Dirac fields on Kerr spacetime and the Hawking radiation IV

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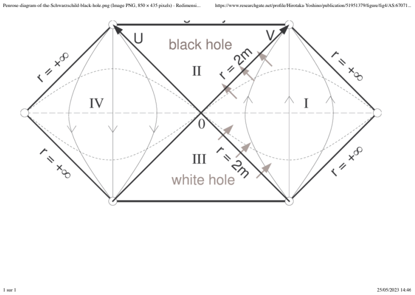
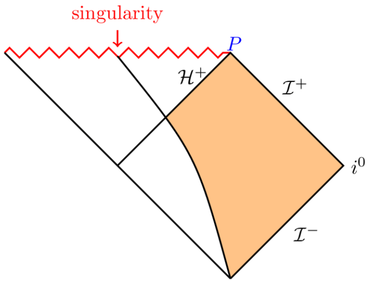
Part IV: The Hawking effect





S. W. Hawking, *Particle creation by black holes*, Comm. Math. Phys. 43 (1975), 199-220

"...quantum mechanical effects cause black holes to create and emit particles as if they were hot bodies with temperature $\frac{h\kappa}{4\pi^2 k} \sim 10^{-6} \left(\frac{M_o}{M}\right) K$, where κ is the surface gravity of the black hole."

"It is now generally believed that, according to classical theory, a gravitational collapse will produce a black hole which will settle down rapidly to a stationary axisymmetric equilibrium state characterized by its mass, angular momentum and electric charge... The Kerr-Newman solution represent one such family of black hole equilibrium states and it seems unlikely that there are any others. ... Because these solutions are stationary there will not be any mixing of positive and negative frequencies and so one would not expect to obtain any particle creation... To understand how the particle creation can arise from mixing of positive and negative frequencies, it is essential to consider not only the quasistationary final state of the black hole but also the time-dependent formation phase."

IV.1 The model of the collapsing star

The model of the collapsing star 1

Assumption : The metric outside the collapsing star is the Kerr metric. Surface at Boyer-Lindquist time $t = 0$: $\mathcal{S}_0$. $x_0 \in \mathcal{S}_0$ moves along certain timelike geodesics γ_p.

(A) $L = 0$,
(B) $\tilde{E} = a^2(E^2 - p) = 0$ (rotational energy vanishes),
(C) $\mathcal{Q} = 0$ (total angular momentum about the axis of symmetry vanishes).

Lemma

Let $\sigma^2 = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta$. *Along* γ_p *:*

$$\frac{\partial\theta}{\partial t} = 0, \ \frac{\partial\varphi}{\partial t} = \frac{2a\mathfrak{m}r}{\sigma^2} \text{(local angular velocity of space-time).}$$

Lemma

There exists a variable $\hat{r}$ *associated to null geodesic* γ *with* $L = \mathcal{Q} = 0$ *(SNG's) s.t.* $\partial_t\hat{r} = \pm 1$ *along* γ *(Bondi-Sachs type).*

The model of the collapsing star 2

Lemma

Along γ_p we have uniformly in θ when $t \to \infty$:

$$\hat{r} = -t - \hat{A}(\theta)e^{-2\kappa_+ t} + \hat{B}(\theta) + \mathcal{O}(e^{-4\kappa_+ t}),\ \hat{A}(\theta) > 0.$$

$$\text{Asymptotic assumption}:\ \hat{B}(\theta) = 0, \tag{1}$$

$$\mathcal{S} = \{(t, \hat{z}(t,\theta), \omega); t \in \mathbb{R},\ \omega \in S^2\}, \tag{2}$$

$$\forall t \le 0, \theta \in [0,\pi] \quad \hat{z}(t,\theta) = \hat{z}(0,\theta) < 0, \tag{3}$$

$$\hat{z}(t,\theta) = -t - \hat{A}(\theta)e^{-2\kappa_+ t} + \mathcal{O}(e^{-4\kappa_+ t}),\ t \to \infty,\ \hat{A}(\theta) > 0. \tag{4}$$

κ_+ : surface gravity at outer horizon.

$$\mathcal{M}_{col} = \bigcup_t \Sigma_t^{col},\ \Sigma_t^{col} = \{(t, \hat{r}, \omega) \in \mathbb{R}_t \times \mathbb{R}_{\hat{r}} \times S^2_\omega;\ \hat{r} \ge \hat{z}(t,\theta)\}.$$

Remark (Dirac equation on $\mathcal{M}_{col}$)

We use a MIT boundary condition. The Dirac equation is then solved by an isometric propagator $U(t,s)$.

IV.2 Dirac quantum fields

Dirac quantum fields (Dimock)

$\mathcal{M}_{col} = \bigcup_{t\in\mathbb{R}} \Sigma_t^{col}, \quad \Sigma_t^{col} = \{(t,\hat{r},\theta,\varphi); \hat{r} \ge \hat{z}(t,\theta)\}.$
Dirac field Ψ_0 and the CAR-algebra $\mathcal{U}(\mathcal{H}_0)$ constructed in the usual way.

$$S_{col} : \begin{array}{ccc} (C_0^\infty(\mathcal{M}_{col}))^4 & \to & \mathcal{H}_0 \\ \Phi & \mapsto & S_{col}\Phi := \int_{\mathbb{R}} U(0,t)\Phi(t)dt \end{array}$$

Quantum spin field :

$$\Psi_{col} : \begin{array}{ccc} (C_0^\infty(\mathcal{M}_{col}))^4 & \to & CAR(\mathcal{H}_0) \\ \Phi & \mapsto & \Psi_{col}(\Phi) := \Psi_0(S_{col}\Phi) \end{array}$$

$\mathcal{U}_{col}(\mathcal{O})$ = algebra generated by $\Psi_{col}^*(\Phi^1)\Psi_{col}(\Phi^2)$, $\operatorname{supp}\Phi^j \subset \mathcal{O}$.

$$\mathcal{U}_{col}(\mathcal{M}_{col}) = \overline{\bigcup_{\mathcal{O}\subset\mathcal{M}_{col}} \mathcal{U}_{col}(\mathcal{O})}.$$

Same procedure on $\mathcal{M}_{BH}$:
$S : \Phi \in (C_0^\infty(\mathcal{M}_{BH}))^4 \mapsto S\Phi := \int_{\mathbb{R}} e^{-itH}\Phi(t)dt.$

States

1. $\mathcal{U}_{col}(\mathcal{M}_{col})$
   Vacuum state :

$$\begin{aligned}\omega_{col}(\Psi^*_{col}(\Phi_1)\Psi_{col}(\Phi_2)) &:= \omega_{vac}(\Psi^*_0(S_{col}\Phi_1)\Psi_0(S_{col}\Phi_2)) \\ &= \langle \mathbf{1}_{\mathbb{R}^+}(H_0)S_{col}\Phi_1, S_{col}\Phi_2\rangle.\end{aligned}$$

2. $\mathcal{U}_{BH}(\mathcal{M}_{BH})$
   1. Vacuum state

$$\omega_{vac}(\Psi^*_{BH}(\Phi_1)\Psi_{BH}(\phi_2)) = \langle \mathbf{1}_{\mathbb{R}^+}(H)S\phi_1, S\phi_2\rangle.$$

   2. Thermal Hawking state

$$\begin{aligned}\omega^{\eta,\sigma}_{Haw}(\Psi^*_{BH}(\Phi_1)\Psi_{BH}(\Phi_2)) &= \langle \mu e^{\sigma H}(1+\mu e^{\sigma H})^{-1}S\Phi_1, S\Phi_2\rangle_{\mathcal{H}} \\ &=: \omega^{\eta,\sigma}_{KMS}(\Psi^*(S\Phi_1)\Psi(S\Phi_2)), \\ T_{Haw} &= \sigma^{-1},\ \mu = e^{\sigma\eta},\ \sigma > 0.\end{aligned}$$

T_{Haw} Hawking temperature, μ chemical potential.

The Hawking effect

$$\Phi \in (C_0^\infty(\mathcal{M}_{col}))^4,\ \Phi^T(t,\hat{r},\omega) = \Phi(t-T,\hat{r},\omega).$$

Theorem (Hawking effect, H '09)

Let $\Phi_j \in (C_0^\infty(\mathcal{M}_{col}))^4$, $j=1,2$. *Then we have*

$$\begin{aligned} &\lim_{T\to\infty} \omega_{col}(\Psi^*_{col}(\Phi_1^T)\Psi_{col}(\Phi_2^T)) \\ &\quad = \omega_{Haw}^{\eta,\sigma}(\Psi^*_{BH}(\mathbf{1}_{\mathbb{R}^+}(P^-)\Phi_1)\Psi_{BH}(\mathbf{1}_{\mathbb{R}^+}(P^-)\Phi_2)) \\ &\quad + \omega_{vac}(\Psi^*_{BH}(\mathbf{1}_{\mathbb{R}^-}(P^-)\Phi_1)\Psi_{BH}(\mathbf{1}_{\mathbb{R}^-}(P^-)\Phi_2)), \\ &\quad T_{Haw} = 1/\sigma = \kappa_+/2\pi,\ \mu = e^{\sigma\eta},\ \eta = \frac{aD_\varphi}{r_+^2+a^2}. \end{aligned}$$

Remarks

Remark

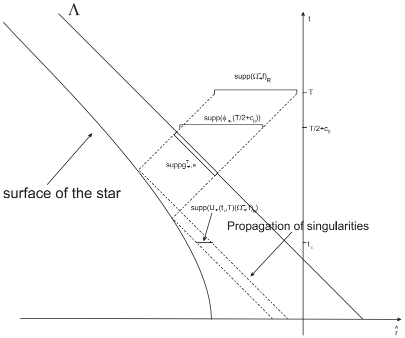
1. *The limit state is the Unruh state.*
2. $S_{col}\Phi^T = \int_a^b U(0,T)U(T,s+T)\Phi(s)ds = U(0,T)S\Phi.$

Remark

Spherical symmetry

- *First theorem by Bachelot ('99) for bosons.*
- *Fermions treated by Melnyk ('04). He also shows that the initial state can be a KMS state with arbitrary temperature.*
- *In the $\Lambda > 0$ case one can show exponentially fast convergence towards the Unruh state : Drouot ('17).*

Explanation



Change in frequencies : mixing of positive and negative frequencies.

Remark

The figure shows a simplified situation where Huygens' principle holds. In the proof this has to be replaced by a "weak Huygens' principle" which follows from suitable propagation estimates.

IV.3 Toy model

Toy model : The moving mirror 1

$$z \in C^2(\mathbb{R}),\ \forall t \leq 0,\ z(t) = 0,$$

$$\forall t \geq 1,\ z(t) = -t - Ae^{-2\kappa t};\ A > 0,\ \kappa > 0,$$

$$\left\{ \begin{array}{rcl} \partial_t \Psi & = & iH\Psi, \\ \psi_1(t, z(t)) & = & -\sqrt{\frac{1-\dot{z}}{1+\dot{z}}}\psi_2(t, z(t)) \\ \Psi(t = s, .) & = & \Psi_s(.) \end{array} \right. ,\ H = \Gamma D_x,\ \Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Solution given by a unitary propagator $U(t, s)$. Conserved L^2 norm : $||\psi||^2_{\mathcal{H}_t} = \int_{z(t)}^{\infty} |\psi|^2(t, x)dx$. Explicit solution :

$$x > z(t),\ \psi_2(t, x) = \psi_2^s(x - t),$$

$$x > z(s) + s - t,\ \psi_1(t, x) = \psi_1^s(x + t - s),$$

$$z(t) < x < z(s) + s - t,$$

$$\psi_1(t, x) = -Z(\tau(x + t))\psi_2^s(x + t + s - 2\tau(x + t)).$$

Toy model : the moving mirror 2

Here, $\tau(x)$ is given by

$$z(\tau(x)) + \tau(x) = x \Leftrightarrow \tau(x) = -\frac{1}{2\kappa}\ln\left(\frac{-x}{A}\right)$$

and $Z(t)$ by $Z(t) = \sqrt{\frac{1-\dot{z}}{1+\dot{z}}}$. We have

$$Z(\tau(x)) = \frac{1}{\sqrt{-\kappa x}} + \mathcal{O}(x),\ x \to 0^-.$$

When $s = T$, we obtain

$$\begin{aligned}\psi_1(t,x) &\sim -\frac{1}{\sqrt{-\kappa(x+t)}}\psi_2^T((x+t+T-2\tau(x+t))\\ &= -\frac{1}{\sqrt{-\kappa(x+t)}}\psi_2^T\left(x+t+T+\frac{1}{\kappa}\ln\left(\frac{-(x+t)}{A}\right)\right)\\ &\sim -\frac{1}{\sqrt{-\kappa(x+t)}}\psi_2^T\left(\frac{1}{\kappa}\ln\left(\frac{-(x+t)}{A}e^{\kappa T}\right)\right)\end{aligned}$$

Toy model : the moving mirror 3

For $f \in C_0^\infty(\mathbb{R})$ we define $f^T(x) = \frac{1}{\sqrt{-\kappa x}} f\left(\frac{1}{\kappa}\ln\left(\frac{-x}{A}e^{\kappa T}\right)\right)$ (geometric optics approximation). Let

$$\mathcal{H}_0 = (L^2(]z(0),\infty[;dx)^2,\ \mathcal{H}_\infty = (L^2(\mathbb{R};dx))^2.$$

Unitary transform $P : \mathcal{H}_0 \to \mathcal{H}_\infty$, $g = Pf$ with

$$x \geq z(0),\ g(x) = f(x),$$
$$x \leq z(0),\ g_1(x) = -f_2(2z(0)-x),\ g_2(x) = -f_1(2z(0)-x).$$

We have $HPf = PHf$ for $f \in \mathcal{H}_0^1$ and H_0 is unitary equivalent to $-\frac{\xi}{\sqrt{4\pi}}\Gamma$ by the Fourier transform. We obtain :

$$\mathbf{1}_{\mathbb{R}^+}(H_0) = P^{-1}\mathcal{F}^{-1}\begin{pmatrix} \mathbf{1}_{\mathbb{R}^+}(\xi) & 0 \\ 0 & \mathbf{1}_{\mathbb{R}^+}(-\xi) \end{pmatrix}\mathcal{F}P.$$

Toy model : the moving mirror 4

Now,

$$
\begin{aligned}
&2\pi\|\mathbf{1}_{\mathbb{R}^+}(H_0)f^T\|^2\\
&= \int_0^\infty |\mathcal{F}(f^T)(\xi)|^2 d\xi\\
&= \lim_{\epsilon\to 0} A\kappa \int_0^\infty \left|\int_{\mathbb{R}} e^{i(A+i\epsilon)\zeta e^{\kappa y}} e^{\kappa/2y} f(y)dy\right|^2 d\zeta.\\
&= \lim_{\epsilon\to 0^+} \int_{\mathbb{R}\times\mathbb{R}} \frac{1}{\epsilon\cosh\left[\frac{\kappa}{2}(y_1-y_2)\right] - iA\sinh\left[\frac{\kappa}{2}(y_1-y_2)\right]} f(y_1)\bar{f}(y_2)dy_1dy_2.\\
&= \lim_{\epsilon\to 0^+} \frac{A\kappa}{4\pi}\int_{\mathbb{R}} |\hat{f}(\xi)|^2 \mathcal{F}\left(\frac{1}{\epsilon\cosh(\frac{\kappa}{2}x) - iA\sinh(\frac{\kappa}{2}x)}\right)(-\xi)d\xi.
\end{aligned}
$$

Let $h(x) = \frac{e^{-ix\xi}}{\epsilon\cosh(\frac{\kappa}{2}x) - iA\sinh(\frac{\kappa}{2}x)}$.

Toy model : the moving mirror 5

Integrating $h(x)$ along a suitable path and using Cauchy's formula gives :

$$\int_{\mathbb{R}} h(x)dx = 2\pi i \sum_{n=1}^{\infty} \rho_n(\epsilon),$$

where $\rho_n(\epsilon)$ are the residues of $h(x)$ at the poles $z_n(\epsilon) \in \{z \in \mathbb{C};\, \mathrm{Im}\, z > 0\}$. One can check that

$$z_n(\epsilon) = \frac{2i}{\kappa}\left(n\pi - \arctan\left(\frac{\epsilon}{A}\right)\right),$$

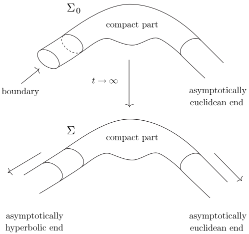
$$\sup_{n\geq 1} \left|\rho_n(\epsilon) - (-1)^n e^{\frac{2\pi n}{\kappa}\xi}\right| \leq C\epsilon,$$

hence we get that for $\xi < 0$ we have

$$\left|\mathcal{F}\left(\frac{1}{\epsilon \cosh(\frac{\kappa}{2}x) - iA\sinh(\frac{\kappa}{2}x)}\right)(\xi) - \frac{4\pi}{A\kappa}e^{\frac{2\pi}{\kappa}\xi}\left(1 + e^{\frac{2\pi}{\kappa}\xi}\right)^{-1}\right| \leq C\epsilon.$$

IV.4 Elements of the proof

The analytic problem



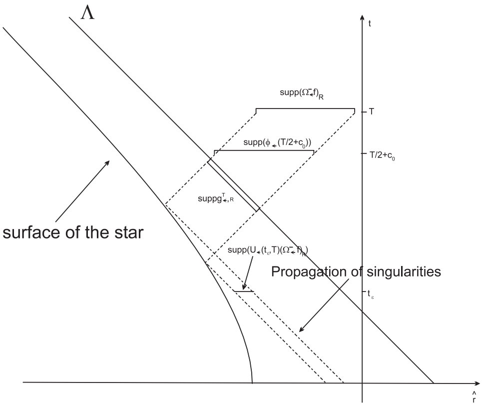
Let $f(r_*, \omega) \in (C_0^\infty(\mathbb{R} \times S^2))^4$. We have to show

$$\begin{aligned} &\lim_{T\to\infty} ||\mathbf{1}_{[0,\infty)}(H_0)U(0,T)f||_0^2 \\ &= \langle \mathbf{1}_{\mathbb{R}^+}(P^-)f, \mu e^{\sigma H}(1+\mu e^{\sigma H})^{-1}\mathbf{1}_{\mathbb{R}^+}(P^-)f\rangle \\ &+ ||\mathbf{1}_{[0,\infty)}(H)\mathbf{1}_{\mathbb{R}^-}(P^-)f||^2, \end{aligned} \tag{5}$$

where

$$\mu = e^{\sigma\eta},\ \eta = \frac{aD_\varphi}{r_+^2 + a^2},\ \sigma = \frac{2\pi}{\kappa_+}.$$

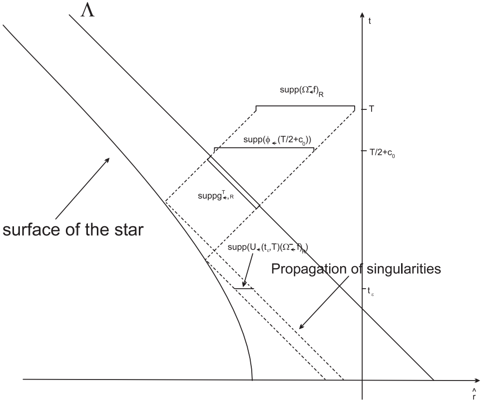
The three time intervals



Collapse of the star.

- $[T/2 + c_0, T]$: Scattering at fixed energy.

The three time intervals



Collapse of the star.

- $[T/2 + c_0, T]$: Scattering at fixed energy.
- $[t_\epsilon, T/2 + c_0]$: High frequency problem : Duhamel formula ?

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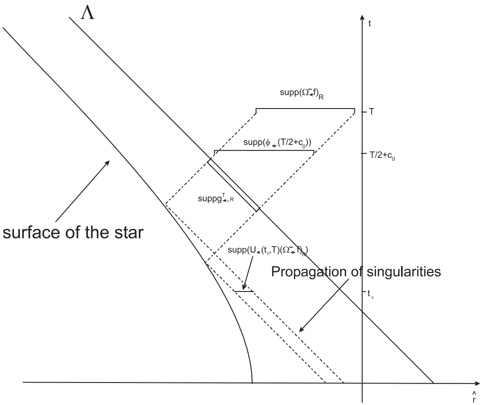

Collapse of the star.

- $[T/2 + c_0, T]$: Scattering at fixed energy.
- $[t_\epsilon, T/2 + c_0]$: High frequency problem : Duhamel formula ?
- $[0, t_\epsilon]$: propagation of singularites.

A new Newman-Penrose tetrad

Lemma

There exists a Newman-Penrose tetrad and a coordinate system $(t, \hat{r}, \omega)$ *s.t. :*
$H = \Gamma D_{\hat{r}} + P_\omega + W$, $\Gamma = Diag(1, -1)$. P_ω *is a differential operator with derivatives only in the angular directions and* W *is a potential.*

In the above lemma n and l are the generators of the simple null geodesics. Comparison dynamics

$$H_{\leftarrow} = \Gamma D_{\hat{r}} - \frac{a}{r_+^2 + a^2} D_\varphi.$$

Outline of the proof 1

- Decouple the problem at infinity from the problem near the horizon.

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- Consider $U(t,T)f$ on Λ. Data : g^T. Consider $U_{\leftarrow}(t,T)\Omega_{\leftarrow}^{-}f$ on Λ. Data : $g_{\leftarrow}^{T}$.

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- Solve characteristic Cauchy problem with data $g_{\leftarrow,R}^{T}$:

$$G(g_{\leftarrow,R}^{T}) = U(0,T/2+c_0)\Phi(T/2+c_0),$$

where $\phi(T/2+c,0)$ is the solution of the characteristic Cauchy problem in the whole space.

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- Consider $U(t,T)f$ on Λ. Data : g^T. Consider $U_{\leftarrow}(t,T)\Omega_{\leftarrow}^{-}f$ on Λ. Data : $g^T_{\leftarrow}$.
- Solve characteristic Cauchy problem with data $g^T_{\leftarrow,R}$:

$$G(g^T_{\leftarrow,R}) = U(0,T/2+c_0)\Phi(T/2+c_0),$$

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- We also consider $G(g^T)$ and $G_{\leftarrow}(g^T_{\leftarrow,R})$.

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  where $\phi(T/2+c,0)$ is the solution of the characteristic Cauchy problem in the whole space.
- We also consider $G(g^T)$ and $G_{\leftarrow}(g_{\leftarrow,R}^T)$.
- Asymptotic completeness :

$$g^T - g_{\leftarrow,R}^T \to 0$$

  Continuous dependence on the data :

$$G(g^T) - G(g_{\leftarrow,R}^T) \to 0.$$

Outline of the proof 2

- We write

$$
\begin{aligned}
& G(g^T_{\leftarrow,R}) - G_{\leftarrow}(g^T_{\leftarrow,R}) \\
&\quad = \quad U(0,T/2+c_0)(\Phi(T/2+c_0) - \Phi_{\leftarrow}(T/2+c_0)) \\
&\quad + \quad (U(0,T/2+c_0) - U_{\leftarrow}(0,T/2+c_0))\Phi_{\leftarrow}(T/2+c_0).
\end{aligned}
$$

The first term becomes small when T becomes large (scattering). New tetrad : $\forall \epsilon > 0$, $\exists t_\epsilon > 0$

$$
||\mathcal{J}_\epsilon(\hat{r})(U(t_\epsilon,T/2+c_0) - U_{\leftarrow}(t_\epsilon,T/2+c_0))\Phi_{\leftarrow}(T/2+c_0)|| < \epsilon
$$

uniformly in T large.

Outline of the proof 3

- Thus

$$\lim_{T\to\infty} ||\mathbf{1}_{[0,\infty)}(H_0)j_-U(0,T)f||_0^2$$
$$\sim \lim_{T\to\infty} ||\mathbf{1}_{[0,\infty)}(H_0)U(0,t_\epsilon)U_\leftarrow(t_\epsilon,T/2+c_0)\Phi_\leftarrow(T/2+c_0)||_0^2,$$

where j_- is a smooth cut-off which equals 1 near the boundary and 0 at infinity.

Outline of the proof 3

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  where j_- is a smooth cut-off which equals 1 near the boundary and 0 at infinity.
- Replace $U_\leftarrow(t_\epsilon, T/2+c_0)\Phi_\leftarrow(T/2+c_0)$ by a geometric optics approximation $F^T_{t_\epsilon}$ with :

$$\mathrm{supp}\, F^T_{t_\epsilon} \subset (-t_\epsilon - |\mathcal{O}(e^{-\kappa_+ T})|, -t_\epsilon),\ F^T_{t_\epsilon} \rightharpoonup 0,\ T\to\infty,$$
$$\forall\lambda>0 \quad Op(\chi(\langle\xi\rangle \le \lambda\langle\alpha\rangle))F^T_{t_\epsilon} \to 0,\ T\to\infty.$$

  Here ξ and α are the dual coordinates to $\hat{r},\theta$ respectively.

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$$\forall \lambda > 0 \quad Op(\chi(\langle\xi\rangle \le \lambda\langle\alpha\rangle))F^T_{t_\epsilon} \to 0,\ T\to\infty.$$

  Here ξ and α are the dual coordinates to $\hat{r}, \theta$ respectively.
- We show that for λ sufficiently large possible singularities of $Op(\chi(\langle\xi\rangle \ge \lambda\langle\alpha\rangle))F^T_{t_\epsilon}$ are transported by the group $e^{-it_\epsilon H}$ in such a way that they always stay away from the surface of the star.

- Let ϕ_δ be a cut-off outside the surface of the star at time 0. If $\phi_\delta = 1$ sufficiently close to the surface of the star at time 0 we see by the previous point that

$$(1-\phi_\delta)e^{-it_\epsilon H}F^T_{t_\epsilon} \to 0,\ T\to\infty. \tag{6}$$

Using (6) we show that (modulo a small error term):

$$(U(0,t_\epsilon)-\phi_\delta e^{-it_\epsilon H})F^T_{t_\epsilon} \to 0,\ T\to\infty.$$

Therefore it remains to consider :

$$\lim_{T\to\infty} ||\mathbf{1}_{[0,\infty)}(H_0)\phi_\delta e^{-it_\epsilon H}F^T_{t_\epsilon}||_0.$$

Outline of the proof 5

- We show that we can replace $\mathbf{1}_{[0,\infty)}(H_0)$ by $\mathbf{1}_{[0,\infty)}(H)$. This will essentially allow to commute the energy cut-off and the group. We then show that we can replace the energy cut-off by $\mathbf{1}_{[0,\infty)}(H_{\leftarrow})$. We end up with :

$$\lim_{T\to\infty} ||\mathbf{1}_{[0,\infty)}(H_{\leftarrow})e^{-it_\epsilon H_{\leftarrow}} F^T_{t_\epsilon}||. \tag{7}$$

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$$\lim_{T\to\infty} ||\mathbf{1}_{[0,\infty)}(H_{\leftarrow})e^{-it_\epsilon H_{\leftarrow}} F_{t_\epsilon}^T||. \tag{7}$$

- We compute the limit in (7) explicitly.

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Quantum and classical fields interacting with geometry

Thematic 6-weeks program at Institut Henri Poincaré, Paris, March 18-April 26 2024.

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Thank you for your attention !