Generic Hecke algebra and theta correspondence over finite fields

Ma, Jia-Jun

(joint with Congling Qiu and Jialiang Zou)

School of Mathematical Sciences
Xiamen University

April 14, 2022

Setting

- $F := \mathbb{F}_q$ a finite field, s.t. $|F| = q$ and $\mathrm{Char} F = p$.
- $F_0 := F$ or $F_0 = \mathbb{F}_{\sqrt{q}}$.
- V: a finite dimensional ϵ-Hermitian space over F.
- $\mathrm{U}(V)$ the isometry group of V.
- (V, V') dual pair of Hermitian spaces ($\epsilon \cdot \epsilon' = -1$)

	$\mathrm{U}(V)$	$\mathrm{U}(V')$	
(A)	unitary gp.	unitary gp.	
(B)	odd orthogonal gp.	"metaplectic" gp.	
(D)	even orthogonal gp.	symplectic gp.	$p \neq 2$
(C)	symplectic gp.	even orthogonal gp.	
$(\widetilde{C})$	"metaplectic" gp.	odd orthogonal gp.	

Theta lifting

- $\mathrm{U}(V) \times \mathrm{U}(V') \longrightarrow \mathrm{U}(V \otimes_F V')$
- $\omega_{V \otimes V'}$ Weil representationo of $\mathrm{U}(V \otimes_F V')$ (by Gérardin)
- ξ the quadratic character of $F^\times$
- (modified) Weil representation

$$\omega_{\psi,V,V'} := \begin{cases} \omega_{V \otimes_F V'} & \text{type } (A), \\ \left((\xi \circ \det_V)^{\frac{1}{2}\dim_F V'} \boxtimes \mathbf{1}\right) \otimes \omega_{V \otimes_F V'} & \text{type } (B)\ (D), \\ \left(\mathbf{1} \boxtimes (\xi \circ \det_{V'})^{\frac{1}{2}\dim_F V}\right) \otimes \omega_{V \otimes_F V'} & \text{type } (C)\ (\widetilde{C}), \end{cases}$$

- Orthogonal gp. acts geometrically in Schrödinger model.
- Theta lifting

$$\begin{array}{rcl} \Theta_{V,V'} \colon \mathrm{Rep}(\mathrm{U}(V)) & \longrightarrow & \mathrm{Rep}(\mathrm{U}(V')) \\ \sigma & \mapsto & \mathrm{Hom}_{\mathrm{U}(V)}(\sigma, \omega_{V,V'}) \end{array}$$

Certain parabolic induction

- $V_l = V \oplus \mathbb{H}^l$ ($\mathbb{H}$ the hyperbolic space)
- Fix a parabolic subgp. P_l of $\mathrm{U}(V_l)$ with Levi

$$L_l = \underbrace{\mathrm{GL}_1(F) \times \cdots \times \mathrm{GL}_1(F)}_{l\text{-terms}} \times \mathrm{U}(V)$$

- Character of $\mathrm{GL}_1(F)$:

$\chi := \xi$ in type $(\widetilde{C})$, or $\mathbf{1}$ otherwise.

- For $\sigma \in \mathrm{Rep}(\mathrm{U}(V))$,

$$\mathfrak{J}_l(\sigma) := \mathrm{Ind}_{P_l}^{\mathrm{U}(V_l)} \underbrace{\chi \boxtimes \cdots \boxtimes \chi}_{l\text{-terms}} \boxtimes \sigma.$$

- σ is *θ-cuspidal*

  $\Leftrightarrow$ σ dose not occur in $\mathfrak{J}_r(\sigma_0)$ for any r and σ_0 where $V \cong V_0 \oplus \mathbb{H}^r$ and $\sigma_0 \in \mathrm{Rep}(\mathrm{U}(V_0))$.
- cuspdial $\Rightarrow$ θ-cuspidal

θ-cuspdial representation

- $\mathcal{V}'$: a Witt tower
- First occurence index

$$n_{V,\mathcal{V}'}(\sigma) := \min \{ \dim V' \mid \Theta_{V,V'}(\sigma) \neq 0, V' \in \mathcal{V}' \}$$

- Assume σ is θ-cuspidal.

  Arguments in Mœglin-Vignéras-Waldspurger $\Rightarrow$

- $\Theta_{V,V'}(\sigma)$ is irreducible and θ-cuspidal at first occurrence.
- $\Theta_{V,V'}(\sigma)$ is irreducible and non-θ-cuspidal after first occurrence.

Conservation relation I

- Assume σ is θ-cuspidal.

  Theorem (Conservation relation) Let $\widetilde{\mathcal{V}}'$ be the companion Witt tower.

$$n_{V,\mathcal{V}'}(\sigma) + n_{V,\widetilde{\mathcal{V}}'}(\sigma) = 2\dim V + \delta$$

- Type (A): $\delta = 1$, parity of $\widetilde{\mathcal{V}}' \neq$ parity of $\mathcal{V}'$.
- Type (C),$(\widetilde{C})$: $\delta = 2$, parity of $\widetilde{\mathcal{V}}' =$ parity of $\mathcal{V}'$
  disc. of $\widetilde{\mathcal{V}}' \neq$ disc. of $\mathcal{V}'$.
- Type (B),(D): $\delta = 0$, $\widetilde{\mathcal{V}}' = \mathcal{V}'$, but define

$$n_{V,\widetilde{\mathcal{V}}'}(\sigma) := \min\{\dim V' \mid \mathrm{Hom}_{\mathrm{U}(V)}(\sigma, \omega_{V,V'} \otimes \det) \neq 0, V' \in \widetilde{\mathcal{V}}'\}$$

- By Sun-Zhu, δ is the max. dim. of an anisotropic ϵ'-Hermition space.

Hekce algebra attached to σ (θ-cuspidal)

- $\mathcal{H}^{op} := \mathrm{End}_{\mathrm{U}(V_l)}(\mathfrak{I}_l(\sigma))$ acts on $\mathrm{Hom}_{\mathrm{U}(V_l)}(\mathfrak{I}_l(\sigma), \omega_{V_l,V'_{l'}})$ by right composition.

  We prefer left action!
- $\mathcal{H} := \{ f \colon G \to \mathrm{End}_{\mathbb{C}}(\sigma_l^\vee) \mid f(p_1 g p_2) = \sigma_l^\vee(p_1) f(g) \sigma_l^\vee(p_2) \}$.
- $W = \mathrm{Norm}_{\mathrm{U}(V_l)}(L_l)/L_l \cong \mathsf{W}_l := \mathsf{S}_l \ltimes \{ \pm 1 \}^l$.
- Tits' lifting: $W \to \mathrm{Norm}_{\mathrm{U}(V_l)}(L_l)$, $w \mapsto \dot{w}$.
- A basis of $\mathcal{H}$ (as a vector space):

  $\{ T_w \mid w \in W \}$ such that

$$\mathrm{Supp}(T_w) = P_l\, \dot{w}\, P_l$$

$$T_w(\dot{w}) = \mathrm{id}_{\sigma_l^\vee}$$

- Howlett-Lehrer: $\mathcal{H}$ is isomorphic to the Hecke algebra of W_l

⚠ with unequal parameters

⚠ upto a 2-cocycle (from the extension of σ_l to $\mathrm{Norm}_{\mathrm{U}(V_l)}(L_l)$).

Lusztig (unipotent repn.), Geck (in general)

Hecke algebra

⚠ 2-cocycle

Howlett-Lehrer (Lemma 6.5):

$\mathcal{H}$ has a 1-dim repn. $\Rightarrow$ 2-cocycle is trivial.

$\mathrm{Hom}_{\mathrm{U}(V_l)\times\mathrm{U}(V')}(\mathfrak{I}_l(\sigma)\otimes\sigma', \omega_{V_l,V'})$ is 1-dim,

if σ' is the first occurrence lifting of σ.

⚠ parameters of $\mathcal{H}$?

The quadratic relation (s is a simple reflection)

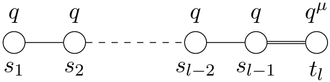
$$(T_s - C_1)(T_s - C_2) = 0$$

can be determined if we consider two different 1-dim repn. of $\mathcal{H}$.

$\exists$ another 1-dim repn. by changing Witt tower!

Hecke algebra of W_l

- Dynkin diagram



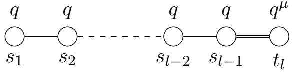
- Hecke algebra $\mathcal{H}_l(q, q^\mu)$ is the algebra over $\mathbb{C}$ with
  1. basis $\{ \mathcal{T}_w \mid w \in \mathsf{W}_l \}$ such that
  2. $\mathcal{T}_{w_1 w_2} = \mathcal{T}_{w_1}\mathcal{T}_{w_2}$ when $l(w_1 w_2) = l(w_1) + l(w_2)$,
  3. $(\mathcal{T}_{\mathsf{s}_i} + 1)(\mathcal{T}_{\mathsf{s}_i} - q) = 0, i = 1, \cdots, l-1$,
  4. $(\mathcal{T}_{\mathsf{t}_l} + 1)(\mathcal{T}_{\mathsf{t}_l} - q^\mu) = 0$.
- generic Hecke algebra $\mathscr{H}_l(\nu, \nu^\mu)$ (free over $\mathbb{Z}[\nu^{\frac{1}{2}}, \nu^{-\frac{1}{2}}]$)

  replace q by an indeterminant ν.

$\mathcal{H}$-action

- Assume $\sigma \overset{\Theta}{\longleftrightarrow} \sigma'$, σ and σ' θ-cuspidal
- $\mathcal{M} := \mathrm{Hom}_{P_l \times \mathrm{U}(V')}(\sigma_l \otimes \sigma', \omega_{V_l, V'})$
- T_w (recall $T_w(\dot{w}) = \mathrm{id}_{\sigma_l^\vee}$) and $\phi \in \mathcal{M}$

$$T_w \cdot \phi = \sum_{u \in U_w} u\dot{w} \circ \phi, \qquad U_w = \prod_{\substack{\alpha>0 \\ w\alpha<0}} U_\alpha$$



- $T_{w_1 w_2} = T_{w_1} T_{w_2}$ if $l(w_1 w_2) = l(w_1) + l(w_2)$
- $(T_{s_k} + 1)(T_{s_k} - q) = 0$ for $k = 1, 2, \cdots, l-1$
- $(T_{t_l})^2 = C\, T_{k_l} - A$

  $A = -\chi(-1)\,|U_{t_l}| = -\chi(-1) q^{\dim V + \frac{1}{2}\delta}$

  $C = C_1 + C_2,\ A = C_1 C_2$

Fourier Transform

- $V_l = V_l^+ \oplus V \oplus V_l^-$.
- Mixed model:

$$\omega_{V_l,V'} = \mathbb{C}[\mathrm{Hom}_F(V_l^+, V')] \otimes \omega_{V,V'} \tag{$\star$}$$

- $\mathcal{I}_0 \in \mathcal{M} \subset \mathbb{C}[\mathrm{Hom}_F(V_l^+, V')] \otimes \mathrm{Hom}_{\mathrm{U}(V)\times\mathrm{U}(V')}(\sigma \otimes \sigma', \omega_{V,V'})$.
- $\mathrm{Supp}\,\mathcal{I}_0 = \{\,0\,\} \subset \mathrm{Hom}_F(V_l^+, V')$.
- $\omega_{V,V'}(\dot{t}_l)$ acts on $(\star)$ by partial Fourier transform.

$$T_{t_l} \cdot \mathcal{I}_0 = \underbrace{\sum_{u \in U_{t_l}} \omega_{V_l,V'}(u)}_{\text{truncation}} \underbrace{\omega_{V_l,V'}(\dot{t}_l)}_{\text{Fourier}} \mathcal{I}_0 = C_1\,\mathcal{I}_0$$

- $C_1 = \underbrace{\gamma_{V'}}_{\text{Weil index}} |U_{t_l}| q^{-\frac{1}{2}\dim V'}$

Normalization of $\mathcal{H}$

- $\widetilde{V}'$ the first occurrence of σ w.r.t. the companion Witt tower $\widetilde{\mathcal{V}}'$.
- $C_1 = \gamma_{V'} |U_{t_l}| q^{-\frac{1}{2}\dim V'}$
- $C_2 = \gamma_{\widetilde{V}'} |U_{t_l}| q^{-\frac{1}{2}\dim \tilde{V}'}$
- $\gamma_{V'} / \gamma_{\widetilde{V}'} = -1$
- $C_1 C_2 = A = -\chi(-1)|U_{t_l}| = -\chi(-1) q^{\dim V + \frac{1}{2}\delta}$

  $\Rightarrow$ Conservation relation

$$n_{V,\mathcal{V}'}(\sigma) + n_{V,\widetilde{\mathcal{V}}'}(\sigma) = 2\dim V + \delta$$

- Normalization: $\mathcal{T}_{t_l} := -C_1^{-1} T_{t_l}$.
- $\mu = \mu(\sigma) = \frac{1}{2}(n_{V,\mathcal{V}'}(\sigma) - n_{V,\widetilde{\mathcal{V}}'}(\sigma))$
- $$\mathcal{M} \overset{\text{Tits deform.}}{\leftrightsquigarrow} \quad \varepsilon_l : \quad \begin{array}{ccc} \mathsf{S}_l \ltimes \{\pm 1\}^l & \rightarrow & \{\pm 1\} \\ (s, (a_1, \cdots, a_l)) & \mapsto & a_1 \cdots a_l \end{array}$$

Main theorem of the general case

- $\mathcal{M} := \mathrm{Hom}_{P_l \times P'_{l'}}(\sigma_l \otimes \sigma'_{l'}, \omega_{V_l, V'_{l'}})$

Goal Understand $\mathcal{M}$ as $\mathcal{H} \times \mathcal{H}'$-module

- $\mathscr{H} := \mathscr{H}_l(\nu, \nu^{\mu(\sigma)})$ an algebra free over $R := \mathbb{Z}[\nu^{\frac{1}{2}}, \nu^{-\frac{1}{2}}]$.
- specialization of $\mathscr{H}$-module $\mathscr{M}$ at $q := \mathscr{M} \otimes_R \mathbb{C}_q$ $(\nu \mapsto q)$.

Main Theorem

There is an $\mathscr{H} \times \mathscr{H}'$-module $\mathscr{M}$, free over R (constructed explicitly) such that

- specialization of $\mathscr{M}$ at $q = \mathcal{M}$
- specialization of $\mathscr{M}$ at $1 \cong \displaystyle\sum_{k=0}^{\min\{l,l'\}} \mathrm{Ind}_{\mathsf{W}_{l-k} \times \triangle \mathsf{W}_k \times \mathsf{W}_{l'-k}}^{\mathsf{W}_l \times \mathsf{W}_{l'}} \varepsilon_{l-k} \boxtimes \varepsilon_k \boxtimes \varepsilon_{l'-k}$.

- $+$ Adams-Moy $\Rightarrow$ Aubert-Michel-Rouquier and Pan
- Lusztig/AMR's normalization: $\mu(\sigma), \mu(\sigma') \geq 0$

Conservation relation in the general case

- Suppose $\sigma \in \mathrm{Irr}(\mathrm{U}(V))$.
- $c(\sigma) = \max \{ l > 0 \mid \exists \sigma'$ s. t. σ occurs in $\mathfrak{J}_l(\sigma') \}$
- $c(\sigma) = 0 \Leftrightarrow \sigma$ is θ-cuspidal

  Conservation relation

$$n_{V,\mathcal{V}'}(\sigma) + n_{V,\widetilde{\mathcal{V}}'}(\sigma) + c(\sigma) = 2\dim V + \delta$$

- Obtained by Pan (2019) via a reduction to the unipotent case.

Kudla's filtration

- Assume $\sigma \overset{\Theta}{\longleftrightarrow} \sigma'$, σ and σ' θ-cuspidal
- $\mathcal{M} := \mathrm{Hom}_{P_l \times P'_{l'}}(\sigma_l \otimes \sigma'_{l'}, \omega_{V_l, V'_{l'}})$
- $V_l = V_l^+ \oplus V \oplus V_l^-$ and $V'_{l'} = V_{l'}^+ \oplus V' \oplus V_{l'}^-$
- Mixed model: $\mathcal{M} \subset \mathbb{C}[\mathrm{Hom}(V_l^+, V'_{l'})] \otimes \mathrm{Hom}_{\mathbb{C}}(\sigma \otimes \sigma', \omega_{V, V'_l})$
- $\mathcal{Z} := \left\{ A \in \mathrm{Hom}(V_l^+, V'_{l'}) \;\middle|\; \begin{array}{l} \mathrm{Im}\, A \text{ is isotropic} \\ \mathrm{Im}\, A \cap V' = 0 \end{array} \right\}$
- $\mathcal{Z}_k := \mathcal{Z} \cap \{ \text{rank } k \text{ maps} \}$
- B'' Borel subgroup of $\mathrm{GL}(V_l^+)$, $\mathcal{H}'' := \mathbb{C}[B'' \backslash \mathrm{GL}(V_l^+)/B'']$.
- Kudla's filtration $\Rightarrow$

$$\bigoplus_k \mathbb{C}[(B'' \times P'_{l'}) \backslash \mathcal{Z}_k] \xrightarrow{\text{vec. sp. iso.}} \bigoplus_k \mathcal{F}_k = \mathcal{M}$$

- $\mathcal{M}$ is a $\mathcal{H}'' \times \mathcal{H}'_{l'}$-module $\rightsquigarrow \mathscr{H}'' \times \mathscr{H}'_{l'}$-module $\mathscr{M}$
- $\mathscr{M}_{\nu=1} = \bigoplus_k \mathrm{Ind}_{\mathsf{S}_{l-k} \times \triangle \mathsf{S}_k \times \mathsf{W}_{l'-k}}^{\mathsf{S}_l \times \mathsf{W}_{l'}} \mathbf{1}_{l-k} \boxtimes \mathbf{1}_k \boxtimes \varepsilon_{l'-k}$

Get hands dirty

- $\mathcal{A}_k$ the minimal dim. orbit in $(B'' \times P'_l)\backslash\mathcal{Z}_k$

$\leftrightsquigarrow$ generater $\mathcal{I}_k$ of $\mathcal{F}_k$ under $\mathcal{H}'' \times \mathcal{H}'_{l'}$-action ($\operatorname{Supp}\mathcal{I}_k = \mathcal{A}_k$).

- $t_k := (1, (1, \cdots, 1, -1, 1, \cdots, 1)) \in \mathsf{W}_l = \mathsf{S}_l \ltimes \{\pm 1\}^l$.
- Compute the $\mathcal{T}_{t_l}$, $\mathcal{T}'_{t'_k}$, and $\mathcal{T}_{t_{l-k}}$-actions on $\mathcal{I}_k$

$$\begin{aligned}\mathcal{T}_{t_l} \cdot \mathcal{I}_k = &- q^{(\dim V' - \dim V + k - l' - \frac{1}{2}\delta)} \mathcal{T}'_{t'_k} \cdot \mathcal{I}_k \\ &+ (1 - q^{-1})(\text{terms in } \mathcal{F}_{k+1} + \mathcal{F}_k + \mathcal{F}_{k-1})\end{aligned}$$

$$\begin{aligned}\mathcal{T}_{t_{l-k}} \cdot \mathcal{I}_k = &- q^{2k-l'} \mathcal{I}_k \\ &+ (q-1)(\text{terms in } \mathcal{F}_k) \\ &+ \text{terms in } \mathcal{F}_{k+1}\end{aligned}$$

- all coefficients are in $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$

Specialization at $\nu = 1$

- σ is in a family of repn. (q varies), see Reeder.

$\Rightarrow$ replace q by ν gives a $\mathscr{H} \times \mathscr{H}'$-module $\mathscr{M}$.

- Specialize at $\nu = 1 \rightsquigarrow$ a filtration of $\mathsf{W}_l \times \mathsf{W}_{l'}$-module

$$0 = \mathcal{F}_{\geq r+1} \subset \mathcal{F}_{\geq r} \subset \mathcal{F}_{\geq r-1} \subset \cdots \subset \mathcal{F}_{\geq 1} \subset \mathcal{F}_{\geq 0} = \mathcal{M}$$

($r = \min\{\, l, l' \,\}$) whose graded piece

$$\mathcal{F}_{\geq k}/\mathcal{F}_{\geq k+1} \cong \mathrm{Ind}_{\mathsf{W}_{l-k} \times \triangle \mathsf{W}_k \times \mathsf{W}_{l'-k}}^{\mathsf{W}_l \times \mathsf{W}_{l'}} \varepsilon_{l-k} \boxtimes \varepsilon_k \boxtimes \varepsilon_{l'-k}.$$

Chongling Qiu

Jia-Jun Ma

Jialiang Zou

Generic Hecke algebra and theta correspondence over finite fields
in preparation

IMS-NUS, Singapore, 2012

Happy birthday Gordan!