Symmetries of CY Hypersurfaces in Toric Four-folds

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Outline

- Introduction
- Symmetries of CY hyper-surfaces in toric ambient spaces
- Earlier results
- Polytope symmetries and indices
- Some new results
- Conclusion and outlook

Introduction

Freely-acting CY symmetries

CY 3-fold X with freely acting discrete symmetry Γ

Then, the quotient X/Γ is a CY 3-fold.

If $\pi_1(X)$ is trivial, then $\pi_1(X/\Gamma) \cong \Gamma$.

Why are we interested?

- $\eta(X/\Gamma) = \eta(X)/|\Gamma|$ -> 3-folds with smaller Euler/Hodge numbers
- non-trivial $\pi_1(X/\Gamma)$ means we can have flat bundles (Wilson lines)
- Wilson lines are crucial to construct physical heterotic models

Slight detour: standard model of particle physics from strings

	group	representation
SM	$SU(3) \times SU(2) \times U(1)$	$3 \times [(3,2)_1 \oplus (\bar{3},1)_{-4} \oplus (\bar{3},1)_2 \oplus$ $(1,2)_{-3} \oplus (1,1)_6)]$
	$\cap$	$\uparrow$
GUT1	$SU(5)$	$3 \times [10 \oplus \bar{5}]$
	$\cap$	$\uparrow$
GUT2	$Spin(10)$	3×16

Wilson line:
preserves index

In geometric string models
counted by an index (cohomology)

CY hyper-surfaces in toric four-folds

based on half a billion (473800776) reflexive 4d polytopes, as classified by Kreuzer and Skarke. M. Kreuzer, H. Skarke, hep-th/0002240

freely-acting toric symmetries classified by Kreuzer and Batyrev: only 16 cases among the half a billion! V. Batyrev, M. Kreuzer, math/0505432

simplest example: quintic in $\mathbb{P}^4$, coordinates $(x_0, \ldots, x_4)$, $\Gamma \cong \mathbb{Z}_5$ generated by $x_a \mapsto \alpha^a x_a$, where $\alpha = \exp(2\pi i/5)$

however: quintic in $\mathbb{P}^4$ has freely-acting symmetry $\Gamma \cong \mathbb{Z}_5 \times \mathbb{Z}_5$, generated by $x_a \mapsto \alpha^a x_a$ and $x_a \mapsto x_{(a+1) \bmod 5}$

Can we classify all freely-acting symmetries of CY hyper-surfaces (which `descend' from the ambient space) in toric four-folds?

Case study: Complete intersections in products of projective spaces

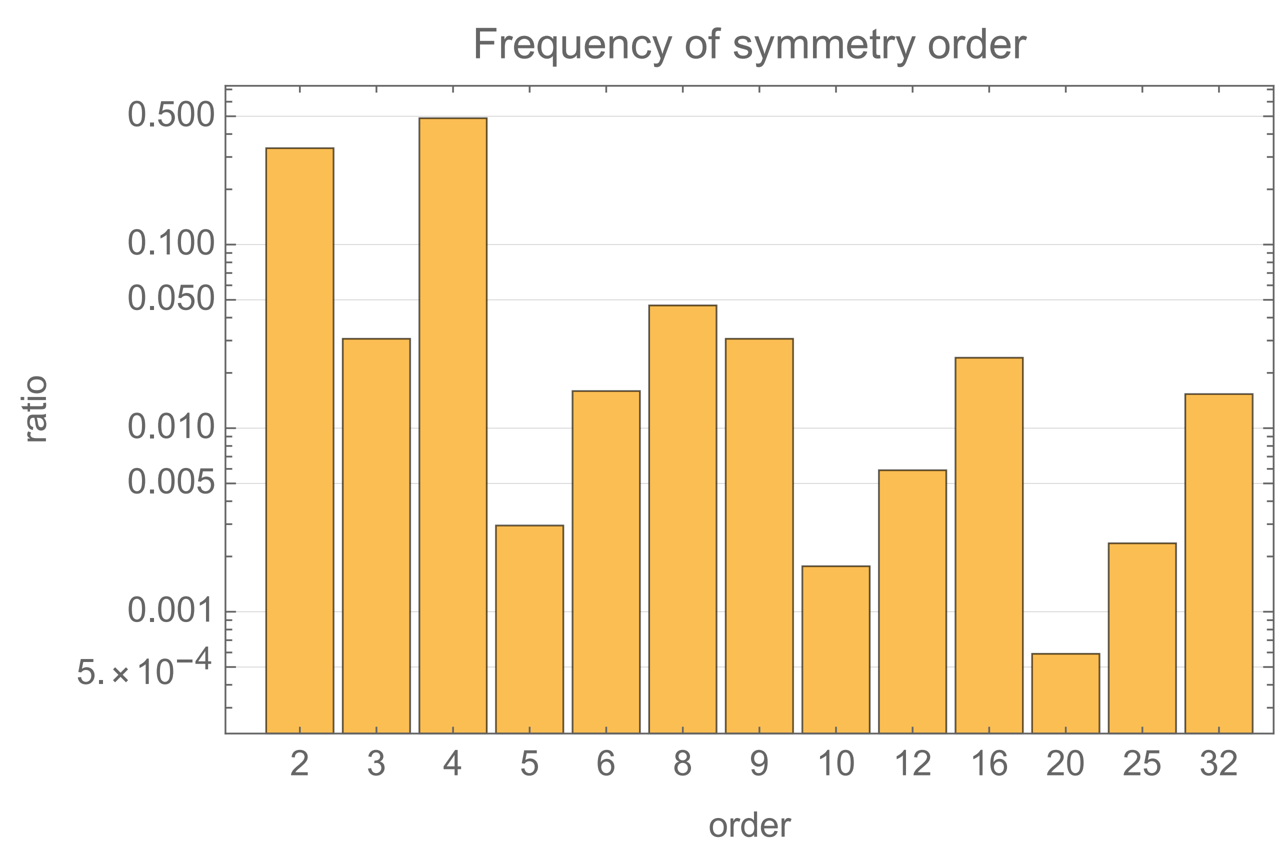
there are 7890 topological types of complete CY intersections in products of projective spaces (CICYs)

Green, Hübsch, CMP 109, 99–108 (1987)
Candelas, Dale, Lütken, Schimmrigk, NPB 298 (1988) 493

Their freely-acting symmetries which descent from linear actions on the ambient space (cover) have been classified by Volker Braun.

Braun, arXiv:1003.3235

There are 1695 symmetries on 195 CICYs (on 2.5% of CICYs).



Symmetries of CY hyper-surfaces in toric ambient spaces

Notation and set-up

face fan Σ of 4d reflexive polytope $\Delta^\circ \subset N \otimes \mathbb{R}$

ray generators $(v_i)_{i=1,\ldots,k}$ with associated coordinates $z = (z_1, \ldots, z_k) \in \mathbb{C}^k$

charge matrix $Q = (q_1, \ldots, q_k)$

zero set

$$\mathbb{P}_\Sigma = \frac{\mathbb{C}^k - Z}{\mathcal{G}} \qquad\qquad \mathcal{G} = (\mathbb{C}^*)^{k-4} \times (\text{finite Abelian})$$

CY $X \subset \mathbb{P}_\Sigma$ defined as zero locus of polynomial

$$p = \sum_{\mathbf{m} \in \Delta \cap M} c_{\mathbf{m}} \prod_{\mathbf{v}_i \in \Delta^\circ \cap N \backslash \{0\}} x_i^{\langle \mathbf{m}, \mathbf{v}_i \rangle + 1}$$

Ambient space symmetry

Q: What is the symmetry group G of $\mathbb{P}_\Sigma$ which descends from $\mathrm{GL}(\mathbb{C}^k)$?

$$\boxed{G = P \ltimes \frac{H}{\mathcal{G}} \qquad\qquad H = \bigotimes_{K \in \mathcal{K}} \mathrm{GL}(K, \mathbb{C})}$$

permutation of blocks

blocks with
same charges

$$\mathcal{K} = \{1, \ldots, k\}/\sim \quad \text{where} \quad i \sim j \quad \Leftrightarrow \quad q_i = q_j$$

Example for ambient space symmetry

$$(v_1,\dots,v_5,v_6,v_7)=\begin{pmatrix} -1 & -1 & 1 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 2 & -2 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad Q = \begin{array}{c} \begin{array}{ccccccc} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array} \quad \Longrightarrow \quad \mathcal{K}=\{\{1,4\},\{2,5\},\{3\},\{6\},\{7\}\}$$

$$\Rightarrow G = P \ltimes \frac{H}{\mathcal{G}}$$ where

$$\mathcal{G} \ni g = \mathrm{diag}(\mathbf{s}^{\mathbf{q}_{\{1,4\}}},\mathbf{s}^{\mathbf{q}_{\{1,4\}}},\mathbf{s}^{\mathbf{q}_{\{2,5\}}},\mathbf{s}^{\mathbf{q}_{\{2,5\}}},\mathbf{s}^{\mathbf{q}_{\{3\}}},\mathbf{s}^{\mathbf{q}_{\{6\}}},\mathbf{s}^{\mathbf{q}_{\{7\}}})$$

$$H = \mathrm{Gl}(\{1,4\})\times\mathrm{Gl}(\{2,5\})\times\mathrm{Gl}(\{3\})\times\mathrm{Gl}(\{6\})\times\mathrm{Gl}(\{7\})$$

$P \cong \mathbb{Z}_2$ generated by $\begin{pmatrix} 0 & \mathbb{1}_2 & 0 & 0 & 0 \\ \mathbb{1}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

CY symmetry

We are looking for group (mono-)morphisms $R : \Gamma \to G$ such that

- $R(\Gamma)$ leaves X (for suitable choices of p) invariant
- $R(\Gamma)$ does not have fix points on X
- X is smooth

-> $X/R(\Gamma)$ is a smooth CY three-fold

The group Γ is constrained since its order $|\Gamma|$ has to divide the indices

Candelas, Lutken, Schimmrigk, NPB 306, 1988.

$$\chi_{k,\ell} := \int_X \mathrm{td}(TX) \wedge \mathrm{ch}(NX^k \otimes TX^\ell)$$

$$\sigma_{k,\ell} := \int_X \mathrm{L}(TX) \wedge \tilde{\mathrm{ch}}(NX^k \otimes TX^\ell)$$

Algorithm to find CY symmetries

- Constrain the group order $|\Gamma|$ from indices
- Generate all finite groups Γ with these allowed group orders
- Determine the ambient space symmetry group G
- Find all group embeddings $R : \Gamma \to G$
- Find all defining polynomial p invariant under $R(\Gamma)$
- Check that the fix point set of $R(\Gamma)$ does not intersect CY $X = \{p = 0\}$
- Check that X is smooth for a generic invariant p

Earlier results

Classification for small Picard number

A. Braun, A. Lukas, C. Sun, 1704.07812

Use database of FRST triangulations of KS database for $h^{1,1}(X) \leq 6$

R. Altmann, J. Gray, Y. H. He, V. Jejjala, B. D. Nelson, 1411.1418

Then, symmetry classification only for $h^{1,1}(X) \leq 3$ (some 350 manifolds)

In this range we found the following examples:

- Quintic in $\mathbb{P}^4$ with $\Gamma = \mathbb{Z}_5$ and $\Gamma = \mathbb{Z}_5 \times \mathbb{Z}_5$
- Bi-cubic in $\mathbb{P}^2 \times \mathbb{P}^2$ with $\Gamma = \mathbb{Z}_3$ and $\Gamma = \mathbb{Z}_3 \times \mathbb{Z}_3$
- Two more cases with $\Gamma = \mathbb{Z}_2$
- One further case with $\Gamma = \mathbb{Z}_2$ and $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$

Well-known, contained in V. Braun's classification

A single new example

toric action, contained in list of 16 by Batyrev and Kreuzer

The new case

- Hodge numbers: $h^{1,1}(X) = 3$, $h^{2,1}(X) = 115$
- Possible symmetry orders from indices: $|\Gamma| \in \{2, 4, 8\}$
- Ray generators of fan Σ:

$$(\mathbf{v}_1, \ldots, \mathbf{v}_7) = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 2 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 2 & 1 & -1 & -1 & 0 & 1 \end{pmatrix}$$

- Charge matrix:

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 & 2 & 4 & 0 \end{pmatrix}$$

- Symmetry $\Gamma = \mathbb{Z}_2 \ltimes \mathbb{Z}_2 = \langle \gamma_1, \gamma_2 \rangle$:

$$R(\gamma_1) = \mathrm{diag}(1, -1, 1, -1, 1, -1, -1)\,, \quad R(\gamma_2) = \mathrm{diag}(\sigma_1, \sigma_1, 1, -1, -1)\,, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Polytope symmetries and indices

Some definitions: automorphisms of fans and polytopes

Idea: Find necessary conditions for CY symmetries which are independent of triangulation.

Definition 1.3. For an invertible map $A : N \to N$, we call A an **automorphism** of Σ if for any $\sigma \in \Sigma$, $A(\sigma) = \sigma'$ is a cone in Σ as well. We denote the automorphism group of Σ by $Aut(N, \Sigma)$.

An automorphism A of Σ permutes the ray generators, so $Av_i = v_{\sigma(i)}$

Definition 1.5. For an invertible map $A : N \to N$, we call A an **automorphism** of a lattice polytope Δ if $A\Delta = \Delta$. This implies that A acts as a permutation on the vertices of Δ. We denote the automorphism group of Δ by $Aut(N, \Delta)$.

Proposition 1.6. For Σ a refinement of the face fan of a reflexive polytope Δ° we have

$$Aut(N, \Sigma) \subseteq Aut(N, \Delta^\circ)\,. \tag{12}$$

An automorphism A of Δ° permutes the lattice points in Δ°, so $Av_i = v_{\sigma(i)}$

Relation to previous discussion: $P = \dfrac{\mathrm{Aut}(N, \Sigma)}{W(N, \Sigma)}$

Conclusion: To have a Γ which acts in a non-toric way it is necessary that $\mathrm{Aut}(N, \Delta^\circ)$ is non-trivial.

Indices: Divisibility of indices $\chi_{k,l}$ and $\sigma_{k,l}$ by $|\Gamma|$ is equivalent to divisibility by $|\Gamma|$ of

$$\int_X \tfrac{1}{2}\mathrm{c}_3(TX) = h^{1,1}(X) - h^{2,1}(X)$$

$$\int_X \tfrac{1}{2}\mathrm{c}_1(NX)\mathrm{c}_2(TX) = \frac{1}{2}\sum_{\Theta^{\circ[1]}} \left(1+\ell^*(\Theta^{\circ[1]})\right)\left[-2+\ell^*(\Theta^{[2]})+\ell(\Theta^{[2]})\right]$$

$$\int_X \tfrac{1}{6}\mathrm{c}_1(NX)^3 + \tfrac{1}{12}\mathrm{c}_1(NX)\mathrm{c}_2(TX)$$

$$\longrightarrow \int_X \mathrm{c}_1(TX)^3 = \sum_{\Theta^{\circ[0]}}\left[13+14\ell^*(\Theta^{[3]})-\ell^*(2\Theta^{[3]})+\sum_{\Theta^{[2]}\subset\Theta^{[3]}}\ell^*(\Theta^{[2]})\right]$$
$$+\sum_{\Theta^{\circ[1]}}\left[-2+2\left(3-\ell^*(\Theta^{\circ[1]})\right)\left(-1+\ell^*(\Theta^{[2]})\right)\right]$$
$$+\sum_{\Theta^{\circ[2]}}\left(1+\ell^*\Theta^{[1]}\right)\left(12-\ell(\Theta^{\circ[2]})+\ell^*(\Theta^{\circ[2]})-\sum_{\Theta^{\circ[0]}\subset\Theta^{\circ[2]}}1\right)$$

Allowed group orders $|\Gamma|$ can be computed from polytope.

Revised strategy

- Find all polytopes with non-trivial automorphism group

- For all those polytopes, compute indices to find allowed group orders $|\Gamma|$

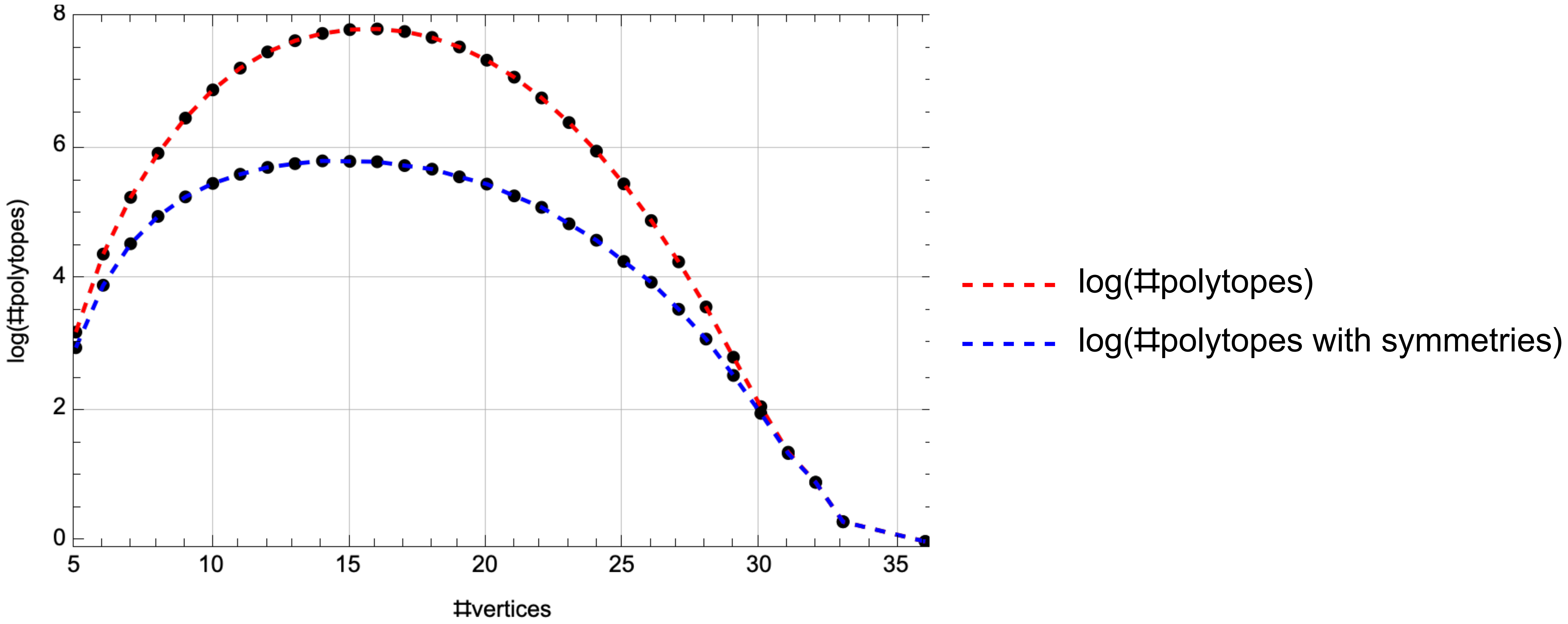
- Find more necessary conditions which are triangulation independent (?)

- Hopefully end up with a sufficiently small number which can then be studied using something similar to the previous algorithm.
  (Plus generating dedicated triangulations which respect the symmetry.) (?)

Find full classification of freely-acting symmetries which descent from the toric ambient space for the entire KS database. (?)

Some new results

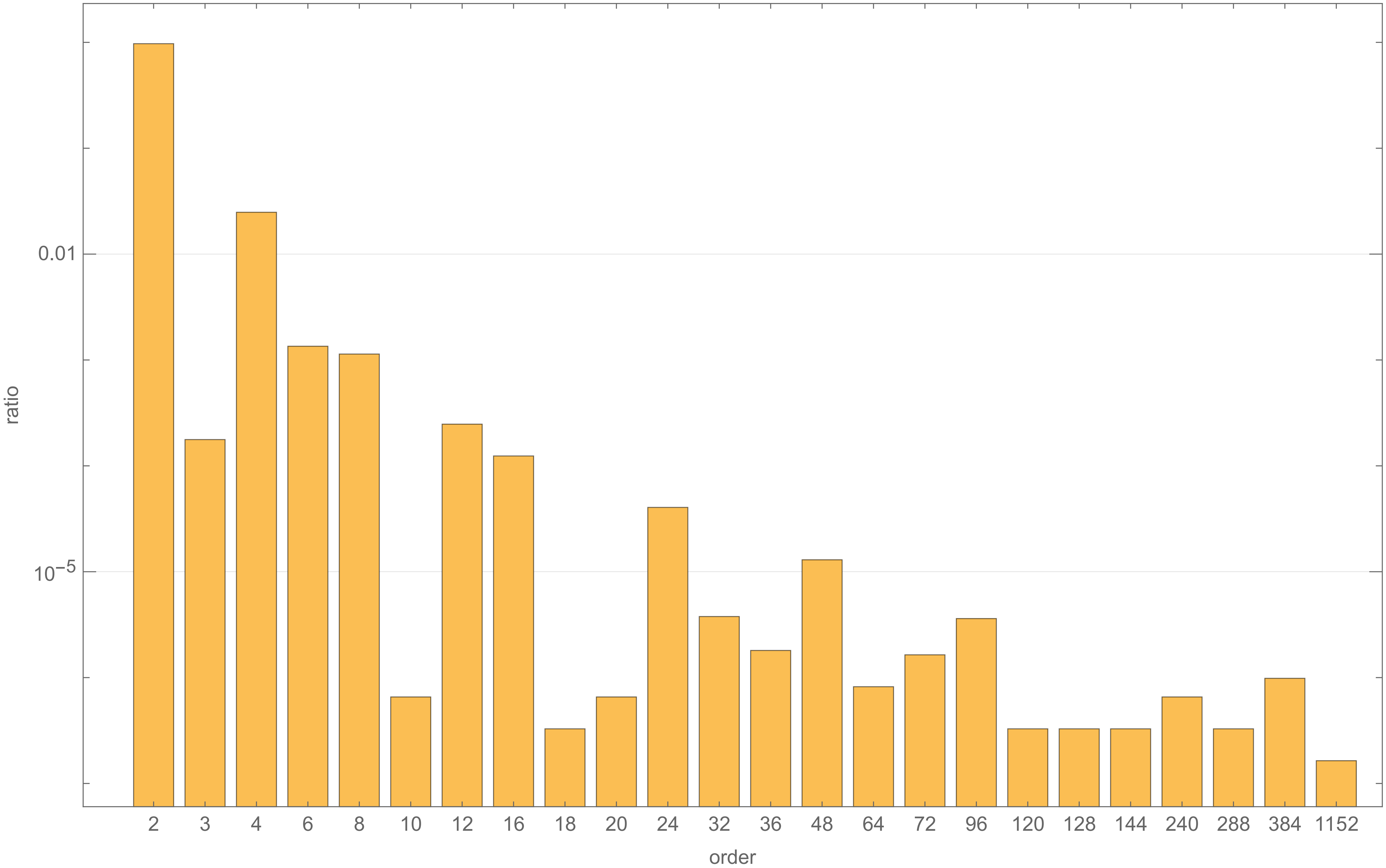
Which polytopes have a non-trivial automorphism?



#polytopes: 473800776

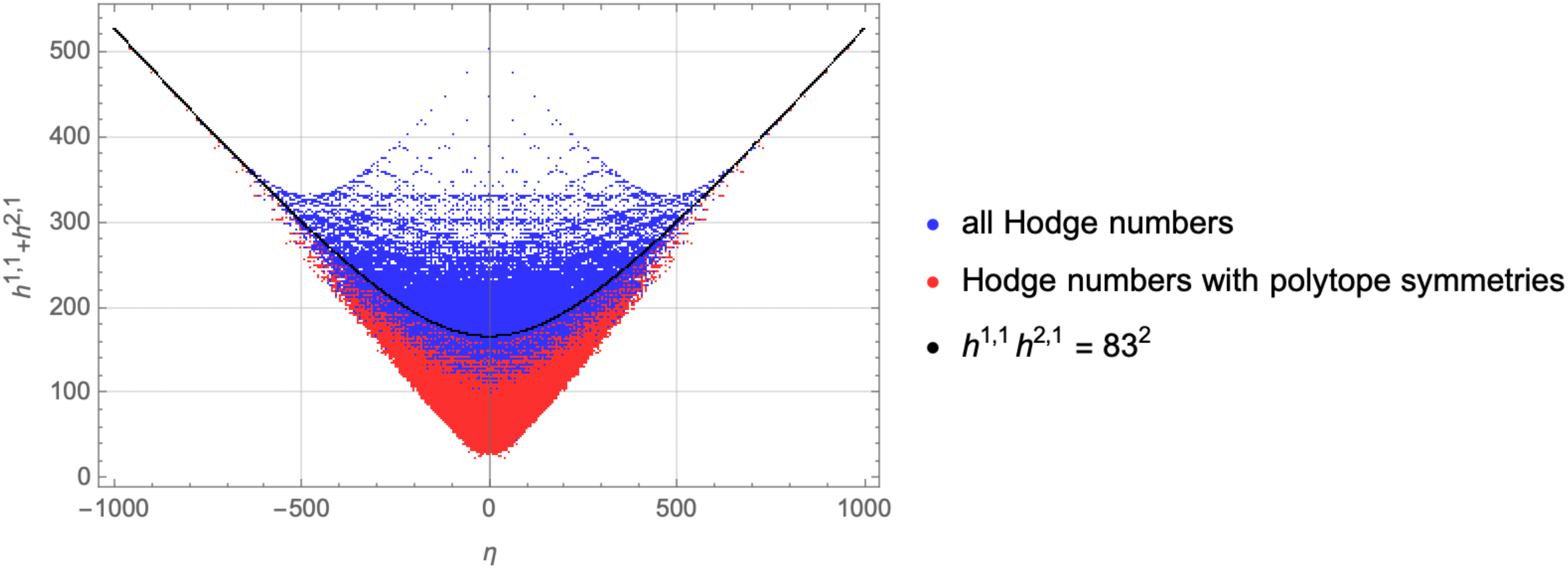
#polytopes with symmetry: 6095530 or about 1.3%

Orders of polytope symmetries



About 97% of symmetries are $\mathbb{Z}_2$

Hodge numbers for polytopes with non-trivial automorphism

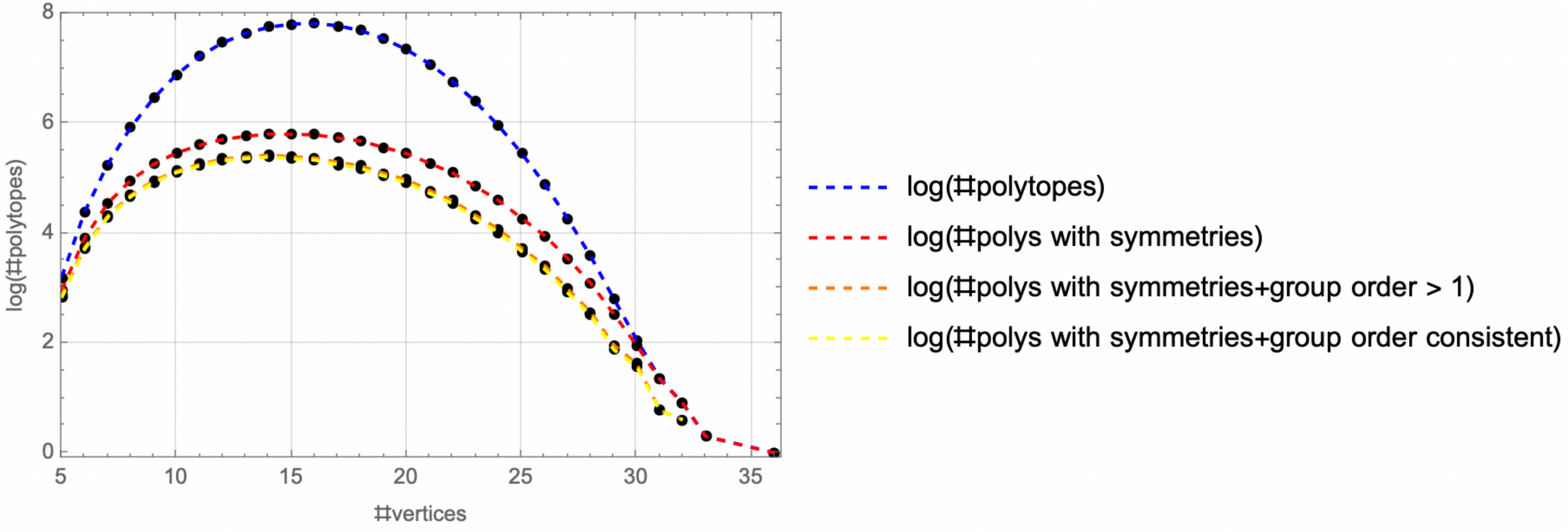


All cases with symmetries satisfy

$$h^{1,1}(X)\,h^{2,1}(X) \leq 83^2$$

Is there a reason for this?

Combining with indices



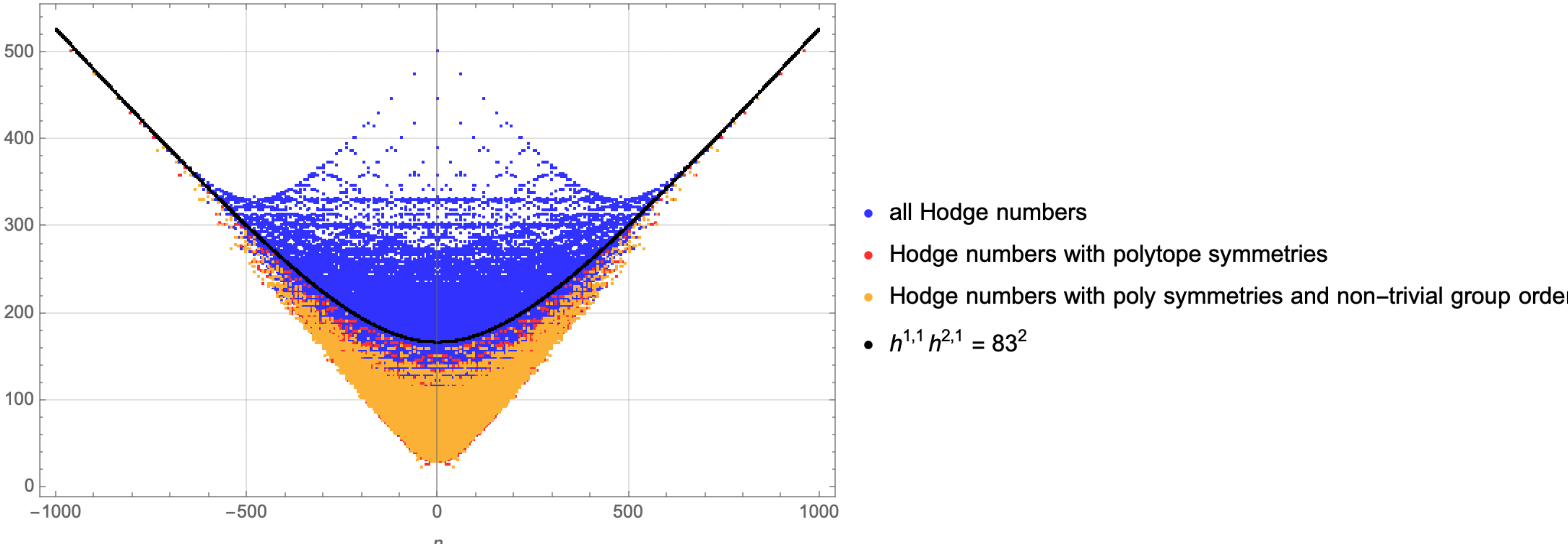
#polytopes: 473800776

#polytopes with symmetries: 6095530

#polytopes with symmetries and group order > 1: 2410049

Like before plus consistency: 2216299 or about 0.5%

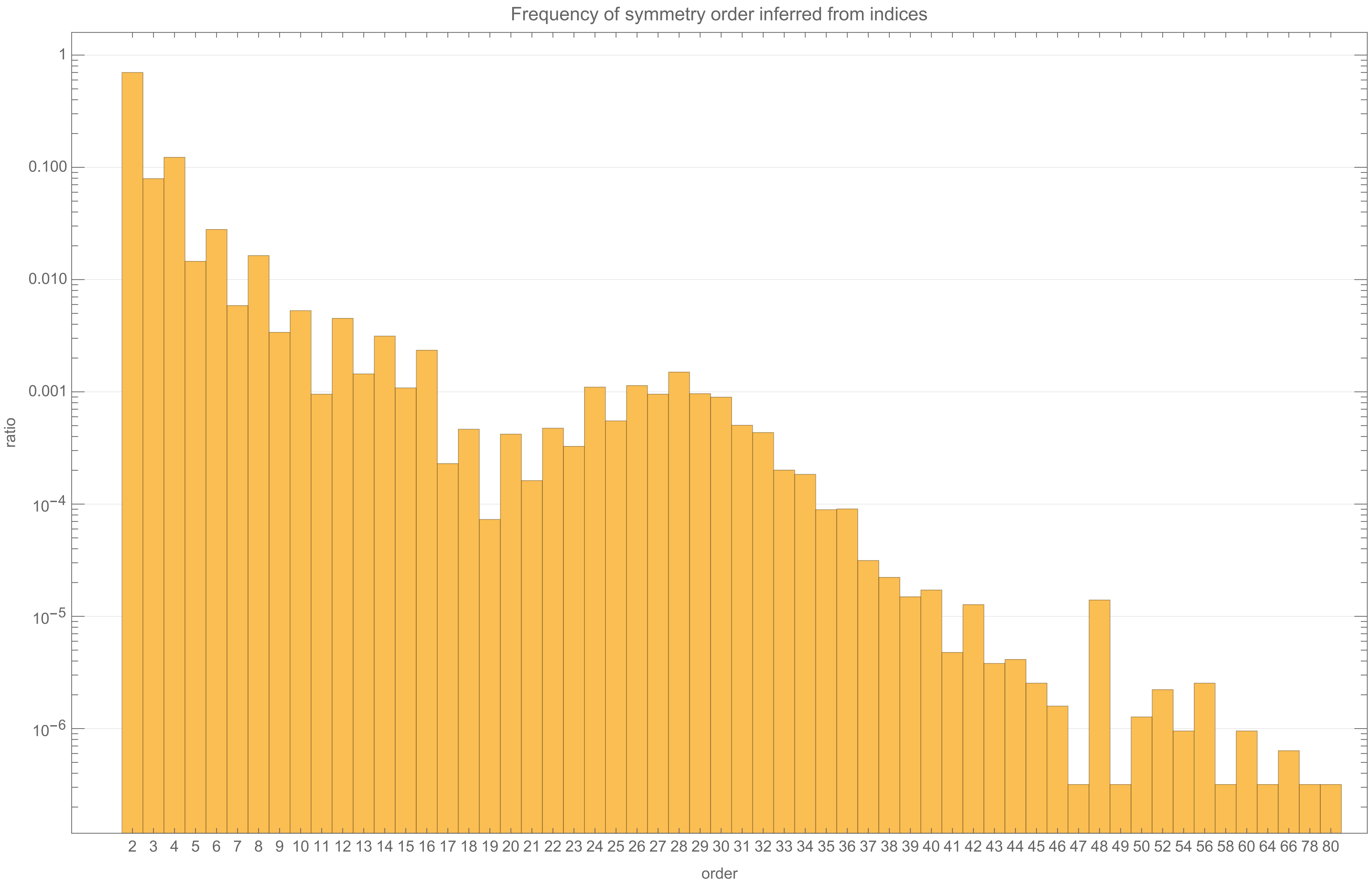
Hodge numbers



All cases with polytope symmetries and non-trivial group order satisfy

$$h^{1,1}(X)\,h^{2,1}(X) \leq 83^2$$

Symmetry orders consistent with indices



Conclusion

- Freely-acting symmetries of CY manifolds - here CY hyper-surfaces in toric 4-folds - are not easy to classify.

- We have looked at some necessary conditions - polytope symmetries and index divisibility - to narrow down the list of KS CYs with possible freely-acting symmetries (which descend from the ambient space).

- About 2.2 million polytopes (0.5%) pass these necessary conditions.

- All these polytopes satisfy $h^{1,1}(X)\, h^{2,1}(X) \leq 83^2$

For the future:

- Find more necessary conditions to reduce this number further.

- Try to deal with the remaining polytopes in a systematic way to get to a full classification.

Thanks