

# Ward Schwinger Dyson equations

---

Enrico I. Russo & Marc P. Bellon

14 Octobre 2020

ESI: Higher Structures Emerging from Renormalisation

# Table of contents

Why WSD

The  $\phi_6^3$  model

General procedure

Corrections

# Why WSD

---

# Introduction

- pQFT gives divergent series
- N-P physics might be hidden in these infinities
- Resurgence needs a differential equation to work well

Idea: SD and RG equations play this role.

# The $\phi_6^3$ model

---

$$\text{---}\bullet\text{---} = \text{---} - \frac{1}{2} \text{---}\bullet\text{---}\bullet\text{---}$$

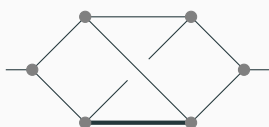
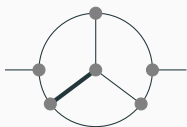
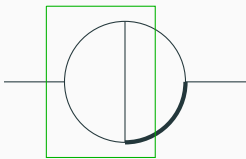
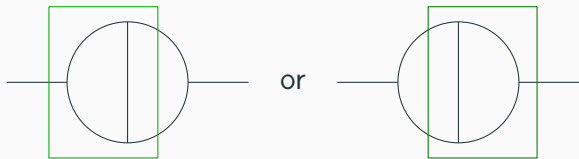
$$\text{---}\bullet\text{---} = \text{---}\text{---} + \text{---}\bullet\text{---}\bullet\text{---}$$

$$\text{---}\bullet\text{---}\bullet\text{---} = \text{---}\bullet\text{---}\bullet\text{---} + \frac{1}{2} \text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---} + \text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---} + \dots$$

$$\text{---} \bullet \text{---} = \text{---} - \frac{1}{2} \text{---} \bullet \text{---} \bullet + \frac{1}{2} \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---}$$

$$\text{---} \bullet \text{---} \bullet = \text{---} \text{---} + 2 \text{---} \text{---} + 3 \text{---} \text{---} + 2 \text{---} \text{---} + \dots$$

$$\text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet = \text{---} \text{---} + 2 \text{---} \text{---} + \text{---} \text{---} + \dots$$





# Ward's trick

$$\text{---} \blacksquare \text{---}^{\mu} := \partial^{\mu} \text{---} \bullet \text{---} = \text{---}^{\mu} \text{---} \blacksquare - \frac{1}{2} \text{---} \blacksquare \text{---} \text{---} \text{---} \text{---} - \frac{1}{2} \text{---} \bullet \text{---} \text{---} \text{---} \text{---} \blacksquare \text{---}$$

$$\text{---} \blacksquare \text{---}^{\mu} = \text{---} \blacksquare \text{---} \text{---} \text{---} \text{---} + \text{---} \bullet \text{---} \text{---} \text{---} \text{---} \text{---} + \text{---} \bullet \text{---} \text{---} \text{---} \text{---} \text{---} \blacksquare \text{---}$$

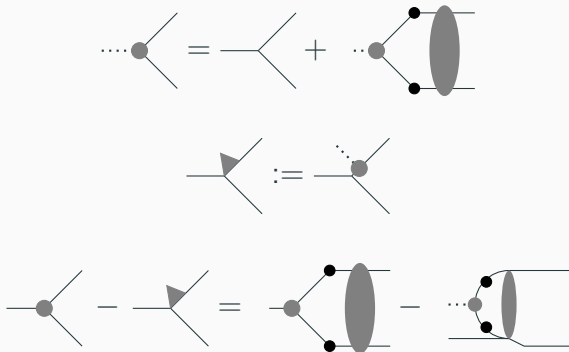
$$\begin{aligned}
 \text{---} \blacksquare \text{---}^\mu &= \text{---}^\mu - \frac{1}{2} \left( \text{---} \blacksquare \text{---}^\mu - \text{---} \blacksquare \text{---}^\mu + \right. \\
 &\quad \left. + \text{---} \bullet \text{---}^\mu - \text{---} \blacksquare \text{---}^\mu \right)
 \end{aligned}$$

The diagrammatic equation shows the expansion of a self-energy correction. On the left, a horizontal line with a black square vertex is labeled with a superscript  $\mu$ . This is equal to the bare line with a superscript  $\mu$  minus a factor of  $\frac{1}{2}$  times a sum of four diagrams in parentheses. The first diagram is a circle with a black square on the left and a black dot on the top, with a superscript  $\mu$  on the right. The second diagram is a fish diagram with a black square on the left and a gray oval in the center, with a superscript  $\mu$  on the right. The third diagram is a circle with a black dot on the top and a black square on the bottom, with a superscript  $\mu$  on the right. The fourth diagram is a fish diagram with a black square on the bottom and a gray oval in the center, with a superscript  $\mu$  on the right.

$$\text{---} \blacksquare \text{---}^\mu = \text{---}^\mu - \frac{1}{2} \text{---} \bullet \text{---}^\mu - \frac{1}{2} \text{---} \blacksquare \text{---}^\mu$$

This diagrammatic equation shows a simplified version of the expansion. The left side is the same as in the first equation. The right side is the bare line with a superscript  $\mu$  minus a factor of  $\frac{1}{2}$  times a circle diagram with a black dot on top and a black square on bottom, with a superscript  $\mu$  on the right, minus another factor of  $\frac{1}{2}$  times a fish diagram with a gray rectangle in the center and a superscript  $\mu$  on the right.

# IR rearrangement



$$\begin{aligned}
 & \text{Diagram 1} = \text{Diagram 2} + 2 \left( \text{Diagram 3} - \text{Diagram 4} \right) + \\
 & + \left( \text{Diagram 5} - \text{Diagram 6} - \text{Diagram 7} + \text{Diagram 8} \right)
 \end{aligned}$$

The diagrammatic equation shows the decomposition of a circle with two dots and a square. The left side is a circle with two dots on the horizontal line and a square at the bottom. The right side is a sum of terms: a circle with two triangles and a square; a term with a coefficient of 2 multiplied by the difference of two diagrams with a vertical oval and a dot; and a term with a coefficient of 1 multiplied by the sum of four diagrams with two vertical ovals and various dot/triangle configurations.

# Renormalised ingredients

$$\begin{aligned}
 \text{---} \blacksquare \text{---}^\mu &= \text{---}^\mu - \frac{T_2}{2} \text{---} \circ \text{---} \\
 \text{---} \bullet \text{---} &= \text{---} \text{---} + T_3 \text{---} \text{---} \text{---}
 \end{aligned}$$

$$\begin{cases}
 G(a, L) := \sum_{n \geq 0} \gamma_n(a) \frac{L^n}{n!} & \text{for } \text{---} \bullet \text{---} \\
 Y(a, L) := \sum_{n \geq 0} \nu_n(a) \frac{L^n}{n!} & \text{for } \text{---} \blacktriangle \text{---} \\
 K^\nu(a, L) := \frac{2p^\nu}{(p^2)^2} \sum_{n \geq 0} (\gamma_{n+1} - \gamma_n) \frac{L^n}{n!} & \text{for } \text{---} \blacksquare \text{---} \\
 \mathbb{F}(a, L) := YGY(a, L) = \sum_{n \geq 0} s_n(a) \frac{L^n}{n!} & \text{for } \text{---} \blacktriangle \bullet \blacktriangle \text{---} \\
 \mathbb{P}(a, L) := GYG(a, L) = \sum_{n \geq 0} q_n(a) \frac{L^n}{n!} & \text{for } \text{---} \bullet \blacktriangle \bullet \text{---}
 \end{cases}$$

$$\partial_L G = (\gamma + \beta a \partial_a) G$$

$$\partial_L Y = (v + \beta a \partial_a) Y$$

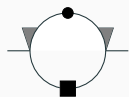
$$\partial_L \Xi = (\gamma + 2v + \beta a \partial_a) \Xi$$

$$\partial_L \Phi = (2\gamma + v + \beta a \partial_a) \Phi$$

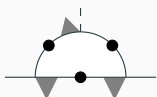
What about  $K^\nu$ ? *WSD* impose a *RG* equation on it and provide a recipe for  $\beta = 3\gamma + 2v$ !

$$L \leftrightarrow \partial \qquad L^n = \partial_x^n e^{xL} \sim \partial_x^n (p^2)^x$$

# The Cat


$$\begin{aligned} &:= a \int_{\mathbb{R}^6} \frac{du}{(2\pi)^6} \mathbb{F}(u) K^\nu(u+p) \\ &= a G_x \mathbb{F}_y \partial^\nu \left( (p^2)^{1+x+y} \frac{\Gamma(-1-x-y)\Gamma(2+x)\Gamma(2+y)}{\Gamma(4+x+y)\Gamma(1-x)\Gamma(1-y)} \right) \\ H^\gamma(x,y) &= \frac{\Gamma(1-x-y)\Gamma(2+x)\Gamma(2+y)}{\Gamma(4+x+y)\Gamma(1-x)\Gamma(1-y)} \end{aligned}$$

# The Walrus



$$:= a \int_{\mathbb{R}^6} \frac{du}{(2\pi)^6} \Phi(u) \Xi(u+p)$$

$$= a \Phi_x \Xi_y e^{(x+y)L} \frac{\Gamma(-x-y)\Gamma(1+x)\Gamma(2+y)}{\Gamma(3+x+y)\Gamma(2-x)\Gamma(1-y)}$$

$$H^v(x, y) = \frac{\Gamma(1-x-y)\Gamma(1+x)\Gamma(2+y)}{\Gamma(3+x+y)\Gamma(2-x)\Gamma(1-y)}$$



# General procedure

---

$$\mathcal{W} \in \{ \text{---}, \text{---}\blacktriangle, \blacktriangle\text{---}, \text{---}\blacktriangle\blacktriangle, \blacktriangle\blacktriangle\text{---}, \blacktriangle\blacktriangle\blacktriangle \}$$

$$\partial_L \mathcal{W} = (w + \beta a \partial_a) \mathcal{W}$$

$$w := \#G \gamma + \#Y v$$

$$\mathcal{O} = a\mathcal{W}_1\mathcal{W}_2$$

$$\mathcal{O}_\bullet = \begin{cases} aG_\bullet \mathbb{F}_\bullet & \text{for } \begin{array}{c} \bullet \\ \text{---} \circ \text{---} \\ \blacktriangle \quad \blacktriangle \\ \text{---} \\ \blacksquare \\ \text{---} \\ \text{---} \end{array} \\ a\Phi_\bullet \mathbb{F}_\bullet & \text{for } \begin{array}{c} \bullet \\ \text{---} \text{---} \text{---} \\ \blacktriangle \quad \blacktriangle \\ \text{---} \\ \bullet \\ \text{---} \\ \blacktriangle \quad \blacktriangle \end{array} \end{cases}$$

$$\gamma_{\mathcal{O}} := \#G \gamma + \#Y v - \beta$$

$$\partial_L \mathcal{O} = (\gamma_{\mathcal{O}} + \beta a \partial_a) \mathcal{O}$$

# The final form of $WSD$

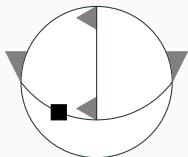
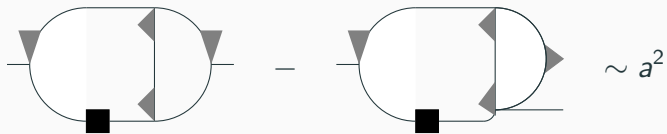
$$\begin{aligned}
 \text{---} \blacksquare \text{---}^\mu &= \text{---}^\mu - \frac{T_2}{2} \text{---} \circlearrowleft \blacksquare \\
 \text{---} \bullet \text{---} &= \text{---} \text{---} \text{---} + T_3 \text{---} \text{---} \text{---} \text{---}
 \end{aligned}$$

$$\gamma(a) = (\gamma - \beta a \partial_a) \gamma + \frac{T_2}{2} \mathcal{O}_\gamma H^\gamma(x, y)$$

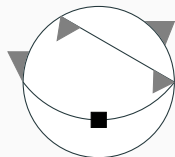
$$v(a) = -a T_3 \mathcal{O}_v H^v(x, y)$$

# Corrections

---



$$\psi_{\tilde{1}} = t_6 \psi_1 + \phi_1$$



$$\psi_{\tilde{2}} = t_6 \psi_2 + \phi_2$$

$$\psi_{\tilde{1}} - \psi_{\tilde{2}} = \phi_1 - \phi_2$$

$$\int dt_1 \dots dt_6 t_5 t_6^3 \left( \frac{C_{\tilde{1}}}{\psi_{\tilde{1}}^4} - \frac{C_{\tilde{2}}}{\psi_{\tilde{2}}^4} \right) \delta_H.$$

$$C_{\tilde{1}} = t_4(t_1 + t_2 + t_3) + t_3 t_2$$

$$C_{\tilde{2}} = t_4(t_1 + t_2 + t_3) + (t_1 + t_2)t_3$$

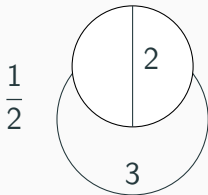
$$\begin{aligned} \frac{C_{\tilde{1}}}{\psi_{\tilde{1}}^4} - \frac{C_{\tilde{2}}}{\psi_{\tilde{2}}^4} &= C_{\tilde{1}} \left( \frac{1}{\psi_{\tilde{1}}^4} - \frac{1}{\psi_{\tilde{2}}^4} \right) - \frac{t_1 t_3}{\psi_{\tilde{2}}^4} \\ &= C_{\tilde{1}} (\psi_{\tilde{2}} - \psi_{\tilde{1}}) \left( \frac{1}{\psi_{\tilde{1}}^4 \psi_{\tilde{2}}} + \frac{1}{\psi_{\tilde{1}} \psi_{\tilde{2}}^4} + \frac{1}{\psi_{\tilde{1}}^2 \psi_{\tilde{2}}^3} + \frac{1}{\psi_{\tilde{1}}^3 \psi_{\tilde{2}}^2} \right) + \\ &\quad - \frac{t_1 t_3}{\psi_{\tilde{2}}^4}. \end{aligned}$$

The diagram shows a sequence of three circles connected by approximation symbols ( $\approx$ ). The first circle is divided into three regions by two curved arcs. The top-left region is labeled '3', the top-right region is labeled '1', and the bottom region is labeled '4'. The two internal arcs are each labeled '2'. This circle is approximately equal to the second circle, which is divided into two regions by a horizontal line. The top region is labeled '3' and the bottom region is labeled '3'. This second circle is approximately equal to the third circle, which is a smaller circle divided into two regions by a horizontal line. The top region is labeled '3' and the bottom region is labeled '1'. The mathematical expression  $\frac{\Gamma(0)}{\Gamma(4)}$  is placed between the first and second circles, and  $\frac{\Gamma(0)}{24}$  is placed between the second and third circles.

$$\text{Circle 1} \approx \frac{\Gamma(0)}{\Gamma(4)} \text{ Circle 2} \approx \frac{\Gamma(0)}{24} \text{ Circle 3}$$

So this contributes to  $\gamma$  with a factor  $-\frac{a^2}{24}$ .





$$\frac{1}{2} \int_{\mathbb{R}_+^6} dt_1 \dots dt_6 \frac{t_5 t_6^2}{\tilde{\psi}^3} \delta_H$$

This contributes to  $v$  with a factor of  $\frac{a^2}{4}$ .

# Result

---

# Results

$$\gamma = \frac{T_2}{12}a + (-11T_2 + 48T_3) \frac{T_2}{432}a^2$$

$$v = -\frac{T_3}{2}a + \left( -\frac{T_5}{4} + \frac{T_3}{16}(T_2 - 6T_3) \right) a^2$$

$$\beta = (T_2 - 4T_3) \frac{a}{4} + (-11T_2^2 + 66T_2T_3 - 108T_3^2 - 72T_5) \frac{a^2}{144}$$

but also trans-series exponents!