KATHOLISCHE UNIVERSITÄT EICHSTÄTT-INGOLSTADT

Variational analysis of auxetic metamaterials of checkerboard-type

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Joint work with:

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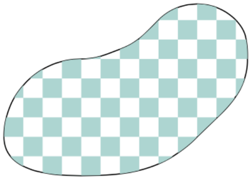
Material class in focus:
Elastic solids with stiff and soft components arranged into checkerboard-type structure in 2d

- **Interplay of two main features**
  - special geometric arrangement of heterogeneities
  - high-contrast: stiff vs. soft components
- **Goals: A variational viewpoint**
  - characterization of macroscopically attainable deformations
  - homogenization via Γ-convergence



Outline

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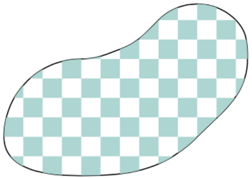
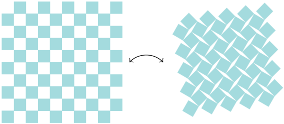
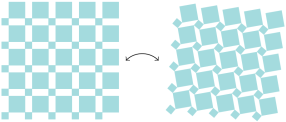


$\rightsquigarrow$ mechanical metamaterial with auxetic deformation behavior

What are auxetic metamaterials?

- **Metamaterial**: fabricated materials designed to have properties that do not naturally occur (**mechanical**, electrical, magnetic, acoustic, etc.)
- **Auxetics**: special case of mechanical metamaterial with **negative Poisson's ratio**, i.e., under stretching in uniaxial direction, thickening in the direction orthogonal to the applied force occurs





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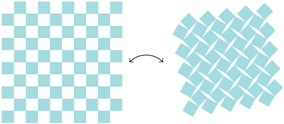
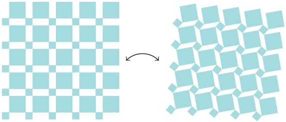


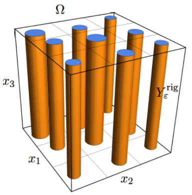


- **Applications**: shock absorbing shoes, kirigami-inspired medical stents, shape memory foams, ...

Some literature on the topic

Selection of related references:

- Engineering viewpoint on auxetics
  [Voigt 1928], [Lakes 1987], [Grima & Evans 2000, 2006], [Grima, Alderson, & Evans 2004], [Milton 2012],...
- Approach from algebraic-geometry for analysis of crystalline structures
  [Borcea & Streinu 2018, 2020],...
- Homogenization problems with high-contrast and stiff inclusions
  [Braides & Garroni 1995], [Cherdantsev & Cherednichenko 2012], [Davoli, Gavioli & Pagliari 2022],...
- Reinforced materials with stiff fibers and layers
  [Pideri & Seppecher 1997], [Bellieud & Bouchitté 1998], [Brillard & El Jarroudi 2001, 2007], [El Jarroudi 2013], [Paroni & Sili 2016], [Bellieud 2013 2017], ...
- Asymptotic rigidity
  [Christowiak & K. 2017, 2020], [Davoli, Ferreira & K. 2021], [Engl, K. & Ritorto 2022], ...



[Engl et al. 2022, M3AS]

Quantitative geometric and asymptotic rigidity

Generalization of Liouville's theorem on smooth local isometries:

Theorem (Reshetnyak 1967, Friesecke, James & Müller 2002)

Let $U \subset \mathbb{R}^2$ be a bounded Lipschitz domain and $p > 1$.

- If $u \in W^{1,p}(U;\mathbb{R}^2)$ with $\nabla u \in SO(2)$ a.e. in U, there exist $R \in SO(2)$ and $b \in \mathbb{R}^2$ such that $u = Rx + b$.
- There exists a constant $C_U > 0$ such that for every $u \in W^{1,p}(U;\mathbb{R}^2)$ there is a rotation $R \in SO(2)$ with

$$\|\nabla u - R\|_{L^p(U;\mathbb{R}^{2\times 2})} \leq C_U \|\text{dist}(\nabla u, SO(2))\|_{L^p(U)}.$$

Observation: The statements fail when U is not connected.

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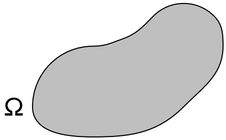
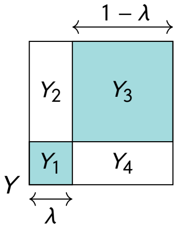
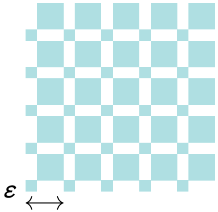
Global effects through specific geometric arrangement of stiff structures on a fine scale

$\rightsquigarrow$ restricted macroscopic material response

Set-up: The geometry







- $\Omega \subset \mathbb{R}^2$ a bounded Lipschitz domain
- $Y = (0,1]^2$ periodicity cell
- $Y_{\text{stiff}} = Y_1 \cup Y_3$ and $Y_{\text{soft}} = Y_2 \cup Y_4$, both extended periodically
- $\lambda \in (0,1)$
- length scale parameter $\varepsilon > 0$

Set-up: Admissible deformations

General assumptions:

- $u \in W^{1,p}(\Omega;\mathbb{R}^2)$ with $p > 2$
- Orientation preservation $$\det(\nabla u) > 0 \qquad \text{a.e. in } \Omega$$
- Ciarlet-Nečas condition $$\int_\Omega |\det(\nabla u)|\, dx \leq |u(\Omega)|$$

$\rightsquigarrow$ general class of **admissible deformations** $\mathcal{A}$

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Condition on stiff components:

Rigidity (for now)

$$\nabla u \in SO(2) \text{ a.e. in } \varepsilon Y_{\text{stiff}} \cap \Omega$$

$\rightsquigarrow \mathcal{A}_\varepsilon^{\text{rig}}$

Characterizing macroscopic deformations - The rigid case

Attainable macroscopic deformations

Set of attainable macroscopic deformations :

$$\mathcal{M}^{\mathrm{rig}} = \{u \in W^{1,p}(\Omega;\mathbb{R}^2) : u_\varepsilon \rightharpoonup u \text{ in } W^{1,p}(\Omega;\mathbb{R}^2) \quad \text{with } u_\varepsilon \in \mathcal{A}_\varepsilon^{\mathrm{rig}}\}$$

Theorem 1 (Düll, Engl & K. '23)

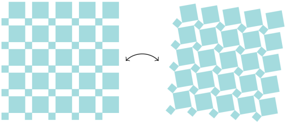
With

$$K := \{\lambda S + (1-\lambda)R : R, S \in \mathrm{SO}(2), Re_1 \cdot Se_1 \geq 0\}$$
$$= \{\alpha Q : Q \in \mathrm{SO}(2), |Y_{\mathrm{stiff}}| \leq \alpha^2 \leq 1\},$$

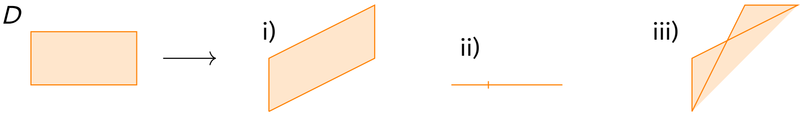
it holds that $\mathcal{M}^{\mathrm{rig}} = \{u : \Omega \to \mathbb{R}^2 : u(x) = Fx + b$ *with* $F \in K$ *and* $b \in \mathbb{R}^2\}$.

Observations:

- Limit deformation are affine conformal contractions
- Poisson ratio $\nu = -1$



Decomposition result at the local level



Let $p > 2$ and $D \subset \mathbb{R}^2$ be an open rectangle with sides $\partial_i D$ for $i = 1, \dots, 4$.
If $u \in W^{1,p}(D;\mathbb{R}^2)$ with $\det \nabla u > 0$ a.e. in D satisfies

$$u|_{\partial_i D} = R_i x + b_i \quad \text{with } R_i \in SO(2) \text{ and } b_i \in \mathbb{R}^2 \text{ for } i = 1, \dots, 4,$$

then there exist $R, S \in SO(2)$ with $\det(Se_1|Se_2) = Se_1 \cdot Re_1 > 0$, $b \in \mathbb{R}^2$ and $\varphi \in W_0^{1,p}(D;\mathbb{R}^2)$ such that

$$\boxed{u(x) = (Se_1|Re_2)x + b + \varphi(x)} \quad \text{for a.e. } x \in D.$$

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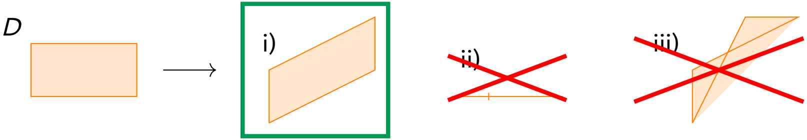

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Proof of Theorem 1: Necessity

To show: If $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^2)$ with $u_\varepsilon \in \mathcal{A}_\varepsilon^{\mathrm{rig}}$, then $\nabla u = F \in K$.

- Apply Reshetnyak's rigidity theorem to u_ε restricted to each rigid component.
- Decomposition result applied to all soft components yields

$$u_\varepsilon = v_\varepsilon + \varphi_\varepsilon \text{ on } \Omega' \Subset \Omega,$$

where $\varphi_\varepsilon \in W^{1,p}(\Omega';\mathbb{R}^2)$ with $\varphi_\varepsilon = 0$ on $\varepsilon Y_{\mathrm{stiff}} \cap \Omega'$ and $v_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$ continuous given by

$$\nabla v_\varepsilon = \begin{cases} S_\varepsilon & \text{on } \varepsilon Y_1, \\ R_\varepsilon & \text{on } \varepsilon Y_3, \\ (S_\varepsilon e_1 | R_\varepsilon e_2) & \text{on } \varepsilon Y_2, \\ (R_\varepsilon e_1 | S_\varepsilon e_2) & \text{on } \varepsilon Y_4, \end{cases} \qquad \text{with } S_\varepsilon, R_\varepsilon \in \mathrm{SO}(2) \text{ such that } S_\varepsilon e_1 \cdot R_\varepsilon e_1 > 0.$$

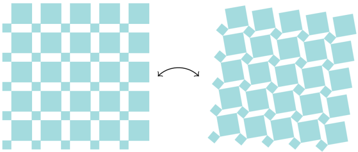
- Observe that $\nabla \varphi_\varepsilon \rightharpoonup 0$ in $L^p(\Omega';\mathbb{R}^{2\times 2})$ and $\nabla v_\varepsilon \rightharpoonup \lambda S + (1-\lambda)R =: F$ for some $S, R \in \mathrm{SO}(2)$ with $Re_1 \cdot Se_1 \geq 0$.

Proof of Theorem 1: Sufficiency

To show: Every affine $u : \Omega \longrightarrow \mathbb{R}^2$ with $\nabla u = F = \lambda S + (1-\lambda)R$ with $S, R \in \mathrm{SO}(2)$ such that $Se_1 \cdot Re_1 \geq 0$ can be approximated weakly in $W^{1,p}(\Omega;\mathbb{R}^2)$ by $u_\varepsilon \in \mathcal{A}_\varepsilon^{\mathrm{rig}}$.

Strategy: Hands-on construction of continuous piecewise affine functions $u_\varepsilon : \Omega \longrightarrow \mathbb{R}^2$:

- if $\det(Se_1|Re_2) = Se_1 \cdot Re_1 > 0$, take u_ε such that $\nabla u_\varepsilon = S$ in εY_1 and $\nabla u_\varepsilon = R$ in εY_3;
- if $\det(Se_1|Re_2) = Se_1 \cdot Re_1 = 0$, then first approximate S by $(S_\varepsilon)_\varepsilon \subset \mathrm{SO}(2)$ such that $S_\varepsilon e_1 \cdot Re_1 > 0$ and use S_ε in place of S.



Then, u_ε is injective and orientation preserving with $\nabla u_\varepsilon \in \mathrm{SO}(2)$ a.e. in $\varepsilon Y_{\mathrm{stiff}} \cap \Omega$, hence, $u_\varepsilon \in \mathcal{A}_\varepsilon^{\mathrm{rig}}$.

Discussion of assumptions

- Dropping orientation preservation or Ciarlet-Nečas condition

  Statement of Theorem 1 still holds. ✓

- Dropping orientation preservation and Ciarlet-Nečas condition

  Similar characterization result for $\lambda = 1/2$ with affine macroscopic deformations $u : \Omega \longrightarrow \mathbb{R}^2$ satisfying

$$\nabla u = F \in \{\alpha Q : \ Q \in SO(2), 0 \leq \alpha \leq 1\} .$$

- Case $p = 2$

  Same characterization result holds ✓ due to trace theorems for curvilinear polygons [Grisvard 1985].

- Case $1 < p < 2$

  Theorem 1 fails ✗, instead: Any affine map $u : \Omega \longrightarrow \mathbb{R}^2$ can be approximated weakly in $W^{1,p}(\Omega; \mathbb{R}^2)$ by a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,p}(\Omega; \mathbb{R}^2)$ with $\int_\Omega u_\varepsilon \, dx = \int_\Omega u \, dx$ and

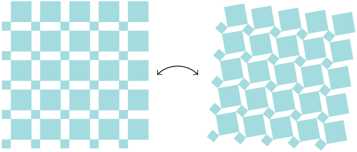
$$\nabla u_\varepsilon \in SO(2) \text{ a.e. in } \Omega \cap \varepsilon Y_{\text{stiff}}.$$

Characterizing macroscopic deformations - The stiff case

From fully rigid to stiff components

Questions:

- How robust are the previous observations to changes in the set-up?
- What is the effect of softening the rigid components by incorporating elastic energy?



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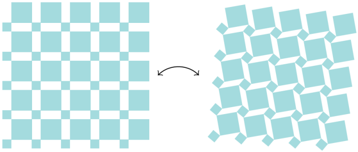

Relaxed assumption:

Rigid components

$$\nabla u \in SO(2) \text{ a.e. in } \varepsilon Y_{\text{stiff}} \cap \Omega$$

$\rightsquigarrow$

Stiff components with diverging elastic constants

$$\int_{\varepsilon Y_{\text{stiff}} \cap \Omega} \operatorname{dist}^p(\nabla u, SO(2))\, dx < C\varepsilon^\beta$$

$\beta > 0$ sufficiently large

Characterization of macroscopic deformations

Theorem 2 (Düll, Engl & K. 2023)

Let $p > 2$, $\beta > 2p - 2$, *and let* $(u_\varepsilon)_\varepsilon \subset \mathcal{A}$ *be a sequence that satisfies*

$$\int_{\varepsilon Y_{\text{stiff}} \cap \Omega} \operatorname{dist}^p(\nabla u_\varepsilon, \mathrm{SO}(2))\, dx \leq C\varepsilon^\beta .$$

If $u_\varepsilon \rightharpoonup u$ *in* $W^{1,p}(\Omega;\mathbb{R}^2)$ *for some* $u \in W^{1,p}(\Omega;\mathbb{R}^2)$, *then*

u *is affine with* $\nabla u \in K = \{\alpha Q : Q \in \mathrm{SO}(2),\ |Y_{\text{stiff}}| \leq \alpha^2 \leq 1\}$.

- Consistency check with rigid case ✓
- Optimality of scaling regime for β is currently open

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If $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^2)$ for some $u \in W^{1,p}(\Omega;\mathbb{R}^2)$, then

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Main proof ingredients:

- **Tool 1:** Quantitative rigidity estimate for cross structures
- **Tool 2:** Poincaré-type inequality for checkerboard structures

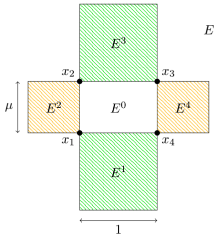
Tool 1: Quantitative rigidity estimate for cross structures

Lemma 1 (Düll, Engl & K. 2023)

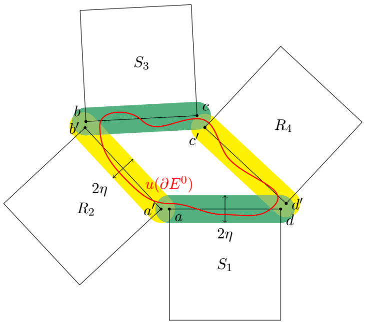
Let $p > 2$. There are constants $C, \delta_0 > 0$ such that for every $u \in W^{1,p}(E;\mathbb{R}^2)$ satisfying the Ciarlet-Nečas condition on $E' := E \setminus E^0$ and $\|\mathrm{dist}(\nabla u, SO(2))\|_{L^p(E')} < \delta_0$, there exist $R, S \in SO(2)$ such that

$$\|\nabla u - S\|_{L^p(E^1\cup E^3;\mathbb{R}^{2\times 2})} + \|\nabla u - R\|_{L^p(E^2\cup E^4;\mathbb{R}^{2\times 2})} \leq C\|\mathrm{dist}(\nabla u, SO(2))\|^{1/2}_{L^p(E')}$$

and $Re_1 \cdot Se_1 \geq -C\|\mathrm{dist}(\nabla u, SO(2))\|^{1/2}_{L^p(E')}$.



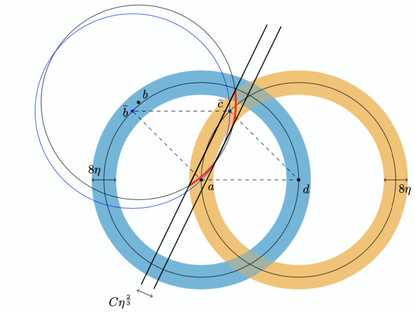
Proof of Lemma 1 for $\mu = 1$ (in pictures)



Step 1: Apply [Friesecke, James & Müller 2002] to each individual rigid square to obtain $S_1, S_3, R_2, R_4 \in \mathrm{SO}(2)$. Set $\eta := C\|\mathrm{dist}(\nabla u, SO(2))\|_{L^p(E')}$.

Step 2: Construct a continuous piecewise affine map $\bar v : E \to \mathbb{R}^2$, determined by the polygon $\bar a \bar b \bar c \bar d$, such that

$$\|u - \bar v\|_{W^{1,p}(E';\mathbb{R}^2)} < C\eta^{1/3}.$$



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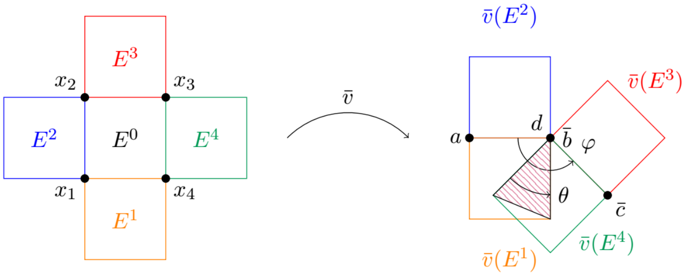

Step 3: Derive estimate for $|S_1 - S_3|$ in terms of $C\eta^{1/3}$, analogously for $|R_2 - R_4|$.

- $\bar{v}(\partial E^0)$ forms parallelogram ✓ by Step 2
- $\bar{v}(\partial E^0)$ is degenerate with sufficiently large angle ✓
- $\bar{v}(\partial E^0)$ is degenerate with small angle ✗ by approximate Ciarlet-Nečas condition on rigid parts

Step 4: Improve estimate to $C\eta^{1/2}$ by repeating Steps 2 and 3 and set $S := S_1$ and $R := R_2$.

Tool 2: Poincaré-type estimate for checkerboard structures

Lemma 2 (Düll, Engl & K. 2023)

Let $p > 2$, $U' \Subset U$ be bounded Lipschitz domains, and $M > 0$. There exists a constant $C > 0$ independent of ε such that for all $u \in W^{1,p}(U;\mathbb{R}^2)$ with

$$\int_{\varepsilon Y_{\mathrm{stiff}} \cap U'} u \, dx = 0$$

and $\|u\|_{L^p(\varepsilon Y_{\mathrm{stiff}} \cap U;\mathbb{R}^2)} \leq M \|u\|_{L^p(\varepsilon Y_{\mathrm{stiff}} \cap U';\mathbb{R}^2)}$, it holds that

$$\boxed{\|u\|_{L^p(\varepsilon Y_{\mathrm{stiff}} \cap U';\mathbb{R}^2)} \leq C \|\nabla u\|_{L^p(\varepsilon Y_{\mathrm{stiff}} \cap U;\mathbb{R}^{2\times 2})}.}$$

Proof of Lemma 2

Step 1: Basic Poincaré inequality for union of Lipschitz domains with path connected closure

Step 2: **Approximate extension** by modification of [Acerbi, Chiadó Piat, Dal Maso & Percivale 1992]:
For $U' \Subset U$ and $r > 0$ sufficiently small, two-step construction of a linear bounded operator

$$\mathcal{E}_r : W^{1,p}(U \cap \varepsilon Y_{\mathrm{stiff}}; \mathbb{R}^2) \cap C^0(\overline{U \cap \varepsilon Y_{\mathrm{stiff}}}; \mathbb{R}^2) \to W^{1,p}(U'; \mathbb{R}^2)$$

such that $\mathcal{E}_r u = u$ a.e. on $U' \cap \varepsilon Y_{\mathrm{stiff}} \setminus \varepsilon B_r$

$$\|\mathcal{E}_r u\|_{L^p(U';\mathbb{R}^2)} \leq C(r)\|u\|_{L^p(U\cap\varepsilon Y_{\mathrm{stiff}};\mathbb{R}^2)}$$ and $$\|\nabla(\mathcal{E}_r u)\|_{L^p(U';\mathbb{R}^{2\times2})} \leq C(r)\|\nabla u\|_{L^p(U\cap\varepsilon Y_{\mathrm{stiff}};\mathbb{R}^{2\times2})}$$

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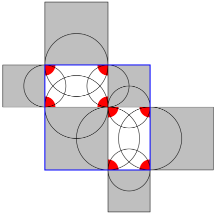

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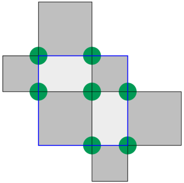


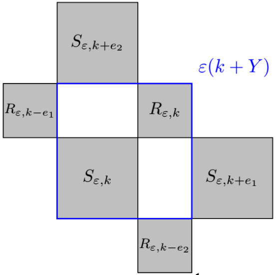
Step 3: Mimick indirect standard proof of Poincaré inequality, with contradiction for r small enough.

Proof of characterization result for macroscopic deformations

Recall Theorem 2: Let $p > 2$, $\beta > 2p - 2$, and let $(u_\varepsilon)_\varepsilon \subset \mathcal{A}$ be a sequence that satisfies

$$\int_{\varepsilon Y_{\text{stiff}} \cap \Omega} \operatorname{dist}^p(\nabla u_\varepsilon, \mathrm{SO}(2))\, dx \le C\varepsilon^\beta$$

If $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^2)$, then u is affine with $\nabla u \in K$.



Proof: Approach shows parallels with

[Friesecke, James & Müller '02, Christowiak & K. '20, Engl, K. & Ritorto '22]

Step 1: Local rigidity argument on each cross structure

Apply **Tool 1** (along with a scaling argument) to find in each cross $E_{\varepsilon,k}$ to find $S_{\varepsilon,k}, R_{\varepsilon,k} \in \mathrm{SO}(2)$ such that

$$\|\nabla u_\varepsilon - S_{\varepsilon,k}\|_{L^p(E^1_{\varepsilon,k}\cup E^3_{\varepsilon,k};\mathbb{R}^{2\times 2})} + \|\nabla u_\varepsilon - R_{\varepsilon,k}\|_{L^p(E^2_{\varepsilon,k}\cup E^4_{\varepsilon,k};\mathbb{R}^{2\times 2})} \le C\varepsilon^{\frac{1}{p}} \|\operatorname{dist}(\nabla u_\varepsilon, \mathrm{SO}(2))\|^{\frac{1}{2}}_{L^p(E'_{\varepsilon,k})}.$$

Define two auxiliary piecewise constant maps $S_\varepsilon, R_\varepsilon : \Omega \to \mathrm{SO}(2)$ as

$$S_\varepsilon := \textstyle\sum_k S_{\varepsilon,k} \mathbb{1}_{\varepsilon(k+Y)} \quad \text{and} \quad R_\varepsilon := \textstyle\sum_k R_{\varepsilon,k} \mathbb{1}_{\varepsilon(k+Y)}$$

Proof of characterization result for macroscopic deformations

Step 2: Strong convergence of the rotation maps For $\Omega' \Subset \Omega$,

$$\int_{\Omega'} |S_\varepsilon(x) - S_\varepsilon(x+\xi)|^p \, dx \leq C\Big(|\xi|^p \varepsilon^{1+\frac{\beta}{2}-p} + \varepsilon^{1+\frac{\beta}{2}}\Big)$$

Key estimate

By **Frechet-Kolmogorov** and due to $\beta > 2p - 2$,

$$S_\varepsilon \longrightarrow S \text{ in } L^p(\Omega'; \mathbb{R}^{2\times 2}) \quad \text{with } S \in SO(2) \text{ constant;}$$

analogously, $R_\varepsilon \longrightarrow R$ in $L^p(\Omega'; \mathbb{R}^{2\times 2})$ and $R \in SO(2)$.

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Step 3: Approximating $(u_\varepsilon)_\varepsilon$ by piecewise affine functions

Let $(w_\varepsilon)_\varepsilon$ piecewise affine with vanishing mean value s.th. $\nabla w_\varepsilon = S$ on εY_1 and $\nabla w_\varepsilon = R$ on εY_3. Then,

$$w_\varepsilon \longrightarrow w \text{ in } L^p(\Omega';\mathbb{R}^2) \quad \text{with } \nabla w = \lambda S + (1-\lambda)R \in K.$$

With Step 2, $\|\nabla u_\varepsilon - \nabla w_\varepsilon\|^p_{L^p(\Omega'\cap\varepsilon Y_{\text{stiff}};\mathbb{R}^{2\times2})} \leq C(\varepsilon^{1+\frac{\beta}{2}} + \|S_\varepsilon - S\|^p_{L^p(\Omega';\mathbb{R}^{2\times2})} + \|R_\varepsilon - R\|^p_{L^p(\Omega';\mathbb{R}^{2\times2})}) \longrightarrow 0$.
Along with **Tool 2**, $\|u_\varepsilon - w_\varepsilon\|_{L^p(\Omega'\cap\varepsilon Y_{\text{stiff}};\mathbb{R}^2)} \longrightarrow 0$, and by a shifting argument, $u_\varepsilon \longrightarrow w$ in $L^p(\Omega';\mathbb{R}^2)$.

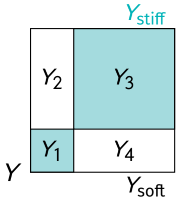
Homogenization via Γ-convergence - An application

Variational problem at finite length scale

Energy functional

For $\varepsilon > 0$, consider $\mathcal{I}_\varepsilon : L^p_0(\Omega;\mathbb{R}^2) \longrightarrow [0,\infty]$ given by

$$\mathcal{I}_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega W_\varepsilon\Big(\frac{x}{\varepsilon}, \nabla u(x)\Big)\,dx & \text{if } u \in \mathcal{A}, \\ \infty & \text{otherwise,} \end{cases}$$



where $W_\varepsilon : \mathbb{R}^2 \times \mathbb{R}^{2\times 2} \longrightarrow [0,\infty]$ is the inhomogeneous energy density

$$W_\varepsilon(y, F) = W_{\text{soft}}(F)\mathbb{1}_{Y_{\text{soft}}}(y) + W_{\text{stiff},\varepsilon}(F)\mathbb{1}_{Y_{\text{stiff}}}(y),$$

and $\mathcal{A} = \{u \in W^{1,p}(\Omega;\mathbb{R}^3)\colon\ \det \nabla u > 0$ a.e. in Ω and u satisfies Ciarlet Nečas$\}$.

Elastic strain densities

Elastic strain density in the soft part, cf. [Conti & Dolzmann '15]

$W_{\text{soft}} : \mathbb{R}^{2\times 2} \longrightarrow [0, \infty]$ is continuous with

- $W_{\text{soft}}(F) = \infty \quad$ if $\det F \leq 0$
- $\frac{1}{C}|F|^p + \frac{1}{C}\theta(\det F) - C \leq W_{\text{soft}}(F) \leq C|F|^p + C\theta(\det F) + C \quad$ if $\det F > 0$
  with $C > 0$ and a convex function $\theta : (0, \infty) \longrightarrow [0, \infty)$ such that $\theta(xy) \leq C(1 + \theta(x))(1 + \theta(y))$ for all $x, y \in (0, \infty)$.

Elastic strain density in the stiff part

$$W_{\text{stiff},\varepsilon} = \varepsilon^{-\beta} W_{\text{stiff}} \qquad \text{for } \beta > 0 \text{ sufficiently large}$$

$W_{\text{stiff}} : \mathbb{R}^{2\times 2} \longrightarrow [0, \infty]$ is continuous with

- $W_{\text{stiff}}(F) = \infty \quad$ if $\det F \leq 0$
- $W_{\text{stiff}} = 0$ on $SO(2)$
- $\frac{1}{C}\operatorname{dist}^p(F, SO(2)) \leq W_{\text{stiff}}(F) \quad$ if $\det F > 0$ with a constant $C > 0$

Homogenization result

Define $\mathcal{I}_{\text{hom}} : L^p_0(\Omega;\mathbb{R}^2) \longrightarrow [0,\infty]$ by

$$\mathcal{I}_{\text{hom}}(u) := \begin{cases} |\Omega| W_{\text{hom}}(F) & \text{if } \nabla u = F \in K, \\ \infty & \text{otherwise,} \end{cases}$$

with $W_{\text{hom}}(F) = \frac{1}{2}|Y_{\text{soft}}| \min_{R,S\in SO(2),\lambda S+(1-\lambda)R=F, Re_1\cdot Se_1\geq 0} W^{\text{qc}}_{\text{soft}}(Se_1|Re_2) + W^{\text{qc}}_{\text{soft}}(Re_1|Se_2).$

Theorem 3 (Düll, E. & Kreisbeck 2023)

If $p > 2$, $\beta > 2p-2$, and $W^{\text{qc}}_{\text{soft}} = W^{\text{pc}}_{\text{soft}}$, then

$$\Gamma(w\text{-}W^{1,p})\text{-}\lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon = \Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon = \mathcal{I}_{\text{hom}}$$

Moreover, any sequence $(u_\varepsilon)_\varepsilon$ with $(u_\varepsilon) \subset L^p_0(\Omega;\mathbb{R}^2)$ and $\sup_\varepsilon \mathcal{I}_\varepsilon(u_\varepsilon) < \infty$ has a subsequence that converges weakly in $W^{1,p}(\Omega;\mathbb{R}^2)$/strongly in $L^p(\Omega;\mathbb{R}^2)$ to an affine function with gradient in K.

Discussion of homogenized energy density

Consider

$$W_{\text{hom}}(F) = \frac{1}{2}|Y_{\text{soft}}| \min_{R,S\in SO(2),\lambda S+(1-\lambda)R=F,Re_1\cdot Se_1\geq 0} W^{\text{qc}}_{\text{soft}}(Se_1|Re_2) + W^{\text{qc}}_{\text{soft}}(Re_1|Se_2)$$

for $F \in K$.

Properties of W_{hom}:

- Simple minimization problem, in fact, at most two choices of R, S to consider for each F
- For W_{soft} frame-indifferent, i.e., $W_{\text{soft}}(QF) = W_{\text{soft}}(F)$ for all $F \in \mathbb{R}^{2\times 2}$ and $Q \in SO(2)$, it holds that

$$W_{\text{hom}}(F) = W_{\text{hom}}(|Fe_1|\,\text{Id}) \quad \text{for } F \in K.$$

- If W_{soft} is frame-indifferent and isotropic, and $\lambda = \frac{1}{2}$, then

$$W_{\text{hom}}(F) = |Y_{\text{soft}}| W^{\text{qc}}_{\text{soft}}\Big(\big(|Fe_1| + \sqrt{1-|Fe_1|^2}\big)\,\text{Id}\Big).$$

Proof of the liminf inequality

Given $(u_\varepsilon)_\varepsilon$ with uniformly bounded energy s.th. $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^2)$ and $\nabla u = \lambda S + (1-\lambda)R \in K$.

Step 1: Exploit $W^{\mathrm{qc}}_{\mathrm{soft}}(F) = W^{\mathrm{pc}}_{\mathrm{soft}}(F) = g(F, \det F)$ with g convex and lower semicontinuous,

$$\mathcal{I}_\varepsilon(u_\varepsilon) \geq \sum_{i\in\{2,4\}} \int_{\Omega'\cap\varepsilon Y_i} W^{\mathrm{qc}}_{\mathrm{soft}}(\nabla u_\varepsilon)\,dx \geq \sum_{i\in\{2,4\}} |\Omega' \cap \varepsilon Y_i|\, g\Big(\fint_{\Omega'\cap\varepsilon Y_i} (\nabla u_\varepsilon, \det \nabla u_\varepsilon)\,dx\Big).$$

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Step 2: Compare with **piecewise affine approximation w_ε** satisfying $\nabla w_\varepsilon = S$ on εY_2 and $\nabla w_\varepsilon = R$ on εY_4 with

$$\Big|\int_{\Omega'\cap\varepsilon Y_i} (\nabla u_\varepsilon, \det\nabla u_\varepsilon) - (\nabla w_\varepsilon, \det\nabla w_\varepsilon)\,dx\Big| \to 0 \quad \text{as } \varepsilon\to 0 \text{ for } i\in\{2,4\}.$$

- Linear extension operator bounded uniformly regarding ε [Grisvard 1985, Lamberti & Provenzano 2020] $L: W^{1,p}(\Omega\cap\varepsilon Y_{\text{stiff}};\mathbb{R}^2)\cap C^0(\overline{\Omega\cap\varepsilon Y_{\text{stiff}}};\mathbb{R}^2) \to W^{1,p}(\Omega';\mathbb{R}^2)$;
- Null-Lagrange property of minors to replace u_ε and w_ε by Lu_ε and Lw_ε on $\Omega'\cap\varepsilon Y_{\text{soft}}$;
- Uniform boundedness of L together with $\|u_\varepsilon - w_\varepsilon\|_{W^{1,p}(\Omega'\cap\varepsilon Y_{\text{stiff}};\mathbb{R}^m)} \to 0$ as $\varepsilon\to 0$.

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Step 3: Conclude $\liminf_{\varepsilon\to 0} \mathcal{I}_\varepsilon(u_\varepsilon) \geq \sum_{i\in\{2,4\}} |\Omega'||Y_i|\, g\Big(\liminf_{\varepsilon\to 0} \fint_{\Omega'\cap\varepsilon Y_i} (\nabla w_\varepsilon, \det\nabla w_\varepsilon)\,dx\Big)$

$$= |\Omega'||Y_2| g\big((Se_1|Re_2), Se_1\cdot Re_1\big) + |\Omega'||Y_4| g\big((Re_1|Se_2), Re_1\cdot Se_1\big)$$

$$= |\Omega'|\frac{|Y_{\mathrm{soft}}|}{2}\big(W^{\mathrm{qc}}_{\mathrm{soft}}(Se_1|Re_2) + W^{\mathrm{qc}}_{\mathrm{soft}}(Re_1|Se_2)\big) \geq |\Omega'| W_{\mathrm{hom}}(F).$$

Construction of recovery sequences

Given an affine map $u(x) = (\lambda R + (1-\lambda)S)x + b$ with $R, S \in SO(2)$, $b \in \mathbb{R}^2$.

Strategy: Construct approximating sequences that are **rigid body motions** on stiff parts.

Step 1: Basic global construction of piecewise affine functions $u_\varepsilon \in \mathcal{A}_\varepsilon^{\mathrm{rig}}$ and $\nabla u_\varepsilon \rightharpoonup \lambda S + (1-\lambda)R$ in $L^p(\tilde{\Omega}; \mathbb{R}^{2\times 2})$ with $\tilde{\Omega} \Supset \Omega$

Step 2: Perturbation in softer components to enforce optimal energy

Orientation preservation via approach through inner perturbations á la [Conti & Dolzmann 2015]
$(u^k_{\varepsilon,j})_j \subset W^{1,p}(\varepsilon(k+Y_i);\mathbb{R}^2)$ such that for $i \in \{2,4\}$

$$u^k_{\varepsilon,j} \rightharpoonup u_\varepsilon \quad \text{in } W^{1,p}(\varepsilon(k+Y_i);\mathbb{R}^2) \text{ as } j\to\infty \quad \text{and} \quad u^k_{\varepsilon,j} = u_\varepsilon \quad \text{on } \partial(\varepsilon(k+Y_i)),$$

and

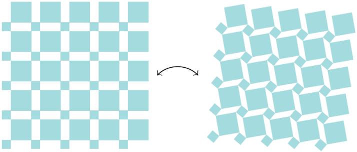
$$\limsup_{j\to\infty} \int_{\varepsilon(k+(Y_2\cup Y_4))} W_{\mathrm{soft}}(\nabla u^k_{\varepsilon,j})\,dx \leq \int_{\varepsilon(k+(Y_2\cup Y_4))} W^{\mathrm{qc}}_{\mathrm{soft}}(\nabla u_\varepsilon)\,dx\,.$$

Ciarlet-Nečas condition is also satisfied by construction [Ball 1981].

Step 3: Diagonalization argument

Summary and outlook

- Characterization of macroscopically attainable deformations as globally affine conformal maps
  - in case of full rigidity
  - for stiff components with diverging elastic constants
- Homogenization result via Γ-convergence



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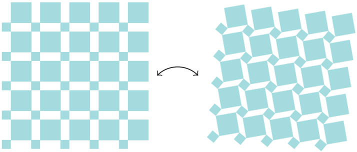

What can be next?

- Optimality of the scaling regime
- Other geometries of stiff components such as triangles
- Higher dimensions and non-periodic structures
- Perturbations in the geometric arrangement, including stochastic effects
- Optimal design of stiff components
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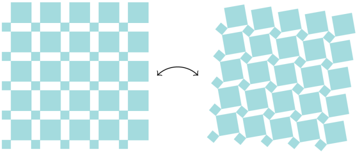

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Thank you!