Advances in Algebraic Quantum Field Theory

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Embedding of physical systems into larger systems requires

- characterization of subsystems
- understanding relations between subsystems (independence versus determinism, correlations, entanglement)

Characterization of subsystems by their material content ("a system of two electrons") turns out to be problematic:

- Particle number not well defined during interaction
- Because of nontrivial particle statistics systems with n particles cannot identified in a natural way as subsystems of a system with $m > n$ particles
  (e.g. Hanbury Brown Twist effect: Photons from different sources are entangled)

Haag (1957): Algebras of observables measurable in some finitely extended region of spacetime are good subsystems.

Needs an a priori notion of spacetime and therefore cannot directly be applied to quantum gravity,

but turned out to cover the essence of relativistic quantum field theory,

in particular due to the stability of this concept in time because of the finite velocity of light.

In nonrelativistic physics the identification of an algebra which is invariant under time evolution for an interesting dynamics is more difficult.

(see no go theorems by Narnhofer and Thirring and the recent work of Buchholz (resolvent algebra) for progress).

Haag-Kastler Axioms

Haag's concept of algebras of local observables can be summarized as follows:

A quantum system is represented by a C*-algebra with unit,

i.e. an algebra with unit over the complex numbers with an antilinear involution $A \mapsto A^*$

and a norm which satisfies the condition $||A^*A|| = ||A||^2$ such that the algebra is complete with respect to the induced topology.

These algebras are isomorphic to norm closed algebras of Hilbert space operators, but might admit mutually inequivalent representations.

The theory is defined in terms of an association of algebras to spacetime regions $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$.

$\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ (the "Haag Kastler net") is assumed to satisfy the conditions

- Inclusion: $\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$
- Local commutativity: If $\mathcal{O}_1, \mathcal{O}_2 \subset \mathcal{O}$ and $\mathcal{O}_1$ is spacelike separated from $\mathcal{O}_2$

  then the commutator $[A_1, A_2] \in \mathfrak{A}(\mathcal{O})$

  vanishes for all $A_1 \in \mathfrak{A}(\mathcal{O}_1), A_2 \in \mathfrak{A}(\mathcal{O}_2)$.
- Covariance If L is a symmetry of the spacetime then there exist isomorphisms $\alpha_{L,\mathcal{O}} : \mathfrak{A}(\mathcal{O}) \to \mathfrak{A}(L\mathcal{O})$ such that

$$\alpha_{L,\mathcal{O}_2}\big|_{\mathfrak{A}(\mathcal{O}_1)} = \alpha_{L,\mathcal{O}_1} \text{ for } \mathcal{O}_1 \subset \mathcal{O}_2$$

  and

$$\alpha_{L_1 L_2,\mathcal{O}} = \alpha_{L_1, L_2\mathcal{O}} \circ \alpha_{L_2,\mathcal{O}} \ .$$

States: positive linear functionals ω with $\omega(1) = 1$ (expectation value)

Representations: homomorphisms $\pi : \mathfrak{A} \to B(H)$ (bounded linear operators on some Hilbert space H)

GNS-construction: $\omega \to (H, \pi, \Omega)$, $\omega(\bullet) = \langle \Omega, \pi(\bullet)\Omega \rangle$

Application to Minkowski space: Poincaré group used for interpretation of states

ω invariant, with positive energy ("vacuum")

induced GNS representation: H vacuum Hilbert space, π vacuum representation, Ω vacuum vector

irreducible representation of Poincaré group: particle states

Haag-Ruelle scattering theory: existence of states which can be interpreted as outging or incoming multi particle states

Superselection sectors and DHR theory

The state space of a typical C*-algebra of local observables is huge. It contains not only states which admit an interpretation in terms of scattering states of particles, but also condensates, thermal equilibrium states and all sorts of nonequilibrium situations. The induced GNS representations are generically inequivalent.

Representation category:

Objects: representations π

Morphisms: intertwiners $T : H \to H'$ with $T\pi(\bullet) = \pi'(\bullet)T$

Treatment impossible (?)

Subclass of representations "of interest for particle physics" (Doplicher, Haag and Roberts 1969-74)

π_0 distinguished irreducible faithful representation ("vacuum")

π DHR representation $\Longleftrightarrow$

π equivalent to π_0 after restriction to the algebra of the spacelike complement $\mathcal{O}'$ of some finitely extended region $\mathcal{O}$

Space of intertwiners $(\pi \leftarrow \pi_0)(\mathcal{O})$ is bimodule over $\mathfrak{A}(\mathcal{O})$

$F, G \in (\pi \leftarrow \pi_0)(\mathcal{O}) \Longrightarrow F^* G \in (\pi_0 \leftarrow \pi_0)(\mathcal{O}) = \pi_0(\mathfrak{A}(\mathcal{O}'))'$

Assumption: $\pi_0(\mathfrak{A}(\mathcal{O}'))' = \pi_0(\mathfrak{A}(\mathcal{O}))$ (Haag duality)

satisfied in typical cases, related to PCT theorem (Bisognano-Wichmann 1974) and to Unruh and Hawking radiation

Consequence: DHR intertwiner spaces are closed under bimodule tensor products:

$F_i, F'_i \in (\pi_i \leftarrow \pi_0)(\mathcal{O})$, $\Phi, \Phi' \in H_0$, $F_2^* F'_2 = \pi_0(A)$, $A \in \mathfrak{A}(\mathcal{O})$

The left module

$$(\pi_2 \leftarrow \pi_0)(\mathcal{O}) \otimes (\pi_1 \leftarrow \pi_0)(\mathcal{O}) \otimes H_0$$

equipped with the scalar product

$$\langle F_2 \otimes F_1 \otimes \Phi, F'_2 \otimes F'_1 \otimes \Phi' \rangle = \langle F_1 \Phi, \pi_1(A) F'_1 \Phi' \rangle$$

defines a new DHR representation $\pi_2 \times \pi_1$ (fusion).

Additional structure from positivity condition on the energy spectrum:

Reeh-Schlieder Theorem and Borchers' property $\Longrightarrow$:

DHR representations are of the form $\pi_0 \circ \rho$
with an endomorphism ρ of $\mathfrak{A}$.

Fusion=multiplication of endomorphisms

Monoidal category with endomorphisms as objects and intertwiners between them as morphisms.

Local commutativity $\Longrightarrow$ braiding structure, trivializes to symmetry if spacelike complement of a point is connected.

In the latter case: DHR category equivalent to representation category of some compact group with a distinguished element of order 2 (corresponding to fermions) (Doplicher, Roberts 1990).

Locally covariant QFT

Generalization to globally hyperbolic spacetimes
(*i.e.* with a Cauchy surface):

Inclusion and local commutativity o.k.
but covariance axiom trivial in generic case

Problems:

- No good concept of vacuum and of particles
- Singularities at different points are unrelated
  $\Longrightarrow$ huge ambiguities in renormalization

Solution: Locally covariant QFT
(Brunetti,KF,Hollands,Kay,Verch,Wald 2000)

A locally covariant QFT is a functor

$$\mathfrak{A} : \mathrm{ghyp} \to \mathrm{Cstar}$$

ghyp category of globally hyperbolic spacetimes of a fixed dimension
(with causality preserving isometric embeddings as morphisms)

Cstar category of unital C*-algebras
(with unital monomorphisms as morphisms)

Axiom

- $\chi_i : M_i \to N$ such that $\chi_1(M_1)$ spacelike to $\chi_2(M_2) \Longrightarrow$

$$[\mathfrak{A}\chi_1(A_1), \mathfrak{A}\chi_2(A_2)] = 0 \ , \ \ A_i \in \mathfrak{A}(M_i) \ .$$

Restriction to globally hyperbolic subregions of a fixed spacetime yields Haag-Kastler net (covariance and inclusion follow from functoriality).

Additional axiom implementing existence of a deterministic dynamical law:

Time slice axiom:

$$\chi : M \to N \text{ with } \chi(M) \supset \Sigma \text{ Cauchy surface of } N$$

$$\implies \mathfrak{A}\chi(\mathfrak{A}(M)) = \mathfrak{A}(N)$$

χ Cauchy morphism (weak equivalence).
Closed paths of weak equivalences induce automorphisms describing intermediate changes of the metric.

Infinitesimal: Energy momentum tensor.

General covariance $\implies$ covariant conservation law

Up to now only generic properties of QFT, no specification of the model.

Now: L Lagrangian of classical field theory of scalar field ϕ, $\mathcal{E} = \mathcal{C}^\infty$ space of smooth field configurations.

$$L(j_x(\phi)) = \left(\frac{1}{2}g^{-1}(d\phi(x), d\phi(x)) - V(\phi(x))\right) d\mu_g(x)$$

Canonical quantization: Foliation by Cauchy surfaces

$$M \equiv \mathbb{R} \times \Sigma \ , \ g(t,x) = a(t,x)^2 dt^2 - h_t(x)$$

$a > 0$, $(h_t)_t$ smooth family of Riemannian metrics on Σ.

$L = \mathcal{L}dt$, $\mathcal{L}$ density on Σ, $\partial_n \phi = a^{-1}\dot{\phi}$ normal derivative,

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_n \phi)} = \dot{\phi} d\mu_{h_t} \text{ canonical momentum}$$

$$[\phi(t,x), \pi(t,y)] = i\delta_x(y) \ , \ \delta_x \text{ Dirac measure at } x$$

canonical commutation relation
($\Longrightarrow$ algebraic structure does not depend on t)

Time evolution by Heisenberg equation

$$\frac{d}{dt}A(t) = i \int [\mathcal{H}(t,x), A(t)]$$

Hamiltonian density

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L}$$

Reformulation:

$F[\phi] = \int f(\phi(t,x),t,x)$ local functional,

$f = 0$ for $(t,x) \notin K$, K compact ("support" of F).

Operation on the system by interaction f, *i.e.* S-matrix $S(F)$)

Dyson series:

$$S(F) = \sum_{n=0}^{\infty} i^n \int_{t_n > \cdots > t_1} f(\phi(t_n,x_n),t_n,x_n)\ldots f(\phi(t_1,x_1),t_1,x_1)$$

ϕ solution of Heisenberg equation for Hamiltonian derived from L.

Boboliubov: construction of field with Lagrangian $L + f$:

Split $F = F_{\leq t} + F_{>t}$ with $F_{\leq t} = \int_{t' \leq t} f$. Then from Dyson's formula

$$S(F) = S(F_{>t})S(F_{\leq t})$$

hence

$$\frac{d}{d\lambda}_{|\lambda=0} S(F)^{-1} S(F + \lambda\phi(t,x)) = S(F_{\leq t})^{-1}\phi(t,x)S(F_{\leq t})$$

solves the modified Heisenberg equation.

S-matrix for local functional G for time evolution from $L + f$:

$$S_F(G) = S(F)^{-1} S(F + G)$$

with the factorization

$$S_F(G) = S_F(G_{>t})S_F(G_{\leq t})$$

hence we obtain the general causal factorization

$$S(F + G + H) = S(F + G)S(G)^{-1}S(G + H)$$

if supp$F$ is in the future and supp$H$ in the past of some Cauchy surface.

Note that this relation does not use any space time splitting. It does not specify the dynamics.

Specification of the dynamics:

A shift $\phi \to \phi + \psi$, ψ with compact support produces an isomorphic theory with the Lagrangian

$$L[\bullet + \psi]$$

The interaction $\delta L(\psi) = \int L[\bullet + \psi] - L$ can be removed and one obtains an automorphism of the original theory

$$\alpha(S(F)) = S(\delta L(\psi))^{-1} S(\delta L(\psi) + F[\bullet + \psi])$$

For suppψ in the future and supp$F$ in the past of some Cauchy surface
$\Longrightarrow \alpha(S(F)) = S(F)$ by causal factorization

$\Longrightarrow \alpha = \mathrm{id}$ (time slice axiom)

Special case: $F = \delta L(\psi') \implies$

$$S(\delta L(\psi))S(\delta L(\psi')) = S(\delta L(\psi + \psi'))$$

i.e. $\lambda \mapsto S(\delta L(\lambda\psi))$ 1-parameter group with generator $\langle \frac{\delta \int L}{\delta \phi}, \psi \rangle$

$$\implies S(\delta L(\psi)) = 1 \text{ (equation of motion)}$$

Proposal (Detlev Buchholz, KF 2020):
The algebra of a scalar quantum field with Lagrangian L is generated by unitaries $S(F)$, F compactly supported local functional of ϕ, $S(0) = 1$. The unitaries satisfy the relations

- $S(F + G + H) = S(F + G)S(G)^{-1}S(G + H)$
  if supp$F$ is in the future and supp$H$ in the past of some Cauchy surface.
- $S(F) = S(F[\bullet + \psi] + \delta L(\psi))$, ψ compactly supported

Perturbative AQFT

Assumption: $F \mapsto S(F)$ is differentiable to all orders.

Dynamical equation

$$0 = \frac{d}{d\lambda}|_{\lambda=0} S(\delta L(\lambda\psi) + F[\bullet + \lambda\psi])$$

$$= \frac{d}{d\lambda}|_{\lambda=0} S(F + \lambda\langle \frac{\delta \int L}{\delta\phi} + F', \psi\rangle)$$

(Schwinger Dyson equation)

Causal factorization: $(G = 0)$

$$\langle S^{(n+m)}(0), F^{\otimes n} \otimes H^{\otimes m}\rangle = \langle S^{(n)}(0), F^{\otimes n}\rangle\langle S^{m}(0), H^{\otimes m}\rangle$$

$\Longrightarrow$ Higher derivatives can be constructed from lower derivatives up to coinciding points.

Starting point for Epstein Glaser renormalization:

$$\langle S^{(1)}(0), F\rangle = :F: \quad \text{normal ordering}$$

for a free (*i.e.* quadratic) Lagrangian.

Inductive construction of higher derivatives in 2 steps:

By causal factorization up to coinciding points and then extension to coinciding points by extending distributions in several variables.

Second step corresponds to renormalization.

Not unique, classification by renormalization group (in the sense of Stückelberg-Petermann)(Stora-Popineau).

Framework (see text books by Kasia Rejzner and Michael Dütsch)

$\mathcal{E}$ space of smooth field configurations

$\mathcal{F}$ functionals on $\mathcal{E}$ with suitable smoothness properties ("microcausal") and compact support.

$\mathcal{F} \supset \mathcal{F}_{\mathrm{loc}}$ real valued local functionals

$$F[\phi] = \int f(j_x(\phi))$$

$j_x(\phi)$ jet of ϕ at x, f density with compact support.

Normal ordering (formally):

$$:F:_H = e^{\frac{1}{2}\hbar\langle H, \frac{\delta^2}{\delta\phi^2}\rangle} F$$

H Hadamard solution of free e.o.m., selects positive frequencies in the commutator function Δ, *e.g.* Wightman 2-point function

H, H' Hadamard $\Longrightarrow$ $H - H'$ smooth $\Longrightarrow$

$$:F:_H = :e^{\frac{1}{2}\hbar\langle (H-H'), \frac{\delta^2}{\delta\phi^2}\rangle} F:_{H'}$$

$\mathcal{A} = \{:F:_H \, | F \in \mathcal{F}\}$ independent of H

$\star$-product of normal ordered (microcausal) functionals

$$:F:_H \star :G:_H = :F \star_H G:_H$$

with

$$(F \star_H G)[\phi] = \left(e^{\hbar\langle H, \frac{\delta^2}{\delta\phi_1\delta\phi_2}\rangle} F[\phi_1] G[\phi_2] \right)_{|\phi_1=\phi_2=\phi}$$

$(\mathcal{A}, \star)$ associative algebra of formal power series.

Time ordered product $\cdot_T$: Commutative product on a subspace of $\mathcal{A}$ containing $\mathcal{A}_{\rm loc}$ with causal factorization

$$:F: \cdot_T :G:=:F: \star :G:$$

if supp$F$ is in the future and supp$G$ in the past of some Cauchy surface.

Example: $F = \int \frac{1}{2} f(x)\phi(x)^2$, $G = \int \frac{1}{2} g(x)\phi(x)^2$

$$:F:_H \cdot_T :G:_H=:F \cdot_{T,H} G:_H$$

$$F \cdot_{T,H} G[\phi] = F[\phi]G[\phi] + \hbar\langle H_F, \phi f \otimes \phi g\rangle + \frac{1}{2}\hbar^2\langle H_F^2, f \otimes g\rangle$$

with $H_F(x,y) = H(x,y)(\theta(t-t') + H(y,x)\theta(t'-t)$ for some time coordinate $t = t(x), t' = t(y)$ and

$H_F^2(x,y) = H^2(x,y)(\theta(t-t') + H^2(y,x)\theta(t'-t)$ for $x \neq y$.

H_F^2 not unique (renormalisation)

Renormalized time ordered product yields S-matrix:

$$S(F) = \exp_T :F:_\mu \ \text{(exponential series w.r.t. } \cdot_T)$$

$:F:_\mu$ locally covariant normal ordering (natural transformation $\mathcal{F}_{\mathrm{loc}} \to \mathcal{A}_{\mathrm{loc}}$), depends on mass scale μ for scalar field in $d = 4$).

C*-algebras

Perturbation theory yields formal power series of Hilbert space operators.

After truncation: Explicit formulas, can be compared to experiments, but little (no?) control over errors.

Alternative: Consider the C*-algebra generated by the S-matrices and their relations (Buchholz, KF 2020)

$\Longrightarrow$ Haag-Kastler net for any given Lagrangian.

Restriction to the free Lagrangian L and affine functionals $F[\phi] = \int f\phi + c$ yields closed algebra.

$K = -(\Box + m^2)$, $\delta L(\psi) = \int (K\psi)\phi + \frac{1}{2}\psi K \psi$

f, g compactly supported $\implies f = f_0 + K\psi$ with ψ compactly supported and suppf_0 does not intersect the past of supp$g$. It follows $\int f\phi = \int f_0\phi + \delta L(\psi) - \frac{1}{2}\int \psi K \psi$
By the dynamical equation

$$S(\int f\phi) = S(\int f_0(\phi - \psi) - \frac{1}{2}\int \psi K \psi)$$

Causal factorization implies

$$S(\int f\phi)S(\int g\phi) = S(\int f_0(\phi - \psi) - \frac{1}{2}\int \psi K \psi + \int g\phi)$$

Applying again the dynamical equation yields

$$S(\int f_0\phi + \int (K\psi)\phi + \int g(\phi + \psi)) = S(\int (f+g)\phi + \int g\psi)$$

Since $\psi = \Delta_R(f - f_0)$ and supp$g \cap$ supp$\Delta_R f_0 = \emptyset$ we arrive at

$$S(\int f\phi)S(\int g\phi) = S(\int (f+g)\phi + \int (\Delta_R f)g)$$

Conclusion: The proposed axioms imply Weyl relations and therefore canonical commutation relations:

1-parameter group

$$\lambda \mapsto S(\int \lambda f\phi + \frac{1}{2}\lambda^2 \int (\Delta_R f)f) = W(\lambda f) \text{ (Weyl operator)}$$

Weyl relation:

$$W(f)W(g) = S(\tfrac{1}{2}\int (g\Delta_R f - f\Delta_R g))W(f+g)$$

Set $S(c) = e^{ic}$ for a constant functional c,

$\phi(x) = \frac{1}{i}\frac{\delta}{\delta f(x)} W(f)|_f = 0 \Longrightarrow$

$$[\phi(x), \phi(y)] = i\Delta(x,y) = i(\Delta_R(x,y) - \Delta_R(y,x))$$

Noether's Theorem in QFT

(Brunetti, Dütsch, KF, Rejzner, in preparation)

Symmetries and conservation laws are intimately connected by Noether's Theorem.

In QFT, renormalization might break this connection (anomalies). Can one understand this connection in the C*-algebraic setting?

Enrich the formalism of locally covariant QFT by interactions: (for an *n*-component scalar field ϕ)

Category Dyn with objects $\mathcal{M} = (M, L, t)$
with a globally hyperbolic spacetime M, a Lagrangian L and a distinguished time orientation t.

Morphisms $\iota \in \mathrm{Hom}(\mathcal{M}, \mathcal{M}')$ are compositions of elementary morphisms

- ι_χ: embedding $\chi : M \to M'$, $\chi^* L' = L$, $\chi^* t' = t$
- ι_Φ: affine field redefinition $\Phi : \phi \mapsto A\phi + \phi_0$, $M' = M$, $L' \circ \Phi = L$, $t' = t$
- $\iota_{V,+}$: retarded interaction $M' = M$, $L' + V = L$, $\mathrm{supp}\, V$ past compact
- $\iota_{V,-}$: advanced interaction $M' = M$, $L' + V = L$, $\mathrm{supp}\, V$ future compact

$$\text{Functor } \mathfrak{A} : \begin{cases} \mathrm{Dyn} & \to & \mathrm{Cstar} \\ \mathcal{M} & \mapsto & \mathfrak{A}(\mathcal{M}) \end{cases}$$

$\mathfrak{A}(\mathcal{M})$ C*-algebra generated by S-matrices $S(F)$ by the relations Causal factorization and Dynamics

Elementary morphism $\iota_\bullet$ mapped to homomorphisms $\mathfrak{A}\iota_\bullet = \alpha_\bullet$:

- $\alpha_\chi(S(F)) = S(\chi_* F)$
- $\alpha_\Phi(S(F)) = S(F \circ \Phi^{-1})$
- $\alpha_{V,+}(S(F)) = S(V(f))^{-1} S(V(f) + F)$
- $\alpha_{V,-}(S(F)) = S(F + V(f)) S(V(f))^{-1}$

f compactly supported and equal to 1 on a sufficiently large region (depending on F)

Special cases:
(1) $M' = M$, χ compactly supported diffeomorphism of M, $\chi^* L' = L$, $\chi^* t' = t$.

Set $\delta_\chi L = \int L' - L$.
Then $\iota_\chi \in \mathrm{Hom}(\mathcal{M}, \mathcal{M}')$, $\iota_{\delta_\chi L,\pm} \in \mathrm{Hom}(\mathcal{M}', \mathcal{M})$ and $\alpha_\pm = \alpha_{\delta_\chi L,\pm} \circ \alpha_\chi$ are automorphisms of $\mathfrak{A}(\mathcal{M})$.

$$\alpha_+(S(F)) = S(\delta_\chi L)^{-1} S(\delta_\chi L + \chi_* F)$$

$$\alpha_-(S(F)) = S(\chi_* F + \delta_\chi L) S(\delta_\chi L)^{-1}$$

Causal factorization + Time slice $\Longrightarrow \alpha_\pm = \mathrm{id}$

(2) $M' = M$, Φ compactly supported affine field redefinition, $L' \circ \Phi = L$, $t' = t$.

Set $\delta_\Phi L = \int L' - L$.
Then $\iota_\Phi \in \mathrm{Hom}(\mathcal{M}, \mathcal{M}')$, $\iota_{\delta_\Phi L,\pm} \in \mathrm{Hom}(\mathcal{M}', \mathcal{M})$
and $\alpha_\pm = \alpha_{\delta_\Phi L,\pm} \circ \alpha_\Phi$ are automorphisms of $\mathfrak{A}(\mathcal{M})$.

$$\alpha_+(S(F)) = S(\delta_\Phi L)^{-1} S(\delta_\Phi L + F \circ \Phi^{-1})$$

$$\alpha_-(S(F)) = S(F \circ \Phi^{-1} + \delta_\Phi L) S(\delta_\Phi L)^{-1}$$

Causal factorization + Time slice $\Longrightarrow \alpha_\pm = \mathrm{id}$

G^c group generated by compactly supported diffeomorphisms and affine field redefinitions
$G \ni g = (A, \phi_0, \chi)$, $g\phi = A(\phi \circ \chi^{-1}) + \phi_0$, $g_* F = F \circ g^{-1}$

$S(F) = S(\delta_g L)^{-1} S(\delta_g L + g_* F)$, $g \in G^c$ (Unitary Master Ward Identity)

(infinitesimal version known in perturbation theory (Dütsch et al))

Application to Noether's theorem:
G generated by diffeomorphisms and affine field redefinitions (without support restrictions),
$G \supset G_L$ subgroup which leaves L invariant.

Let $g \in G_L$ and $h \in G^c$ such that $(g^{-1}\phi)(x) = (h^{-1}\phi)(x)$ for $x \in J_\cap(\mathrm{supp}F)$. $(J_\cap = J_+ \cap J_-)$

Then $g_*F = h_*F$ and $\mathrm{supp}\delta_h L \cap \mathrm{supp}F = \emptyset$.
Decompose $\delta_h L = Q_+ + Q_-$ such that $\mathrm{supp}Q_+ \cap J_-(\mathrm{supp}F) = \emptyset$ and $\mathrm{supp}Q_- \cap J_+(\mathrm{supp}F) = \emptyset$. Then by the unitary master Ward identity

$$S(F) = S(Q_+ + Q_-)^{-1} S(Q_+ + Q_- + g_*F) = S(Q_-)^{-1} S(g_*F) S(Q_-)$$

hence the automorphism $S(F) \mapsto S(g_*F)$ is implemented by the unitary $S(Q_-)$.

Outlook

- Concepts of AQFT, combined with ideas from category theory, solve longstanding problems of QFT.
- Problem of existence of QFT's reduced to search for suitable states.
- Formalism has to be generalized to Fermi fields and gauge theories.
- Perturbatively more or less understood by BV-BRST formalism, but C*-algebraic formulation unknown.
- See the talk of Alexander Schenkel next week for further progress, in particular for gauge theories.