Discussion on Alessandra Frabetti's talk: Direct connections on jet groupoids

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Direct connections and jets: motivation and strategy

- Underlying motivation: understand regularity structures better
- First step (today):
  understand the underlying geometric framework better.
- Regularity structures enable us to attack SPDEs via fixed-point problems, by considering 'Taylor expansions' of solutions.
- Direct connections are a generalisation of the notion of connection that, in particular, can be defined
  "where differentiability is not available".
- **Example:** direct connections relate the corresponding Taylor expansions around different points.
- Jets, on the other hand, can be thought of as a coordinate-free approach to Taylor expansions.

Lie groupoids

- Recall: a groupoid G is a "group-with-a-set-of-basepoints" S,

$$(s,t) : G \rightrightarrows S, \quad m : G \times_S G \to G, \quad e : S \to G, \quad i : G \to G$$

(or a "category-with-invertible-arrows"),

Group of symmetries of an object
$\rightsquigarrow$ groupoid of symmetries of some objects.

- Smooth version:

A Lie groupoid $\mathcal{G}$ is a "Lie-group-with-a-manifold-of-basepoints" M,

$$(s,t) : \mathcal{G} \rightrightarrows M, \quad m : \mathcal{G} \times_M \mathcal{G} \to \mathcal{G}, \quad e : M \to \mathcal{G}, \quad i : \mathcal{G} \to \mathcal{G}$$

($\mathcal{G}, M$ smooth manifolds, s, t surjective submersions).

- We may identify M with the submanifold of identities $e(M)$ in $\mathcal{G}$.
We assume M is a connected manifold in the following.

Gauge Lie groupoids

- Consider principal bundles $\pi : P \to M$ with structure group G and hence:

  associated vector bundles of the form $E = P \times_G V$ over M
  and the gauge groups $\hat{G} = \mathrm{Aut}_M(P)$

- The gauge groupoid $\mathcal{G}(\pi) \rightrightarrows M$ of a principal bundle $\pi : P \to M$ is

$$P \times P/_{(p_1,p_2)\sim(p_1 g, p_2 g)} \;\rightrightarrows\; M$$

  with composition, identities and inverses induced by

$$(p_1, p_2)(p_2, p_3) = (p_1, p_3)$$
$$e(\pi(p)) = (p, p)$$
$$(p_1, p_2)^{-1} = (p_2, p_1)$$

- A Lie groupoid $\mathcal{G} \rightrightarrows M$ arises as a gauge groupoid $\mathcal{G}(\pi)$ $\iff$ $\mathcal{G}$ is transitive (i.e., 'connected' as a graph).

Examples: the pair and the frame gauge groupoids

- The identity id: $M \to M$ is a principal bundle with gauge groupoid

$$\mathrm{Pair}(M) = M \times M \rightrightarrows M$$

Arrows are $(x, y) : x \to y$ with the obvious composition.
The submanifold of identities is the diagonal $\Delta(M) = \{(x, x)\}$.

- Consider a vector bundle $E \to M$ and its frame bundle $F(E) \to M$.
The fibres of the frame bundle are just the frames on the fibres:

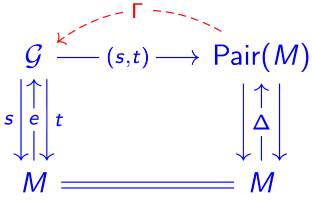
$$F(E)_x = \mathrm{Iso}(\mathbb{R}^r, E_x) \qquad r = \mathrm{rank(E)}, x \in M$$

- The frame bundle is a principal $\mathrm{GL}_r(\mathbb{R})$-bundle with gauge groupoid

$$\mathcal{G}(F(E)) = F(E) \times F(E)_{/\sim} \cong \mathrm{Iso}(E) = \bigcup_{x,y \in M} \mathrm{Iso}(E_y, E_x) \rightrightarrows M$$

Direct connections on a Lie groupoid $\mathcal{G} \rightrightarrows M$

- The anchor $s, t \colon \mathcal{G} \to \mathrm{Pair}(M)$ is a canonical Lie groupoid morphism (preserves source, target, identites, inverses, and composition)



- A **direct connection** Γ is a local section relative to M of the anchor.
- section of the anchor: preserves source & target, $(s,t)\Gamma(x,y) = (x,y)$
- relative to M: preserves the submanifold of identities, $\Gamma\Delta(x) = e(x)$
- local: domain is an open neighbourhood of the submanifold ΔM

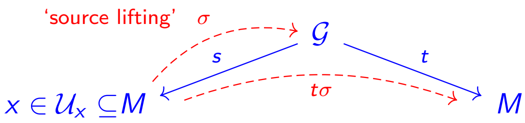
- **Curvature** $R^\Gamma \colon M^3 \dashrightarrow \mathcal{G}$ on an open neighbourhood of the 3-diagonal

$$\Gamma(x,z) \cdot R^\Gamma(x,y,z) \;=\; \Gamma(x,y) \cdot \Gamma(y,z)$$

A direct connection Γ is flat if it is a (partial) groupoid morphism.

Local bisections and jet prolongation of Lie groupoids

- Local bisections $(\sigma, \mathcal{U}_x)$ of $\mathcal{G}$ are sections of $s : \mathcal{G} \to M$ defined in $\mathcal{U}_x$ around $x \in M$, such that $t\sigma$ is a local diffeomorphism on M



diffeomorphism

$t\sigma : \mathcal{U}_x \;\cong\; \mathcal{V}_y$

$t\sigma(x) \;=\; y$

- Local bisections form a groupoid with objects $\mathcal{U}_x$, arrows $\mathcal{U}_x \xrightarrow{\sigma} \mathcal{V}_y$, identities $\mathcal{U}_x \xrightarrow{e_{|\mathcal{U}_x}} \mathcal{U}_x$. The composition $\mathcal{U}_x \xrightarrow{\sigma} \mathcal{V}_y \xrightarrow{\tau} \mathcal{W}_z$ is the local bisection $M \to \mathcal{G}$ sending p in $\mathcal{U}_x$ to $p \xrightarrow{\sigma p} t\sigma p \xrightarrow{\tau t\sigma p} t\tau t\sigma p$ in $\mathcal{G}$.

- Consider n-jets: equivalence classes $j^n_x \sigma = [\sigma, \mathcal{U}_x]$ of local bisections whose derivatives evaluated at x coincide for orders $0 \leq k \leq n$

- The groupoid structure on local bisections induces, on equivalence classes, the n-jet prolongation Lie groupoid $J^n\mathcal{G} \rightrightarrows M$ of $\mathcal{G} \rightrightarrows M$

Jet prolongation of direct connections on Lie groupoids

- Let us start simply, considering some direct connection $\Gamma^{(n)}$ defined on the jet prolongation $J^n\mathcal{G} \rightrightarrows M$ of a Lie groupoid $\mathcal{G} \rightrightarrows M$, i.e. a local section of the anchor $J^n\mathcal{G} \to \mathrm{Pair}(M)$, $[\sigma, \mathcal{U}_x] \mapsto (x, t\sigma x)$ extending $\Gamma^n(x, x) = [e_{|\mathcal{U}_x}]$ to a neighbourhood of the diagonal.

- A principal bundle $P \to M$ has an n-principal prolongation $W^nP \to M$. The gauge groupoid of the prolongation $\mathcal{G}(W^nP) \rightrightarrows M$ coincides with the prolongation of the gauge groupoid $J^n\mathcal{G}(P) \rightrightarrows M$.

- Therefore we can generalise the result that any Lie groupoid admitting a direct connection arises as a gauge groupoid of a principal bundle: If $J^n\mathcal{G} \rightrightarrows M$ has a direct connection $\Gamma^{(n)}$, then its constant term (evaluate each j_x at x) defines a local section $\Gamma^{(0)}$ of the anchor of $\mathcal{G}$. So $\mathcal{G}$ has a direct connection and is a gauge groupoid of some $P \to M$ Thus: a prolonged groupoid $J^n\mathcal{G}$ with a direct connection is the gauge groupoid of a prolonged principal bundle W^nP.

Infinitesimal structure of Lie groupoids $\mathcal{G} \rightrightarrows M$

- To define the Lie algebroid $\mathcal{L}\mathcal{G} \xrightarrow{q} M$ of a Lie groupoid $\mathcal{G}$, consider
  the tangent maps $ds, dt : T\mathcal{G} \rightrightarrows TM$,
  the subbundle $\ker ds \to \mathcal{G}$ of $T\mathcal{G} \to \mathcal{G}$,
  and its pullback $\mathcal{L}\mathcal{G}$ along $e : M \to \mathcal{G}$.

$$\begin{array}{ccc} \mathcal{L}(G) & \xrightarrow{i} & \ker ds \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow \\ M & \xrightarrow{e} & \mathcal{G} \end{array}$$

- The pullback just says $(\mathcal{L}\mathcal{G})_x \cong T_{e(x)} s^{-1}(x)$ for each $x \in M$.
- The Lie bracket of vector fields on $\mathcal{G}$ restricts to induce, by right translation, a bracket on sections of q, and the anchor is

$$a : \mathcal{L}\mathcal{G} \xrightarrow{i} \ker ds \hookrightarrow T\mathcal{G} \xrightarrow{dt} TM.$$

- The tangent bundle $TP \xrightarrow{q} P$ with the bracket of vector fields can be seen as the Lie algebroid of the gauge groupoid $\mathrm{Pair}(P) \rightrightarrows P$
- The Atiyah Lie algebroid $A(P) \xrightarrow{q} M$ on a principal bundle $P \to M$ is the Lie algebroid $\mathcal{L}\mathcal{G}(P)$ of the gauge Lie groupoid $\mathcal{G}(P) \rightrightarrows M$, or alternatively is the quotient TP/G of the tangent bundle TP by G.

Jet prolongation of direct connections on Lie groupoids II

- Suppose M has an affine connection and we have parallel transport $\tau^M(y,x) : T_xM \longrightarrow T_yM$ (along unique small geodesics).
- One can then define local bisections $(\sigma, \mathcal{U}_x) : \mathcal{U}_x \to \mathcal{V}_y$ for Pair(M)

$$\sigma(z) = \Big(z,\ \exp_z(\tau^M(z,x)(\exp_x^{-1}(y)))\Big)$$

whose jets define a direct connection on $J^n\text{Pair}(M)$.

- Given a bundle $E \to M$ and principal bundle $P \subset F(E)$, and a direct connection Γ on the gauge groupoid $\mathcal{G}(P) \rightrightarrows M$, one constructs a **model** $(\Pi^n, \hat{\Gamma}^n)$ using 'short geodesics' $x \overset{\gamma}{\rightsquigarrow} z$:
- The direct connection $\{\hat{\Gamma}^n(y,x)\}$ on $J^n\mathcal{G}(P)$ sends $j^n_x f$ to $j^n_y(\Pi^n_x j^n_x f)$.
- Here $\Pi^n_x : J^n_x E \to \mathcal{D}'(\mathcal{U}_x, E)$ sends a jet $j^n_x f$ to the distribution

$$z \longmapsto \Gamma(z,x) \sum_{k=0} \frac{1}{k!} \frac{d^k}{dt^k} \Gamma(x,\gamma(t)) f(\gamma(t)) \Big|_{t=0}$$

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Thank you!