Z}(\mathfrak{g}) f < \infty.$$

The space $\mathcal{A}(G)$ of **(classical) automorphic forms** on $G(\mathbb{A})$ consists of the functions $f \in \mathcal{A}^\infty(G)$ that are additionally K_∞-finite, i.e., satisfy

$$\dim_{\mathbb{C}} \mathrm{span}_{\mathbb{C}} R(K_\infty) f < \infty.$$

Note:

- $\mathcal{A}^\infty(G)$ is a $G(\mathbb{A})$-invariant subspace of $C^\infty_{umg}(G)$.
- $\mathcal{A}(G)$ is "just" a $(\mathfrak{g}_\infty, K_\infty, G(\mathbb{A}_f))$-module.

From now on, we fix an ideal $\mathcal{J}$ of finite codimension in $\mathcal{Z}(\mathfrak{g})$. We will define an LF-topology on the space

$$\mathcal{A}^\infty_{\mathcal{J}}(G) := \{f \in \mathcal{A}^\infty(G) : \mathcal{J}^n f = 0 \quad \text{for some } n \in \mathbb{Z}_{>0}\}.$$

Lemma 1. There exists $d = d(\mathcal{J}) \in \mathbb{Z}_{\geq 0}$ such that

$$\mathcal{A}^\infty_{\mathcal{J}}(G) \subseteq C^\infty_{umg,d}(G).$$

Let us fix d as in Lemma 1.

For every $n \in \mathbb{Z}_{>0}$,

$$\mathcal{A}^\infty(G)^{K_n,\mathcal{J}^n} := C^\infty_{umg,d}(G)^{K_n,\mathcal{J}^n}$$

is a closed G_∞-invariant subspace of the Fréchet space $C^\infty_{umg,d}(G)^{K_n}$.

We define the LF-space

$$\mathcal{A}^\infty_{\mathcal{J}}(G) = \varinjlim_n \mathcal{A}^\infty(G)^{K_n,\mathcal{J}^n}.$$

Let V be a complete complex Hausdorff locally convex space, and let (π, V) be a continuous representation of $G(\mathbb{A})$.

We equip the subspace

$$V^{\infty_{\mathbb{R}}} = \{v \in V : \text{the orbit map } \pi(\,\cdot\,)v : G_\infty \to V \text{ is smooth}\}$$

of G_∞-smooth vectors in V with the locally convex topology defined by the seminorms of the form

$$p \circ \pi(X),$$

where p is a continuous seminorm on V and $X \in \mathcal{U}(\mathfrak{g})$.

We define the space $V^{\infty_{\mathbb{A}}}$ of $G(\mathbb{A})$-smooth vectors in V by

$$V^{\infty_{\mathbb{A}}} := \varinjlim_n \left(V^{\infty_{\mathbb{R}}}\right)^{K_n}.$$

We say that (π, V) is a **smooth representation of** $G(\mathbb{A})$ if

$$V = V^{\infty_{\mathbb{A}}} \qquad \text{or equivalently} \qquad V^{\infty_{\mathbb{R}}} = V = \varinjlim_n V^{K_n}.$$

The space $\mathcal{A}^\infty_\mathcal{J}(G)$

Lemma 2. Acted upon by right translations, the space

$$\mathcal{A}^\infty_\mathcal{J}(G) = \varinjlim_n \mathcal{A}^\infty(G)^{K_n,\mathcal{J}^n}$$

is a smooth representation of $G(\mathbb{A})$.

Let

$$\begin{aligned}\mathcal{A}_{\mathcal{J}}(G) &= \{f \in \mathcal{A}(G) : \mathcal{J}^n f = 0 \text{ for some } n \in \mathbb{Z}_{>0}\} \\ &= \mathcal{A}_{\mathcal{J}}^{\infty}(G)_{(K_\infty)}.\end{aligned}$$

In this talk:

- A **classical automorphic subrepresentation** is a $(\mathfrak{g}_\infty, K_\infty, G(\mathbb{A}_f))$-submodule of $\mathcal{A}_{\mathcal{J}}(G)$.
- A **classical automorphic representation** is a $(\mathfrak{g}_\infty, K_\infty, G(\mathbb{A}_f))$-subquotient of $\mathcal{A}_{\mathcal{J}}(G)$.

In this talk:

- A **smooth-automorphic subrepresentation** is a closed $G(\mathbb{A})$-invariant subspace U of $\mathcal{A}^{\infty}_{\mathcal{J}}(G)$.
- A **smooth-automorphic representation** is a quotient representation U/W for some closed $G(\mathbb{A})$-invariant subspaces $W \subseteq U$ of $\mathcal{A}^{\infty}_{\mathcal{J}}(G)$.

⚠ A closed subspace U of an LF-space V is not necessarily an LF-space!

⚠ The quotient U/W of a complete Hausdorff locally convex space U by a closed subspace W is not necessarily complete!

A natural question

Does the assignment $V \mapsto V_{(K_\infty)}$ define a 1-1 correspondence

smooth-automorphic (sub)representations	$\leftrightarrow$	classical automorphic (sub)representations?

Proposition 3. If V_0 is a classical automorphic subrepresentation, then

$$V := \mathrm{Cl}_{\mathcal{A}^\infty_{\mathcal{J}}(G)} V_0$$

is a smooth-automorphic subrepresentation.

Theorem 4. The assignments

$$V \mapsto V_{(K_\infty)} \qquad \text{and} \qquad \mathrm{Cl}_{\mathcal{A}^\infty_{\mathcal{J}}(G)} V_0 \leftarrow\!\shortmid V_0$$

define a 1-1 correspondence

$$\left\{ \begin{array}{c} \text{admissible} \\ \text{smooth-automorphic} \\ \text{subrepresentations } V \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{admissible} \\ \text{classical automorphic} \\ \text{subrepresentations } V_0 \end{array} \right\}$$

which restricts to a 1-1 correspondence

$$\left\{ \begin{array}{c} \text{irreducible} \\ \text{smooth-automorphic} \\ \text{subrepresentations } V \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{irreducible} \\ \text{classical automorphic} \\ \text{subrepresentations } V_0 \end{array} \right\}.$$

Casselman-Wallach representations

A representation (π, V) of G_∞ on a Fréchet space V is **of moderate growth** if for every continuous seminorm p on V, there exist $m \in \mathbb{Z}_{>0}$ and a continuous seminorm q on V such that

$$p(\pi(g)v) \leq \|g\|^m q(v), \qquad g \in G_\infty, v \in V.$$

A **Casselman-Wallach representation of** G_∞ is a smooth (Fréchet) representation (π, V) of G_∞ of moderate growth such that the $(\mathfrak{g}_\infty, K_\infty)$-module $V_{(K_\infty)}$ is admissible and finitely generated.

A **Casselman-Wallach representation of** $G(\mathbb{A})$ is a smooth representation (π, V) of $G(\mathbb{A})$ such that for every $n \in \mathbb{Z}_{>0}$, V^{K_n} is a Casselman-Wallach representation of G_∞.

Casselman-Wallach representations

Wallach's results on Casselman-Wallach representations of G_∞ easily imply:

Lemma 5.

(i) If U is a closed $G(\mathbb{A})$-invariant subspace of a Casselman-Wallach representation (π, V) of $G(\mathbb{A})$, then U and V/U are Casselman-Wallach representations of $G(\mathbb{A})$.

(ii) Two Casselman-Wallach representations (π, V) and (σ, W) of $G(\mathbb{A})$ are equivalent if and only if the $(\mathfrak{g}_\infty, K_\infty, G(\mathbb{A}_f))$-modules $V_{(K_\infty)}$ and $W_{(K_\infty)}$ are isomorphic.

Lemma 6. We have

$$\left\{\begin{array}{c}\text{irreducible}\\ \text{smooth-automorphic}\\ \text{subrepresentations}\end{array}\right\} \subseteq \left\{\begin{array}{c}\text{finitely generated}\\ \text{smooth-automorphic}\\ \text{subrepresentations}\end{array}\right\}$$
$$\subseteq \left\{\begin{array}{c}\text{Casselman-Wallach}\\ \text{representations}\\ \text{of } G(\mathbb{A})\end{array}\right\} \subseteq \left\{\begin{array}{c}\text{admissible}\\ \text{representations}\\ \text{of } G(\mathbb{A})\end{array}\right\}.$$

Proposition 7.

(i) If V is an irreducible smooth-automorphic representation, then $V_{(K_\infty)}$ is an irreducible classical automorphic representation.

(ii) Every irreducible smooth-automorphic representation V is admissible.

Theorem 8. The assignment

$$[V] \mapsto [V_{(K_\infty)}]$$

defines a bijection

$$\left\{ \begin{array}{c} \text{equivalence classes} \\ \text{of irreducible} \\ \text{Casselman-Wallach} \\ \text{smooth-automorphic} \\ \text{representations} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{equivalence classes} \\ \text{of irreducible} \\ \text{classical automorphic} \\ \text{representations} \end{array} \right\}.$$

Tensor product theorem

Theorem 9. Let (π, V) be an irreducible smooth-automorphic Casselman-Wallach representation of $G(\mathbb{A})$. Then, for each $\mathrm{v} \in S$, there exists a unique irreducible smooth admissible representation $(\pi_\mathrm{v}, V_\mathrm{v})$ of $G(F_\mathrm{v})$, which is of moderate growth if $\mathrm{v} \in S_\infty$, such that as $G(\mathbb{A})$-representations,

$$\pi \;\cong\; \overline{\bigotimes_{\mathrm{v}\in S_\infty}}_{\mathrm{pr}} \pi_\mathrm{v} \;\; \overline{\otimes_{\mathrm{in}}} \;\; \bigotimes_{\mathrm{v}\in S_f}{}' \pi_\mathrm{v},$$

where the restricted tensor product $\bigotimes'_{\mathrm{v}\in S_f} \pi_\mathrm{v}$ is equipped with the finest locally convex topology.

Another natural question

Do the famous direct sum decompositions of $\mathcal{A}_{\mathcal{J}}(G)$ along parabolic and cuspidal support have their smooth-automorphic counterparts?

LF-compatible smooth-automorphic subrepresentations

We say that a smooth-automorphic subrepresentations V is **LF-compatible** if

$$V = \varinjlim_{n} V^{K_n, \mathcal{J}^n}.$$

Examples of LF-compatible smooth-automorphic subrepresentations:

- $\mathcal{A}_{\mathcal{J}}^{\infty}(G)$
- $\mathcal{A}_{\mathcal{J}}^{\infty}([G]) := \mathcal{A}_{\mathcal{J}}^{\infty}(G) \cap C^{\infty}(G(F)A_G^{\mathbb{R}} \backslash G(\mathbb{A}))$
- smooth-automorphic subrepresentations annihilated by a power of $\mathcal{J}$.

Crucial proposition

Proposition 10. Let V be an LF-compatible smooth-automorphic subrepresentation. Suppose that

$$V_{(K_\infty)} = \bigoplus_{i \in I} V_{0,i}$$

for some $(\mathfrak{g}_\infty, K_\infty, G(\mathbb{A}_f))$-submodules $V_{0,i} \subseteq V_0$. Then, we have the following decomposition into a locally convex direct sum of LF-compatible smooth-automorphic subrepresentations:

$$V = \bigoplus_{i \in I} \mathrm{Cl}_{\mathcal{A}^\infty_{\mathcal{J}}(G)} V_{0,i}.$$

Parabolic support of a smooth-automorphic form

We say a function $f \in C^\infty_{umg}(G)$ is **negligible** along a parabolic F-subgroup $P = L \ltimes N$ of G, if

$$\int_{L(F)\backslash L(\mathbb{A})^1} f_P(lg)\,\overline{\varphi(l)}\,dl = 0, \qquad g \in G(\mathbb{A}),\ \varphi \in \mathcal{A}^{(\infty)}_{cusp}([L]),$$

where

$$f_P(g) = \int_{N(F)\backslash N(\mathbb{A})} f(ng)\,dn, \qquad g \in G(\mathbb{A}).$$

Denoting by $\{P\}$ the associate class of P, let:

- $C^\infty_{umg,\{P\}}(G) = \left\{ f \in C^\infty_{umg}(G) : f \text{ is negligible along all } Q \notin \{P\} \right\}$
- $\mathcal{A}^\infty_{\mathcal{J},\{P\}}([G]) = \mathcal{A}^\infty_{\mathcal{J}}([G]) \cap C^\infty_{umg,\{P\}}(G)$
- $\mathcal{A}_{\mathcal{J},\{P\}}([G]) = \mathcal{A}_{\mathcal{J}}([G]) \cap C^\infty_{umg,\{P\}}(G).$

Langlands's algebraic direct sum decomposition

$$C_{umg}^{\infty}(G) = \bigoplus_{\{P\}} C_{umg,\{P\}}^{\infty}(G)$$

easily implies the algebraic direct sum decomposition

$$\mathcal{A}_{\mathcal{J}}([G]) = \bigoplus_{\{P\}} \mathcal{A}_{\mathcal{J},\{P\}}([G]),$$

which, applying Prop. 10, implies

Theorem 11. We have the following decomposition of $\mathcal{A}_{\mathcal{J}}^{\infty}([G])$ into a locally convex direct sum of LF-compatible smooth-automorphic subrepresentations:

$$\mathcal{A}_{\mathcal{J}}^{\infty}([G]) = \bigoplus_{\{P\}} \mathcal{A}_{\mathcal{J},\{P\}}^{\infty}([G]).$$

Decomposition of $\mathcal{A}^{\infty}_{\mathcal{J},cusp}([G]) = \mathcal{A}^{\infty}_{\mathcal{J},\{G\}}([G])$

Gelfand–Piatetski-Shapiro: We have an orthogonal sum decomposition

$$L^2_{cusp}([G]) = \widehat{\bigoplus_{i \in I}} \mathcal{H}_i$$

into irreducible closed $G(\mathbb{A})$-invariant subspaces.

$\rightsquigarrow$ There exists a subset $I(\mathcal{J}) \subseteq I$ such that

$$\mathcal{A}_{\mathcal{J},cusp}([G]) = \bigoplus_{i \in I(\mathcal{J})} \mathcal{H}^{\infty_{\mathbb{A}}}_{i,(K_\infty)},$$

which, applying Prop. 10 and results on CW-reps, easily implies

Theorem 12. We have the following decomposition of $\mathcal{A}^{\infty}_{\mathcal{J},cusp}([G])$ into a locally convex direct sum of irreducible smooth-automorphic subrepresentations:

$$\mathcal{A}^{\infty}_{\mathcal{J},cusp}([G]) = \bigoplus_{i \in I(\mathcal{J})} \mathcal{H}^{\infty_{\mathbb{A}}}_{i}.$$

Decomposition of $\mathcal{A}^\infty_{\mathcal{J},\{P\}}([G])$

Franke–Schwermer: For every $\{P\}$, we have a direct sum decomposition

$$\mathcal{A}_{\mathcal{J},\{P\}}([G]) = \bigoplus_{\varphi\in\Phi_{\mathcal{J},\{P\}}} \mathcal{A}_{\mathcal{J},\{P\},\varphi}([G])$$

into $(\mathfrak{g}_\infty, K_\infty, G(\mathbb{A}_f))$-submodules, where:

- $\Phi_{\mathcal{J},\{P\}}$ is the set of suitable associate classes φ of irreducible cuspidal (classical or smooth-) automorphic subrepresentations of Levi components of elements of $\{P\}$
- $\mathcal{A}_{\mathcal{J},\{P\},\varphi}([G])$ is spanned by the derivatives of regularized values of Eisenstein series attached to the associate class φ.

Theorem 13. For every $\{P\}$, we have the following decomposition of $\mathcal{A}^\infty_{\mathcal{J},\{P\}}([G])$ into a locally convex direct sum of LF-compatible smooth-automorphic subrepresentations:

$$\mathcal{A}^\infty_{\mathcal{J},\{P\}}([G]) = \bigoplus_{\varphi\in\Phi_{\mathcal{J},\{P\}}} \mathrm{Cl}_{\mathcal{A}^\infty_{\mathcal{J}}([G])}\, \mathcal{A}_{\mathcal{J},\{P\},\varphi}([G]).$$

Happy birthday, Prof. Savin!