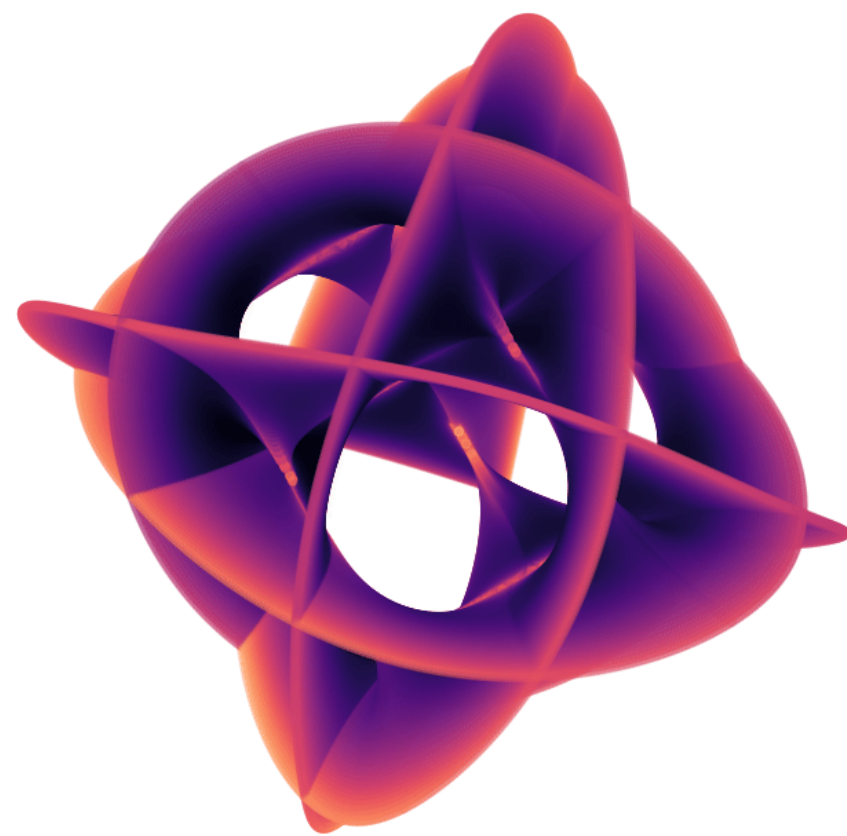


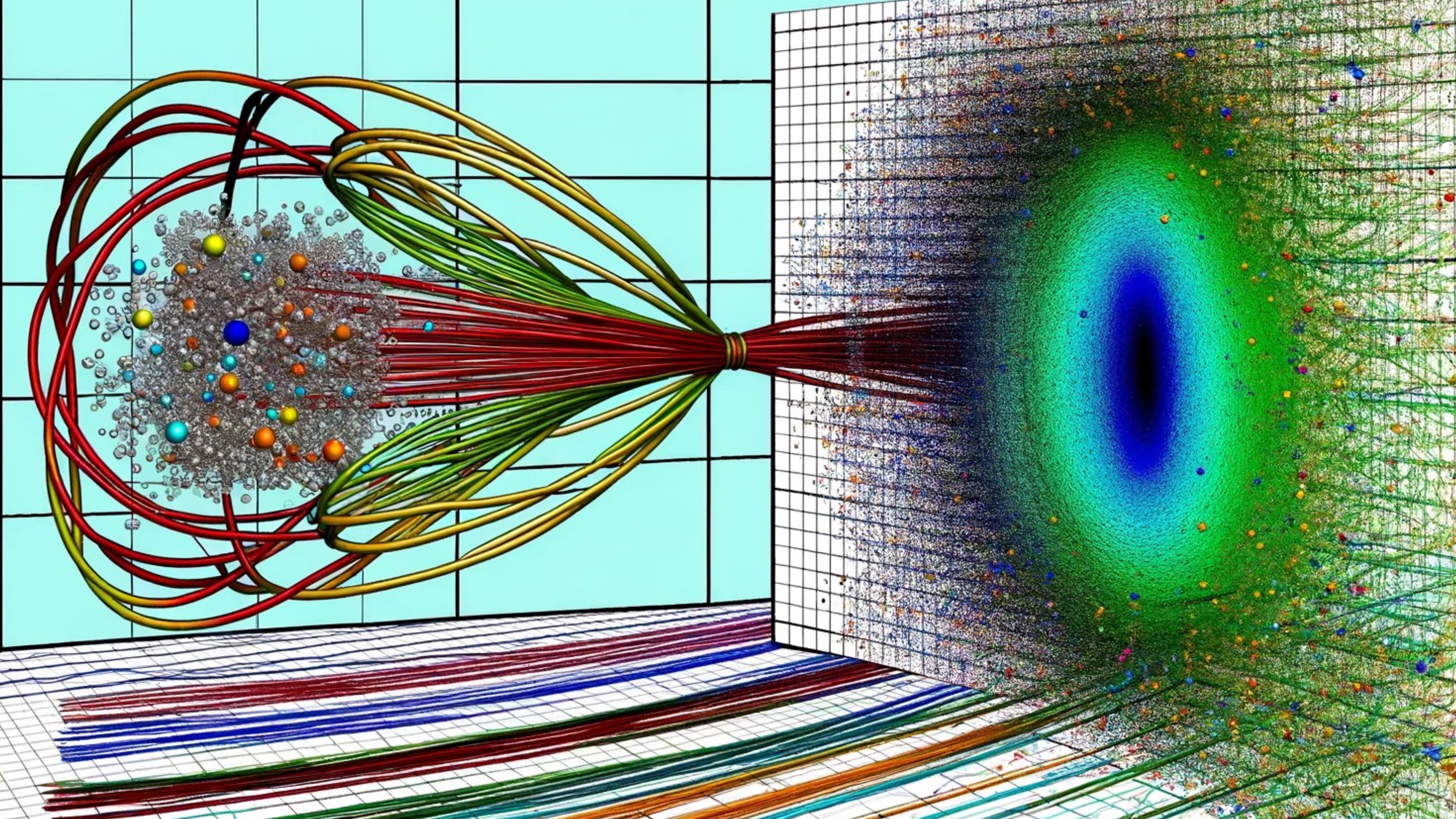
MACHINE AND HUMAN LEARNING THE GEOMETRY OF CALABI–YAU MANIFOLDS

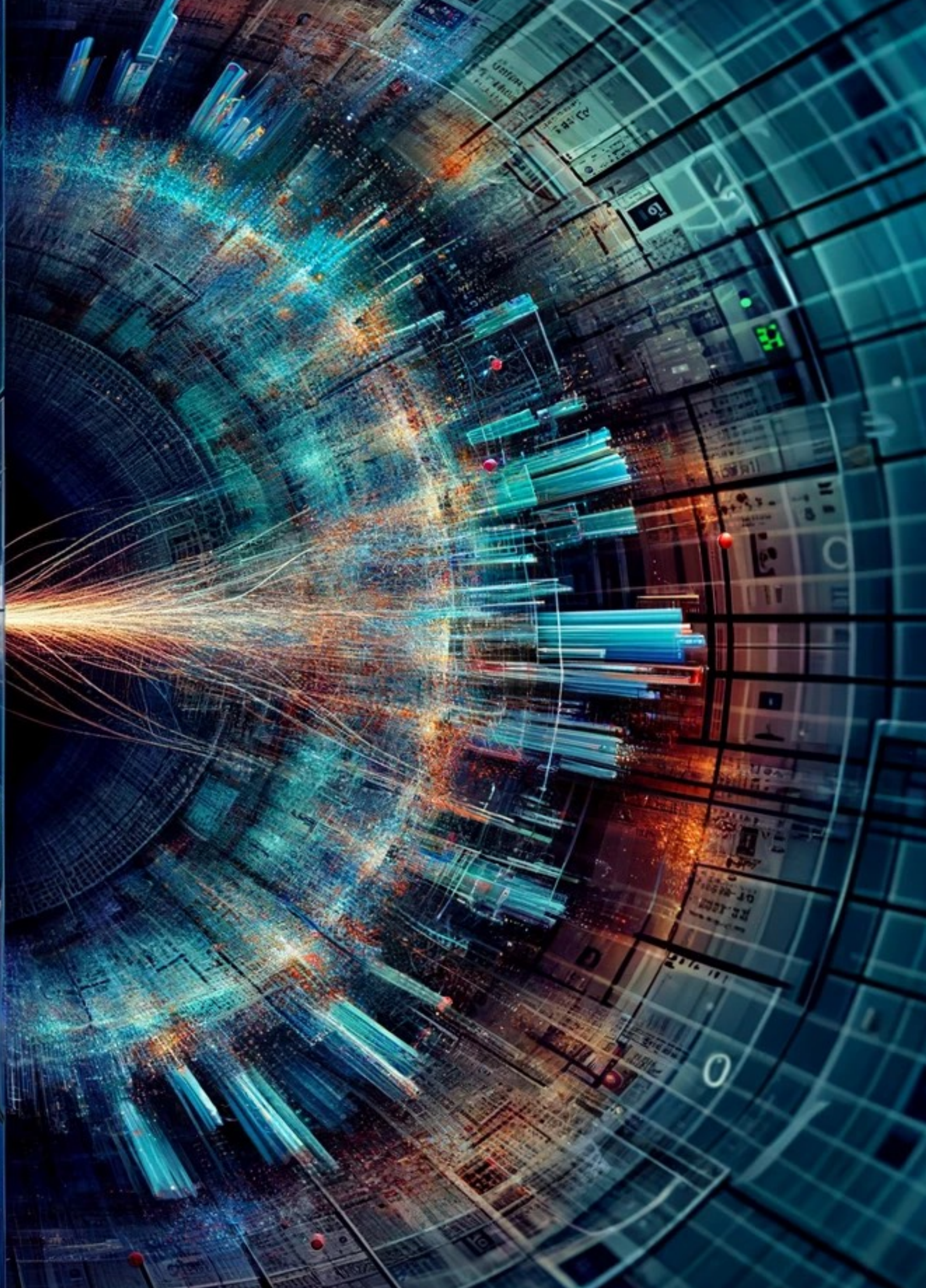


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ERWIN SCHRÖDINGER INTERNATIONAL INSTITUTE FOR MATHEMATICS AND PHYSICS
“THE UNREASONABLE EFFECTIVENESS OF TORIC GEOMETRY”

6 JULY 2026





String Compactification

- Four-dimensional $\mathcal{N} = 1$ effective field theories with chiral matter from compactification of $E_8 \times E_8$ heterotic string on Calabi–Yau threefold

- Complex, Kähler manifold admits a Ricci-flat metric when $c_1 = 0$

Calabi (1957)
Yau (1977)

- Depending on rank, vector bundle can break E_8 to E_6 or $SO(10)$ or $SU(5)$

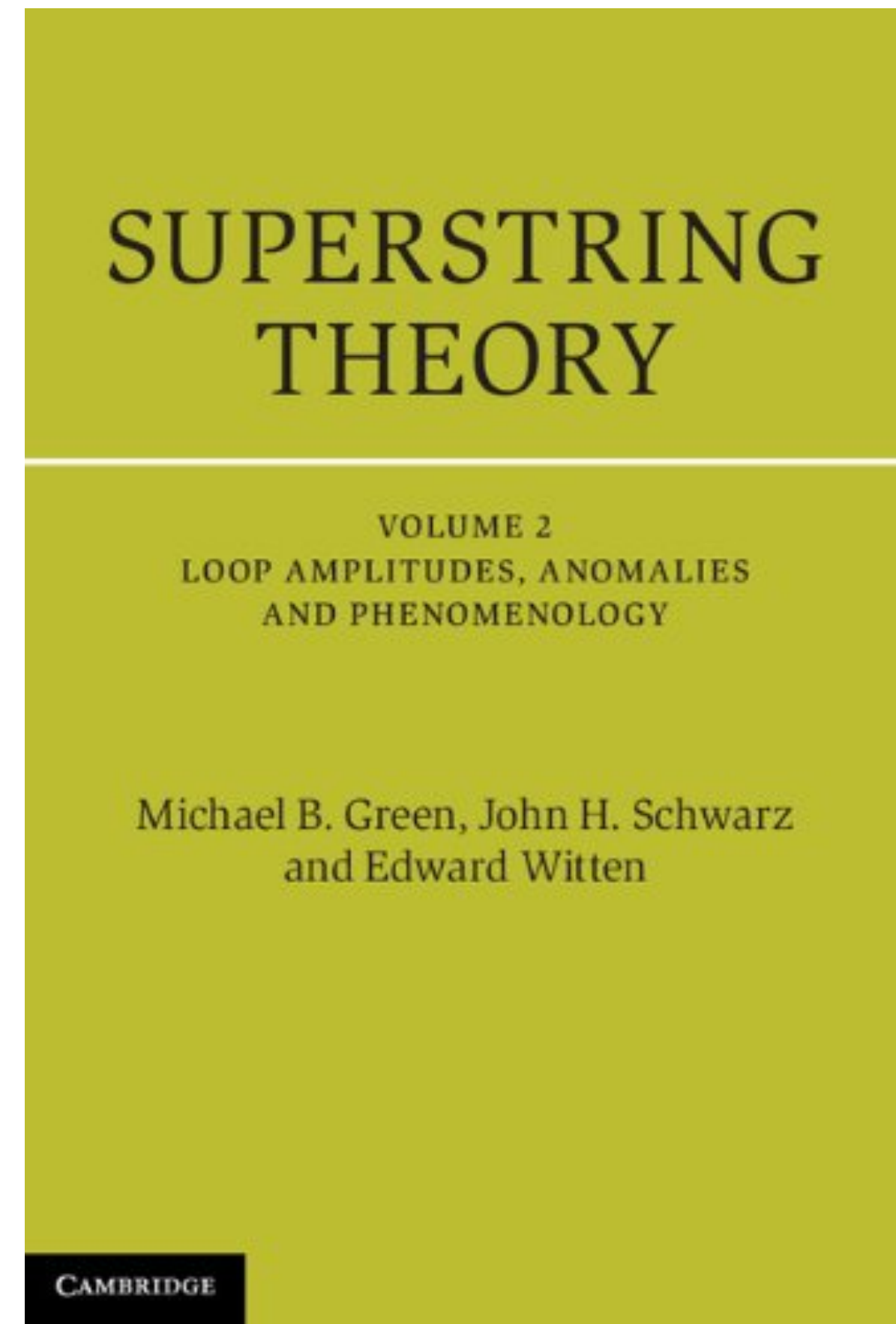
- Break gauge symmetry further by introducing freely acting Wilson lines

- Textbook paradigm for model building in particle physics

Candelas, Horowitz, Strominger, Witten (1985)

- 1/2 billion 4d reflexive polytopes

Kreuzer, Skarke (2000)



Effective Field Theory

- The Lagrangian: $\mathcal{L}_D = \int d^4\theta \, K(\Phi, \bar{\Phi}) \supset N_{i\bar{j}} \Phi^i \bar{\Phi}^{\bar{j}}$

$$\mathcal{L}_F = \int d^2\theta \, W(\Phi) + \text{h.c.} \supset \lambda_{ijk} \Phi^i \Phi^j \Phi^k$$

- Topological data gives us fields in low-energy theory plus the types of interactions that are present

Braun, He, Ovrut, Pantev (2005)
Bouchard, Donagi (2005)

- λ_{ijk} are Yukawa couplings, which tell us how strongly fields interact

- To calculate these in a general construction, need to know Calabi–Yau, X

holomorphic vector bundle, V

Ricci-flat metric

- Ricci-flat metric is unique for any complex structure, Kähler class

Why a Ricci-flat Metric?

- Ricci-flat metric, $g_{a\bar{b}} = \partial_a \partial_{\bar{b}} K$, $R_{a\bar{b}} = 0$, solves leading order vacuum equations of motion
- For purposes of phenomenology, the Ricci-flat metric needed for the following computations
 - normalize kinetic terms from Kähler potential for matter
 - Yukawa couplings in general heterotic compactifications
 - moduli stabilization
 - α' corrections
 - supersymmetry breaking

- Ricci-flatness is a system of fourth order, non-linear partial differential equations, *viz.*,

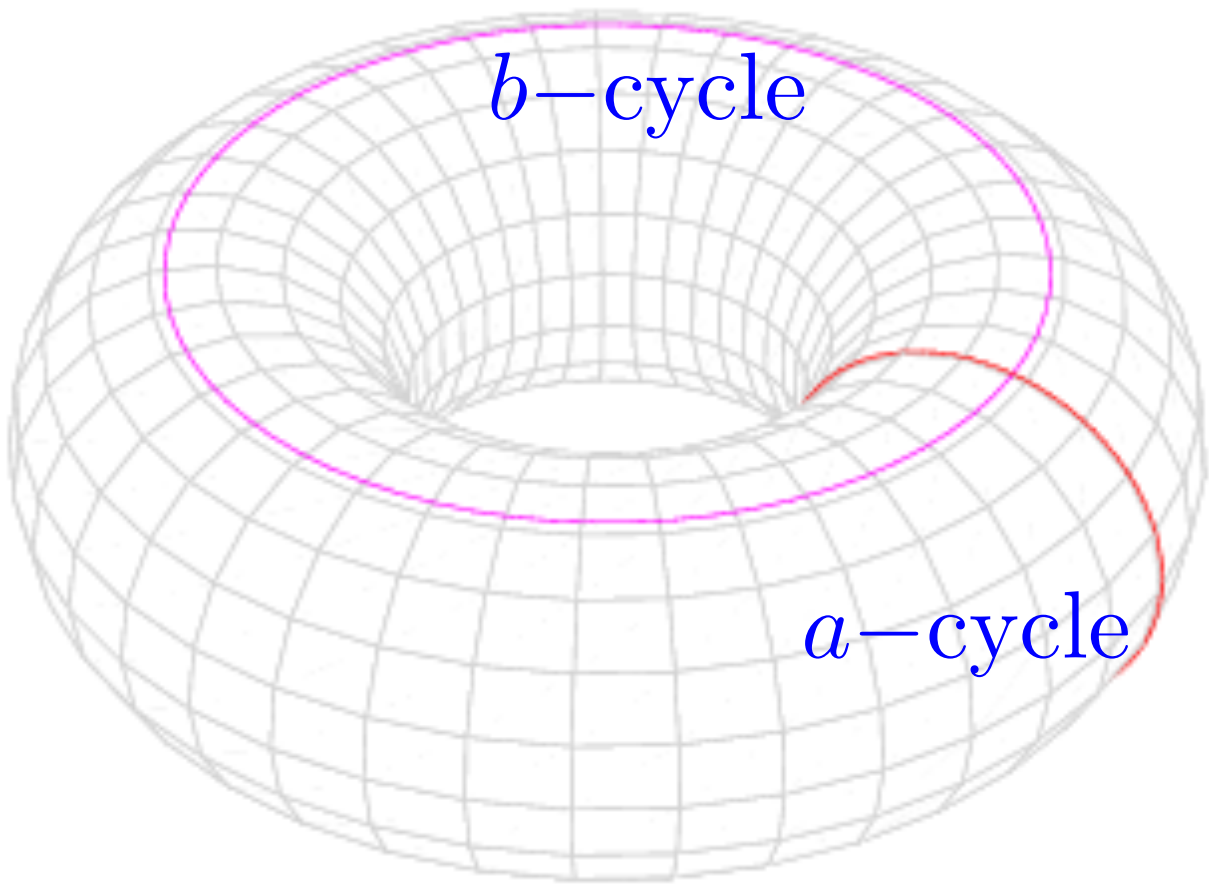
$R_{a\bar{b}} = \partial_a \partial_{\bar{b}} \log \det g = 0 \implies$ equivalent to second order Monge–Ampère equation for φ

$$\det (g_{a\bar{b}} + \partial_a \partial_{\bar{b}} \varphi) = e^f \det g_{a\bar{b}}$$

Yau (1977)

- To date, best we can do is a numerical approximation for complex dimension $n \geq 2$

Analytic Ricci-flat Metrics

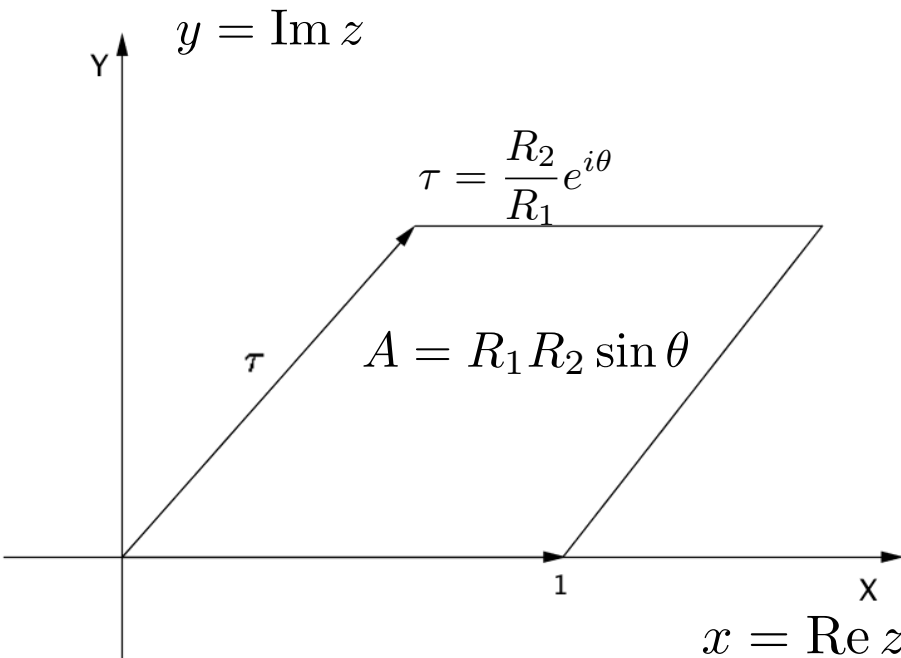
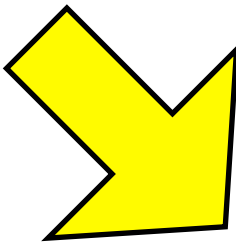


Flat, but has non-trivial homotopy

There are non-contractible cycles

Kähler parameter: area A **size**

complex structure parameter: τ **shape**



1-fold: $z_0^3 + z_1^3 + z_2^3 = 0 \subset \mathbb{P}^2$

1

1

1

$\chi = 2 - 2g = 0$

Clifford (1873)

All K3 surfaces are diffeomorphic
as smooth, real four manifolds

1

0

0

1

0

20

0

1

1

0

0

1

$\chi = 24$

$$\sum_{i=1}^4 z_i^4 - \frac{1}{3} \left(\sum_{i=1}^4 z_i^2 \right)^2 = 0 \subset \mathbb{P}^3$$

quartic in $\mathbb{P}^3$ is singular

Kummer surface $T^4/\mathbb{Z}_2$

Kachru, Tripathy, Zimet (2018, 2020)

How to Approximate Ricci-flat Metrics

- Solve Monge–Ampère directly

Headrick, Wiseman (2005)

$$K(z, \bar{z}) = \frac{1}{\pi} \log |z|^2, \quad |z|^2 = \sum_{a=1}^{n+1} z^a \bar{z}^a$$

- Donaldson's algorithm

– start from Fubini–Study metric on ambient projective space, pullback to manifold

– iterative procedure to construct “balanced metric”

– $k \rightarrow \infty$ limit gives Ricci-flat metric

Tian (1990)
Donaldson (2005)

– numerical implementation expensive, slow convergence

– baseline for comparison

Douglas, Karp, Lukic, Reinbacher (2006)
Braun, Brelidze, Douglas, Ovrut (2007)

$$g_{a\bar{b}} = \partial_a \partial_{\bar{b}} K = \frac{|z|^2 \delta^{a\bar{b}} - z^a \bar{z}^b}{\pi |z|^4}$$

$$K^{(k)}(z, \bar{z}) = \frac{1}{k\pi} \log(h^{a\bar{b}} s_a \bar{s}_b)$$

$$H_{a\bar{b}} = \frac{N_k}{\text{Vol}_\Omega} \int_X d\text{Vol}_\Omega \frac{s_a \bar{s}_b}{h^{c\bar{d}} s_c \bar{s}_d}$$

$$f(x) = \int dy \, w(y) B_w(x, y) f(y)$$

$$w(y) = \det H \implies$$

$$e^{-K^{(k)}} B_w(x, x) = \text{constant}$$

$$h^{a\bar{b}} = (H_{a\bar{b}})^{-1}$$

- Neural network

– engineer a loss function that vanishes when Ricci-flat metric is realized

Ashmore, He, Ovrut (2019)
Anderson, Gerdes, Gray, Krippendorf, Raghuram, Ruehle (2020)
Douglas, Lakshminarasimhan, Qi (2020)
VJ, Mayorga Peña, Mishra (2020)
Douglas (2021)
Larfors, Lukas, Ruehle, Schneider (2021, 2022)
Berglund, Butbaia, Hübsch, VJ, Mayorga Peña, Mishra, Tan (2022)
Gerdes, Krippendorf (2022)
...

Neural Network Approximation

- Loss function: $\mathcal{L} \sim \sum_{\text{points}} \sum_{\text{conditions}} |\text{LHS} - \text{RHS}|$
- One of the conditions is flatness: $\sum_{a,\bar{b}} |R_{a\bar{b}}|$ $\longleftarrow$ derivatives amplify error

$$\text{Vol}_\Omega = \int_X \Omega \wedge \bar{\Omega} \quad \longleftarrow \text{metric independent}$$

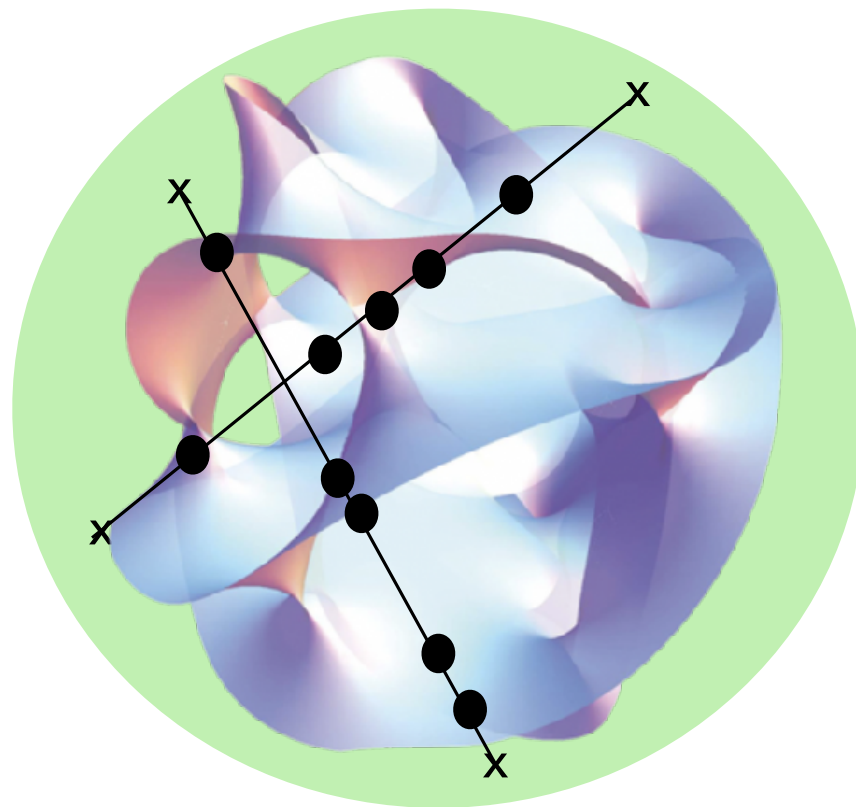
$$\text{Vol}_J = \int_X J \wedge J \wedge J \quad \longleftarrow \text{metric dependent: } J = \frac{i}{2} g_{a\bar{b}} dz^a \wedge d\bar{z}^{\bar{b}}$$

$$dJ = 0, \quad J \wedge \Omega = 0$$
- Other conditions ensure closedness of Kähler form, compatibility between coordinate patches, that we remain in the same Kähler class
- Last can be achieved with Kähler–Ricci flow or Phi model

$$\left. \begin{aligned} J^{\text{CY}} &= J^{\text{FS}} + i \partial \bar{\partial} \phi \\ g^{\text{CY}} &= g^{\text{FS}} + \partial \bar{\partial} \phi \\ \phi &\text{ globally defined} \end{aligned} \right\} \begin{array}{l} \text{implemented in} \\ \text{cymetric} \end{array}$$

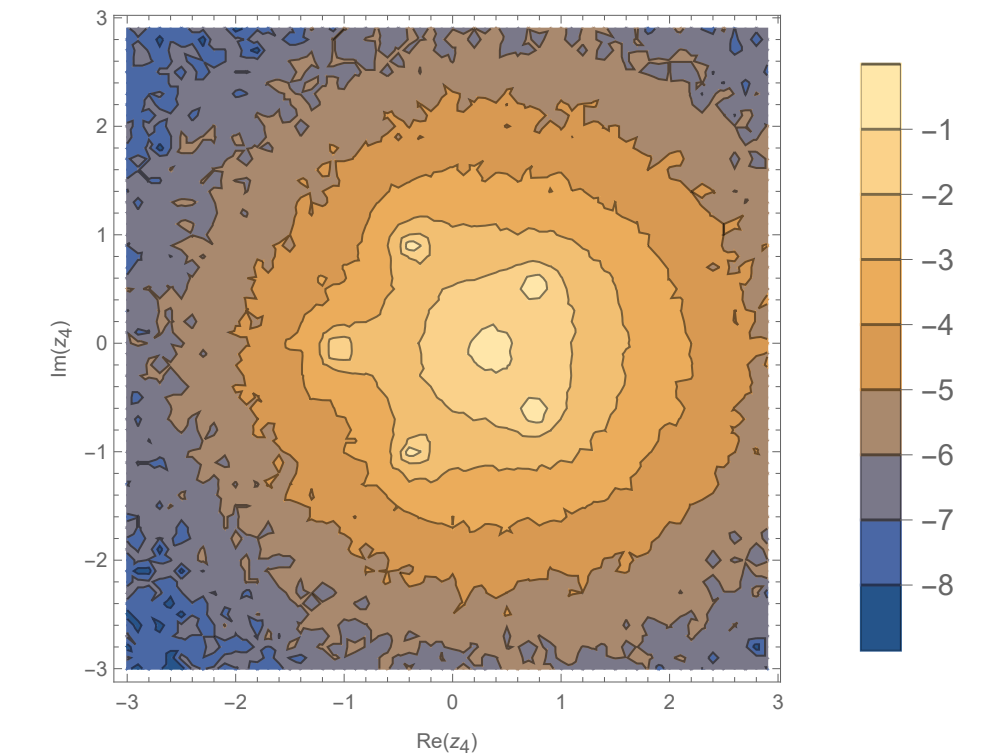
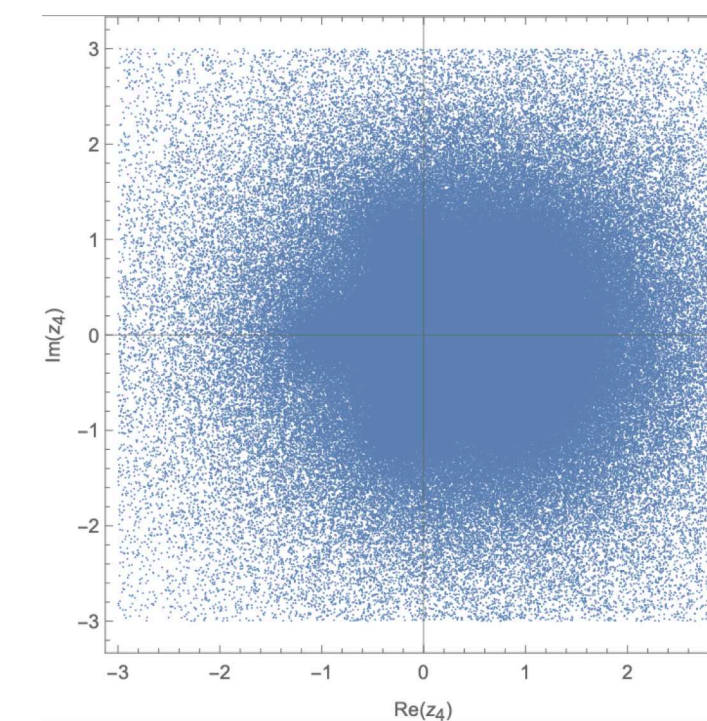
Point Selection

- Lines in $\mathbb{P}^n$ uniformly distributed with respect to Fubini–Study metric
- For CICY hypersurfaces, choose points in ambient space and see where it intersects manifold

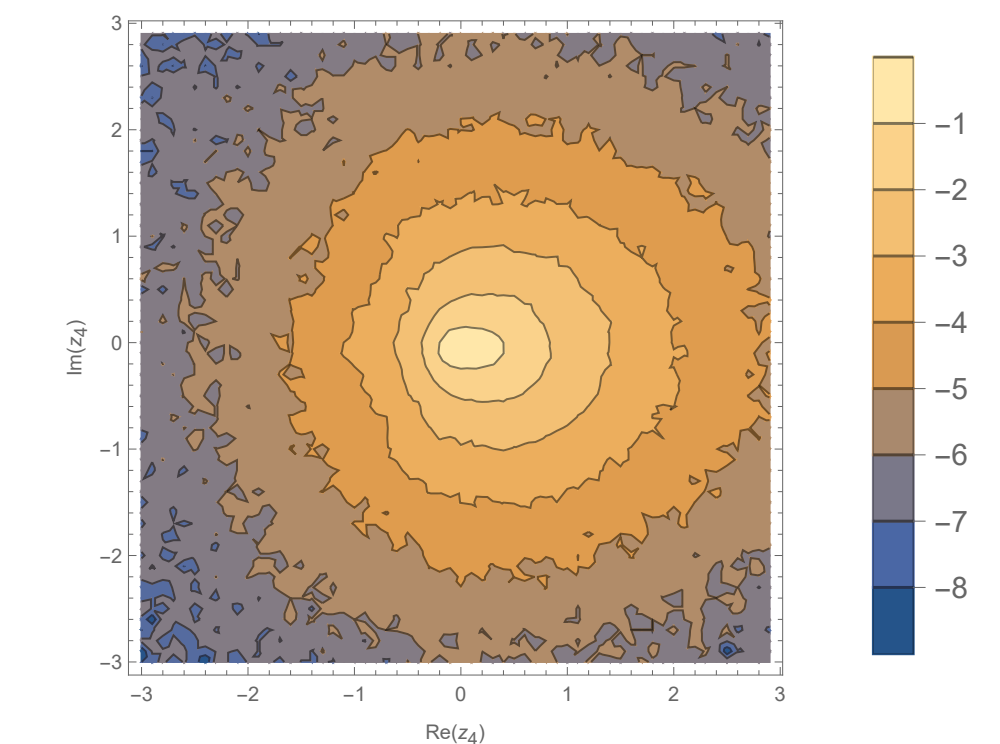
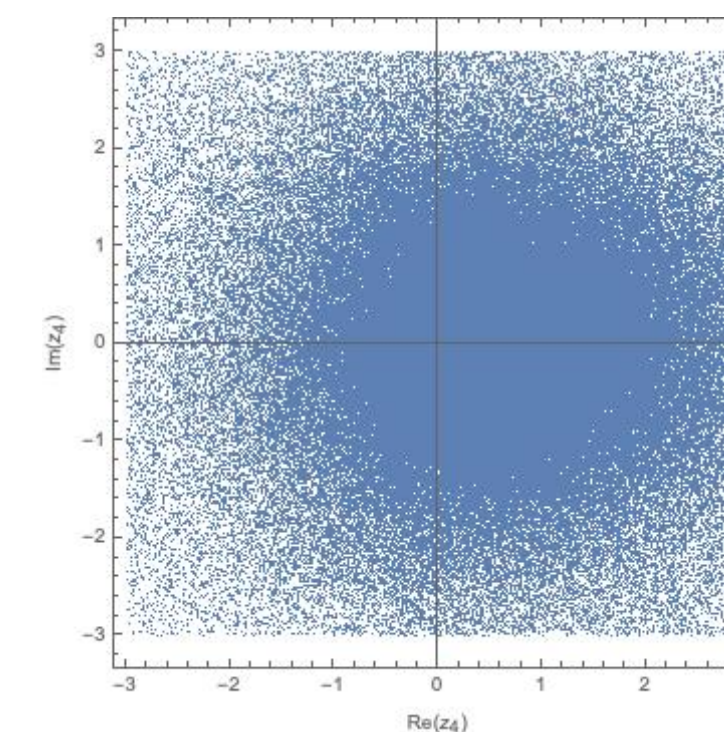


Shiffman, Zelditch (1998)
 Braun, Brelidze, Douglas, Ovrut (2007)
 Anderson, Braun, Karp, Ovrut (2010)
 Ashmore, He, Ovrut (2019)

$$\sum_{i=1}^5 z_i^5 = 0 \subset \mathbb{P}^4$$



$$\sum_{i=1}^5 z_i^5 + \prod_{i=1}^5 z_i = 0 \subset \mathbb{P}^4$$

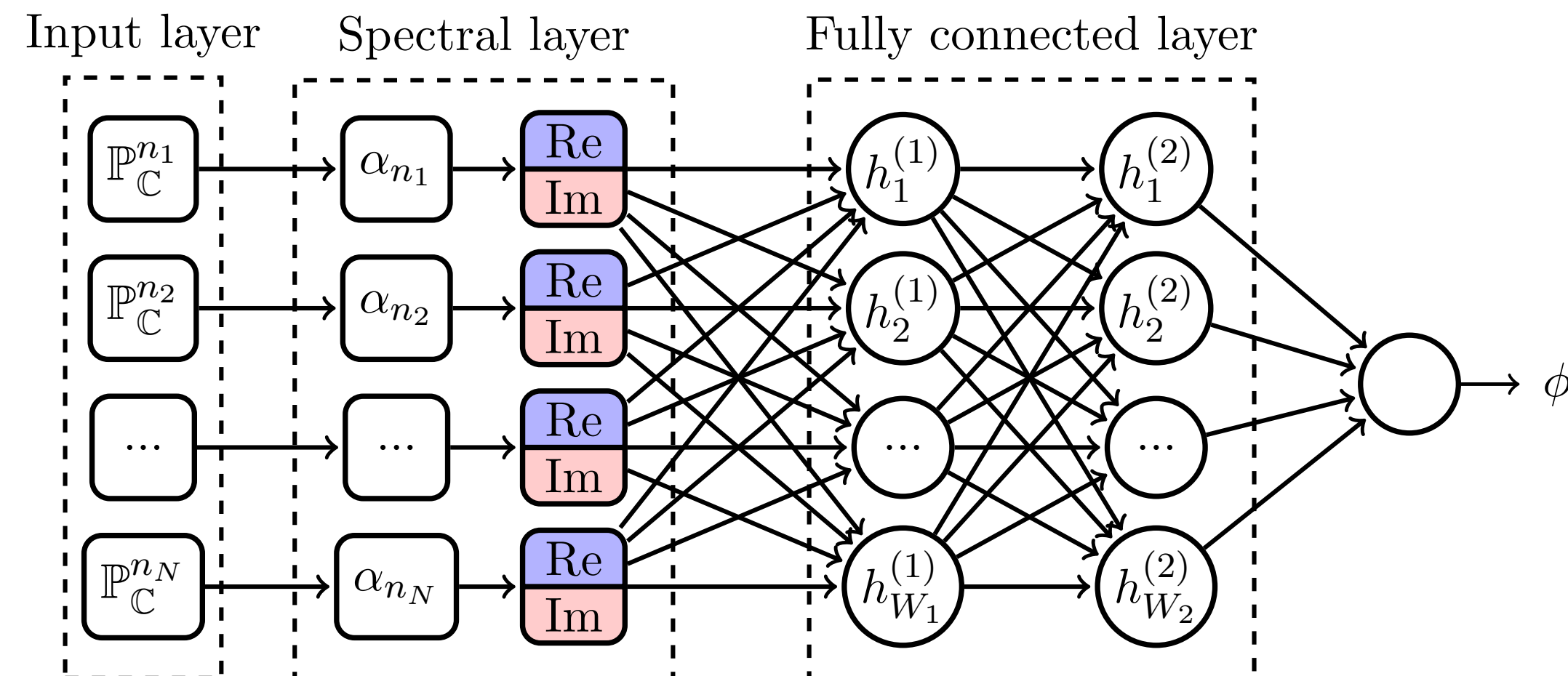


Spectral Network

- Define map $\alpha_n(p) : \mathbb{P}^n \rightarrow \mathbb{C}^{n+1, n+1}$

$$\alpha_n(p) = \begin{bmatrix} \frac{Z_0 \overline{Z_0}}{|Z|^2} & \frac{Z_0 \overline{Z_1}}{|Z|^2} & \cdots & \frac{Z_0 \overline{Z_n}}{|Z|^2} \\ \frac{Z_1 \overline{Z_0}}{|Z|^2} & \frac{Z_1 \overline{Z_1}}{|Z|^2} & \cdots & \frac{Z_1 \overline{Z_n}}{|Z|^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{Z_n \overline{Z_0}}{|Z|^2} & \frac{Z_n \overline{Z_1}}{|Z|^2} & \cdots & \frac{Z_n \overline{Z_n}}{|Z|^2} \end{bmatrix}$$

$$Z \rightarrow \lambda Z, \quad \lambda \in \mathbb{C}^* \quad \text{invariant}$$



$$g^{\text{CY}} = g^{\text{FS}} + \partial \bar{\partial} \phi$$

Calabi–Yau Twofold

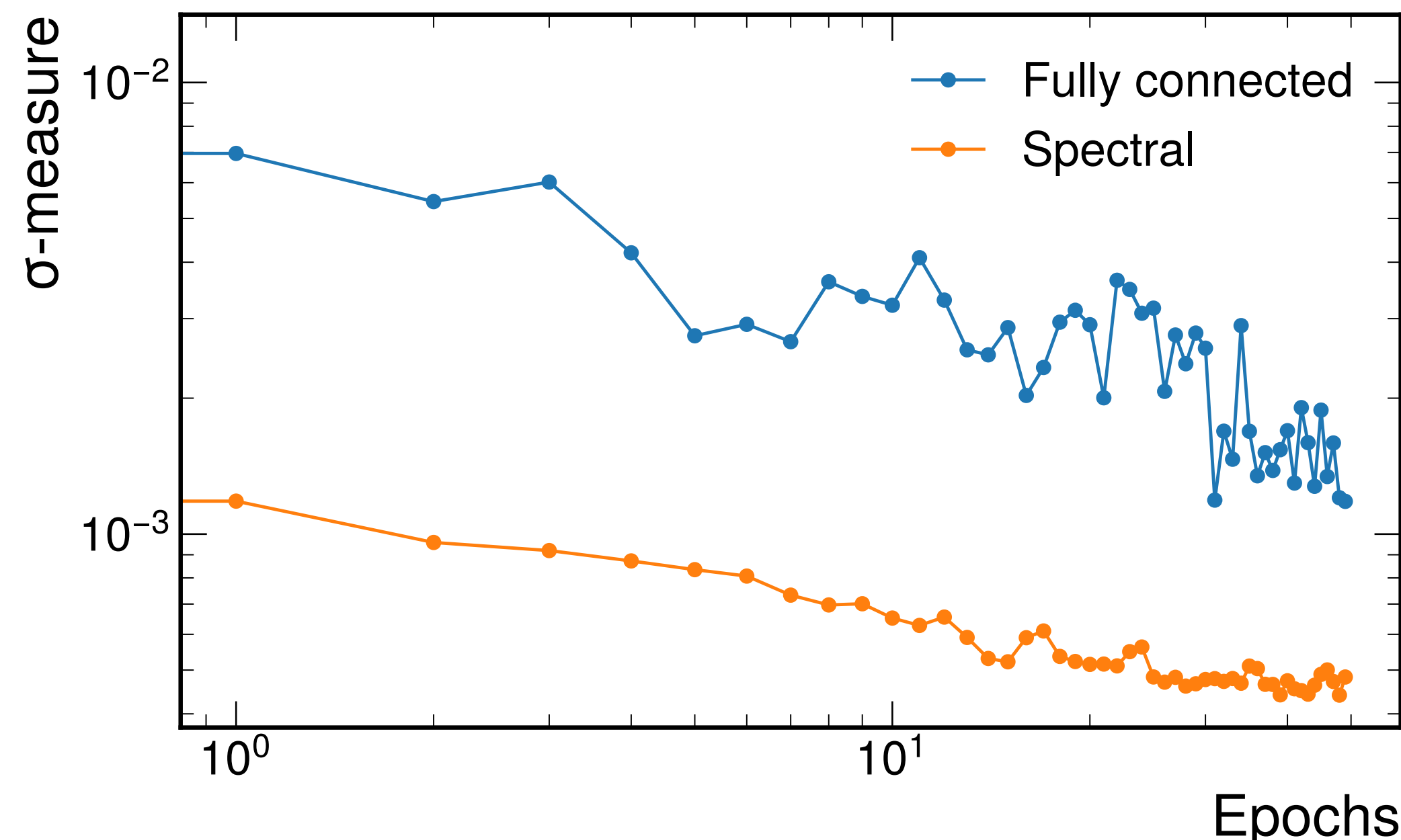
- Consider Cefalú quartic: $p_\lambda(z) = \sum_{i=1}^4 z_i^4 - \frac{\lambda}{3} \left(\sum_{i=1}^4 z_i^2 \right)^2 \subset \mathbb{P}^3$

singular when $\lambda \rightarrow \lambda^\# \in \left\{ \frac{3}{4}, 1, \frac{3}{2}, 3 \right\} \leftarrow X_{\frac{3}{4}} \simeq X_0/\mathbb{Z}_2, \quad X_1 \simeq T^4/\mathbb{Z}_2$

polynomial fails to be transverse
 $\frac{\partial p_\lambda}{\partial z_i} = 0 \quad \forall i$

respectively, $\{8, 16, 12, 4\}$ singularities of A_1 -type
 at $\lambda^\# + \varepsilon$: singular point becomes $\mathbb{P}^1$

- $\lambda = 0$:



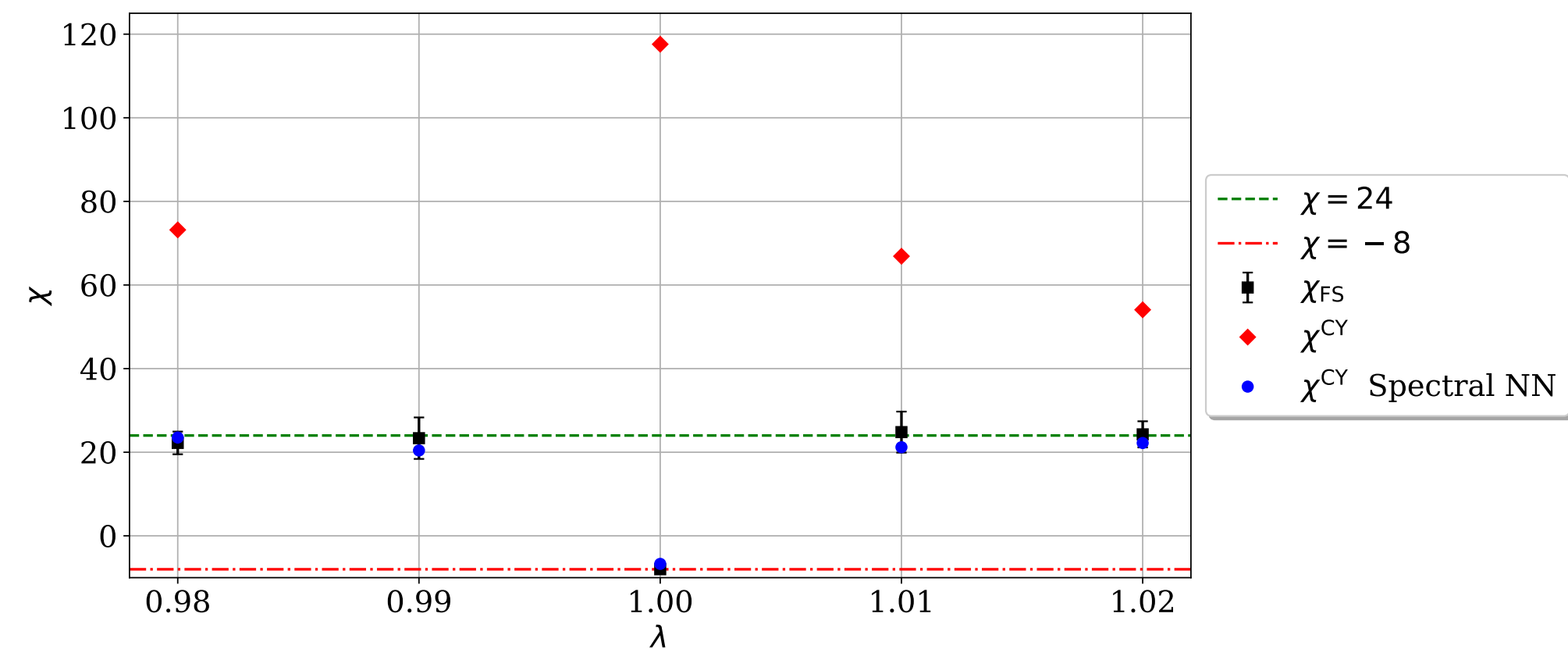
10^5 points for training
 performance equivalent to $k = 8$
 in Donaldson's algorithm

cf. Headrick, Nasser (2009)

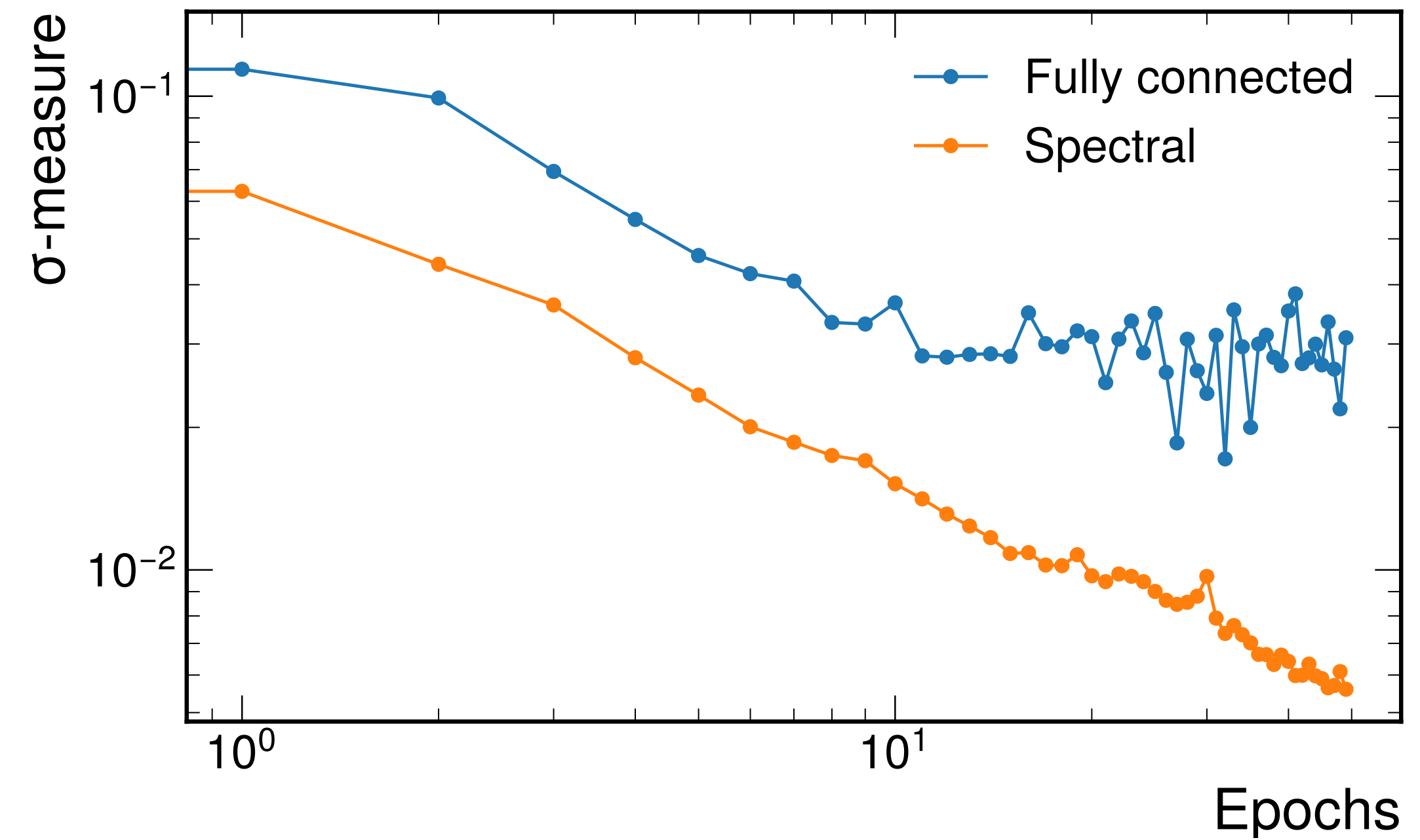
Kähler Geometry

- Non-vanishing Christoffel symbols are $\Gamma_{bc}^a = \overline{\Gamma_{\bar{b}\bar{c}}^{\bar{a}}} = (\partial_b g_{c\bar{d}}) g^{\bar{d}a}$
- Non-vanishing components of Riemann tensor $R_{b\bar{c}d}^a = -\bar{\partial}_{\bar{c}} \Gamma_{bd}^a$, $R_{\bar{b}cd}^a = -\bar{\partial}_{\bar{b}} \Gamma_{cd}^a$
 $R_{b\bar{c}d}^a = \overline{R_{bcd}^{\bar{a}}}$, $R_{\bar{b}cd}^a = \overline{R_{b\bar{c}d}^{\bar{a}}}$
- Ricci tensor is $R_{a\bar{b}} = -R_{ca\bar{b}}^c$
 Ricci form is $\text{Ric}(J) = -i R_{a\bar{b}} dz^a \wedge d\bar{z}^{\bar{b}}$, closed by construction and exact when $c_1(J) = 0$
- Curvature form is $\mathcal{R}_b^a = R_{b\bar{c}d}^a dz^c \wedge d\bar{z}^{\bar{d}}$
- Can compute curvature invariants, e.g. $R_{a\bar{b}cd} R^{a\bar{b}cd}$
- Euler density is a function of Riemann tensor that integrates to Euler character

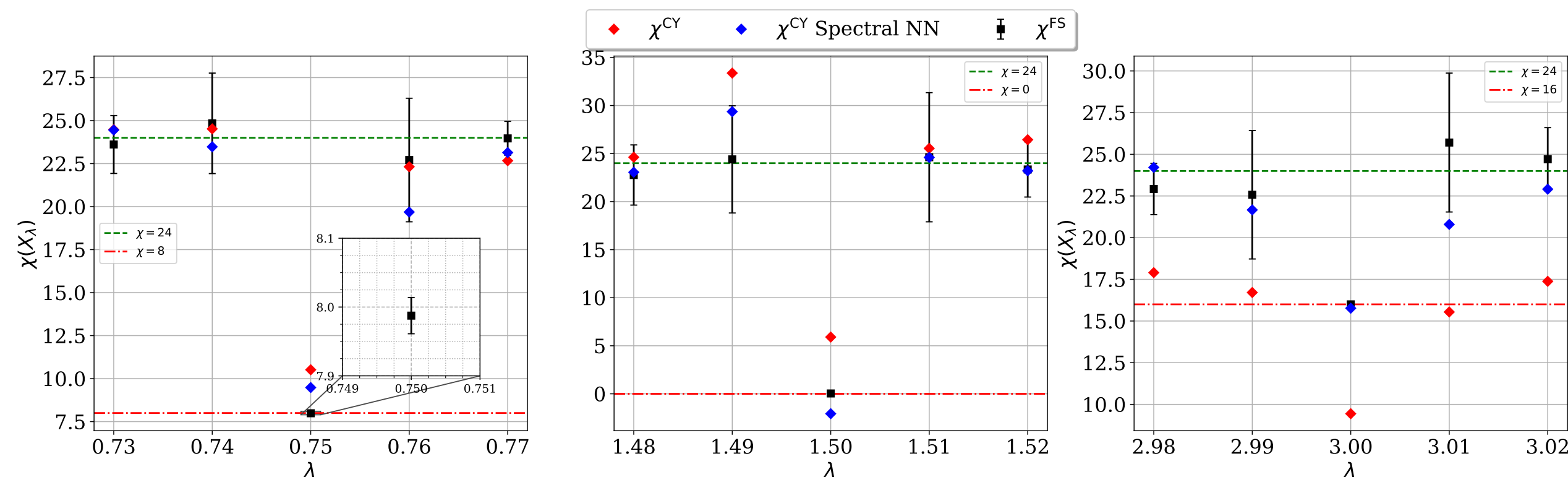
Calabi–Yau Twofold



$\lambda = 1$



$\lambda = 0.98$



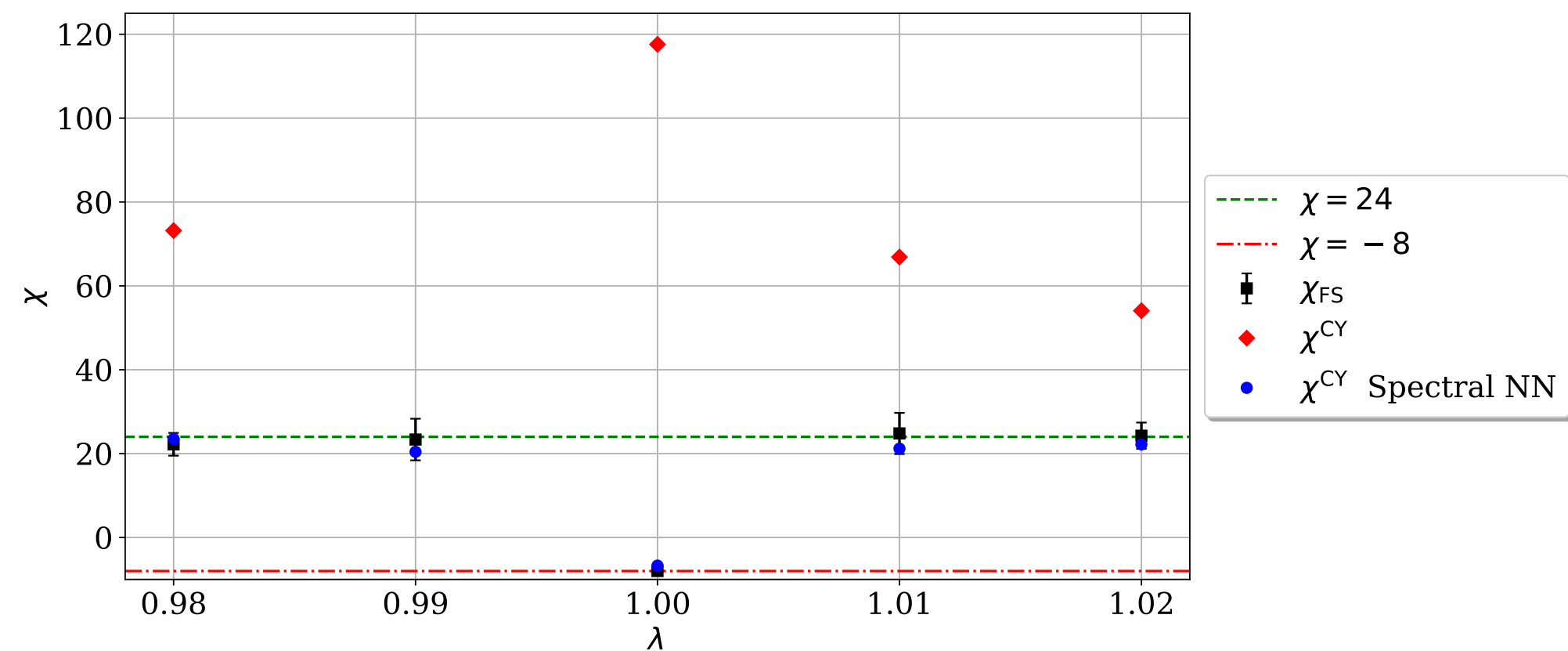
$\lambda = 0.75$

$\lambda = 1.5$

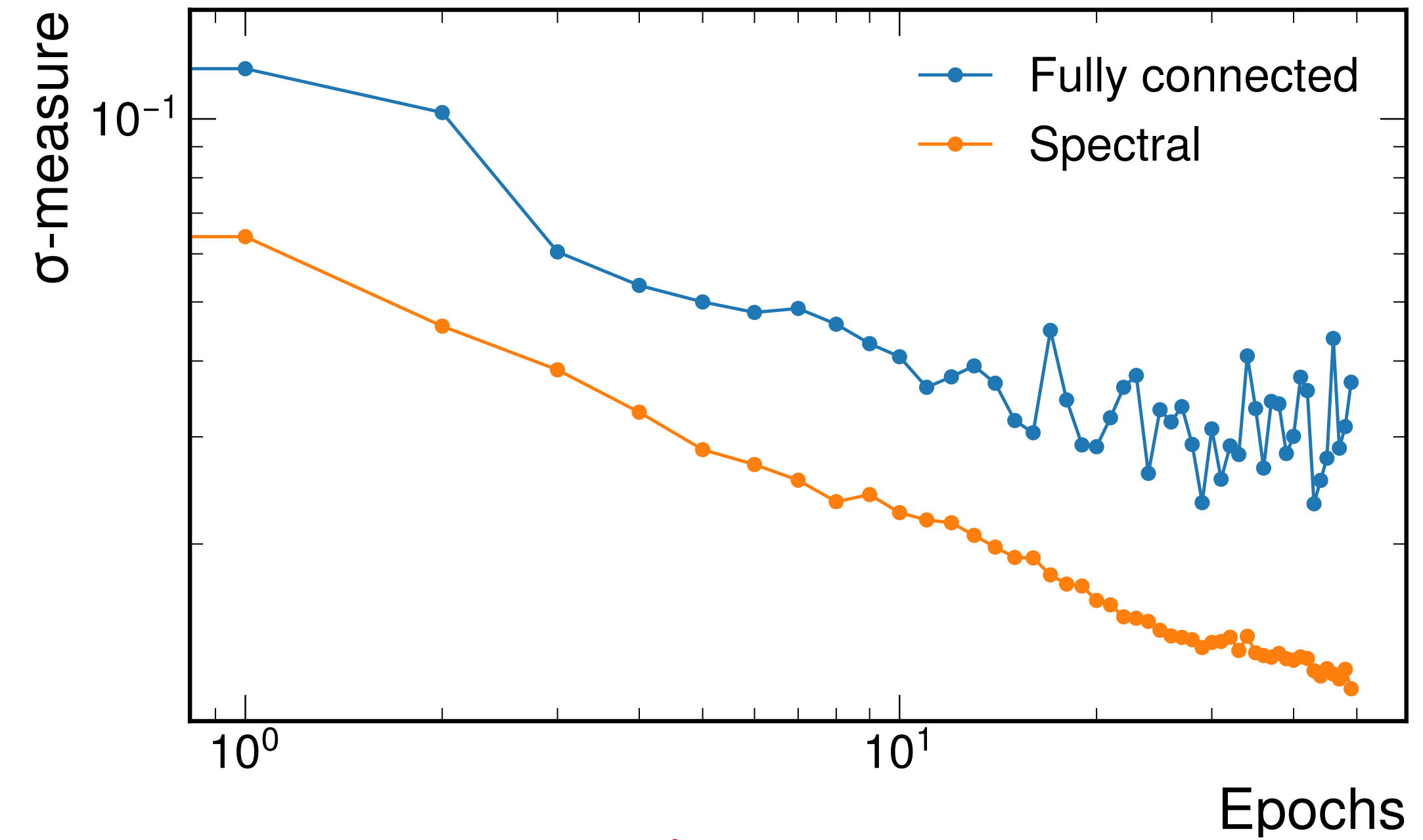
$\lambda = 3$

$$\chi = \int_{X_s} e(J) + 2 |\text{Sing}(X)| = 24 .$$

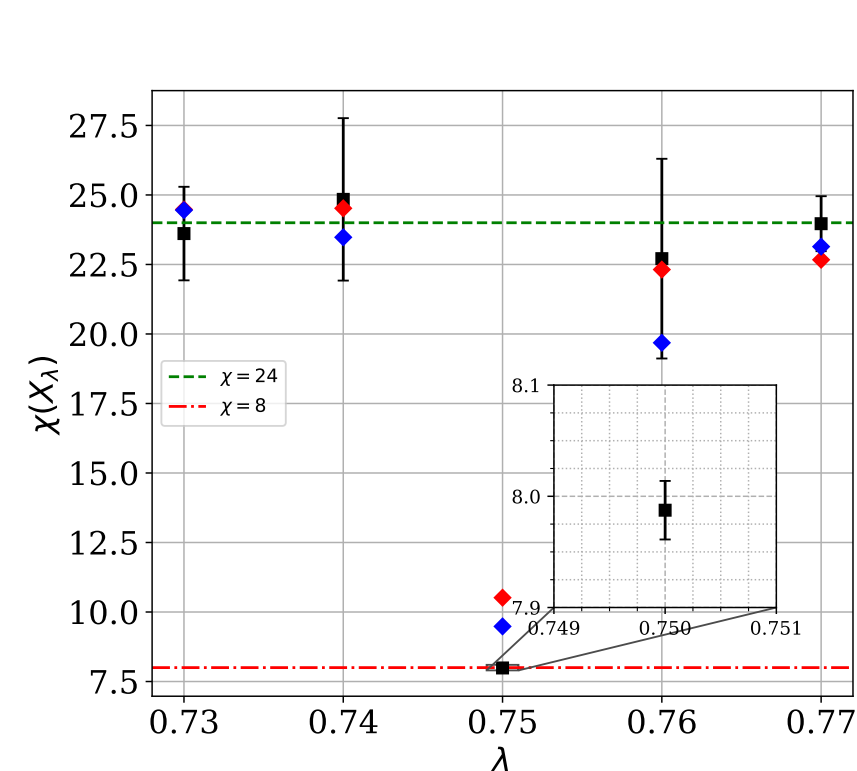
Calabi–Yau Twofold



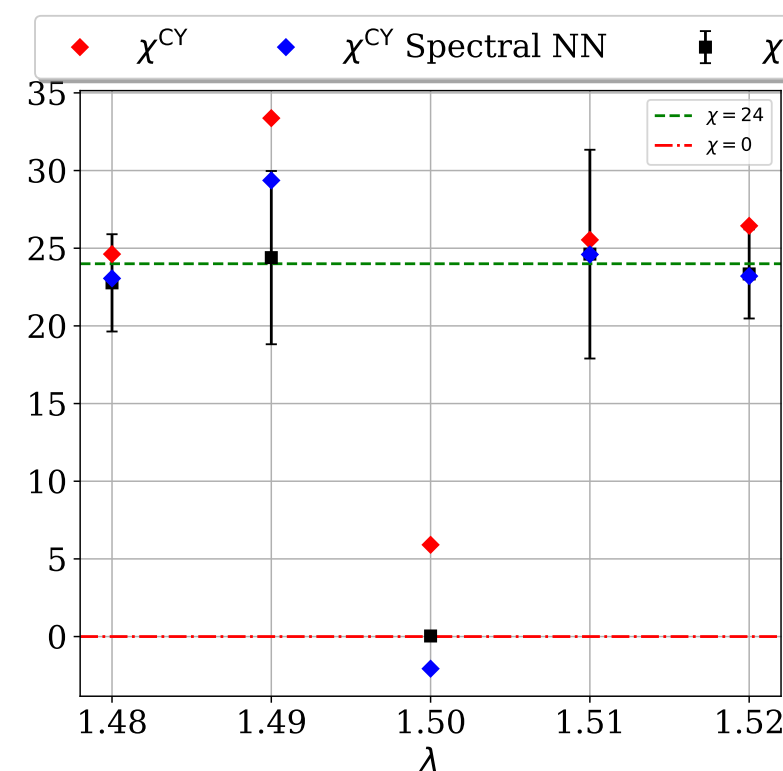
$\lambda = 1$



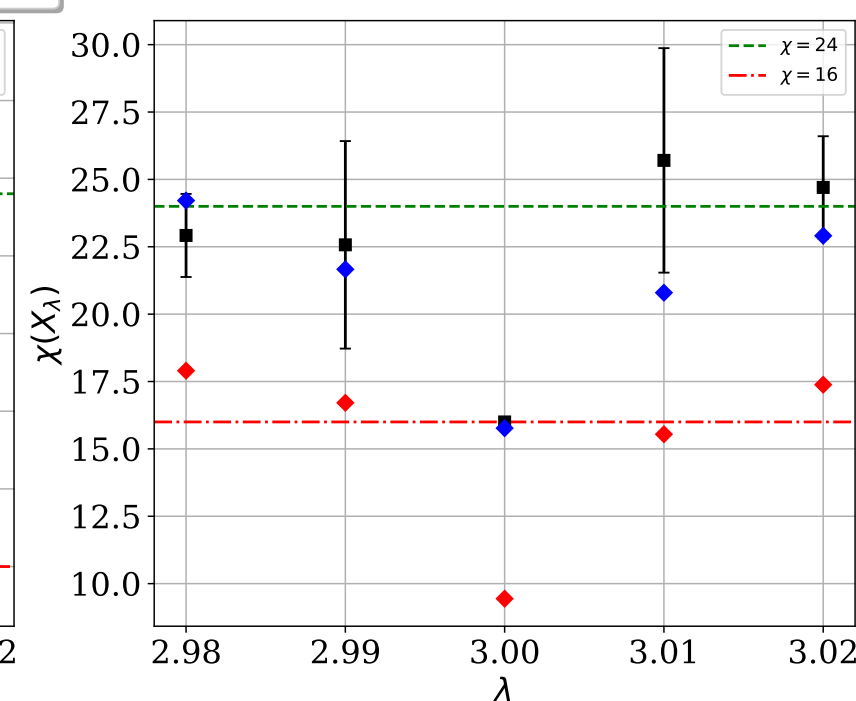
$\lambda = 0.99$



$\lambda = 0.75$



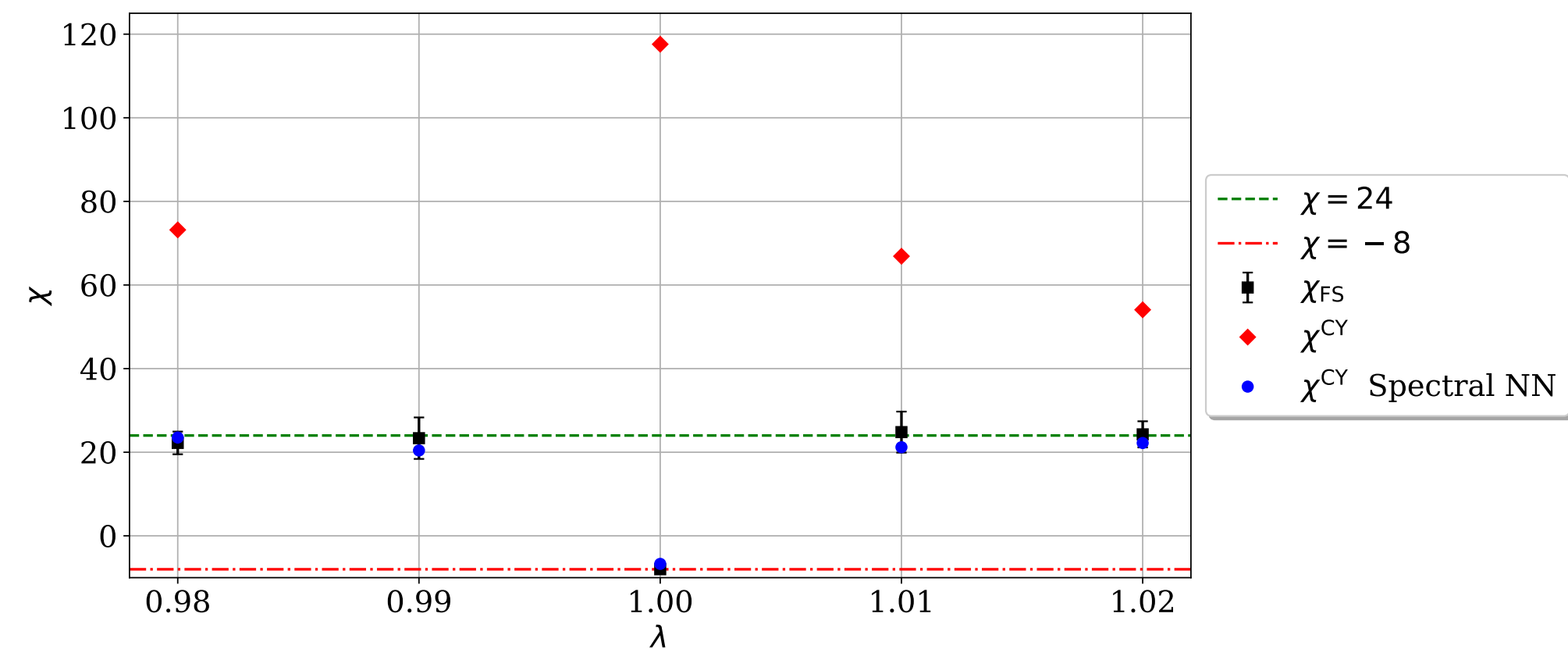
$\lambda = 1.5$



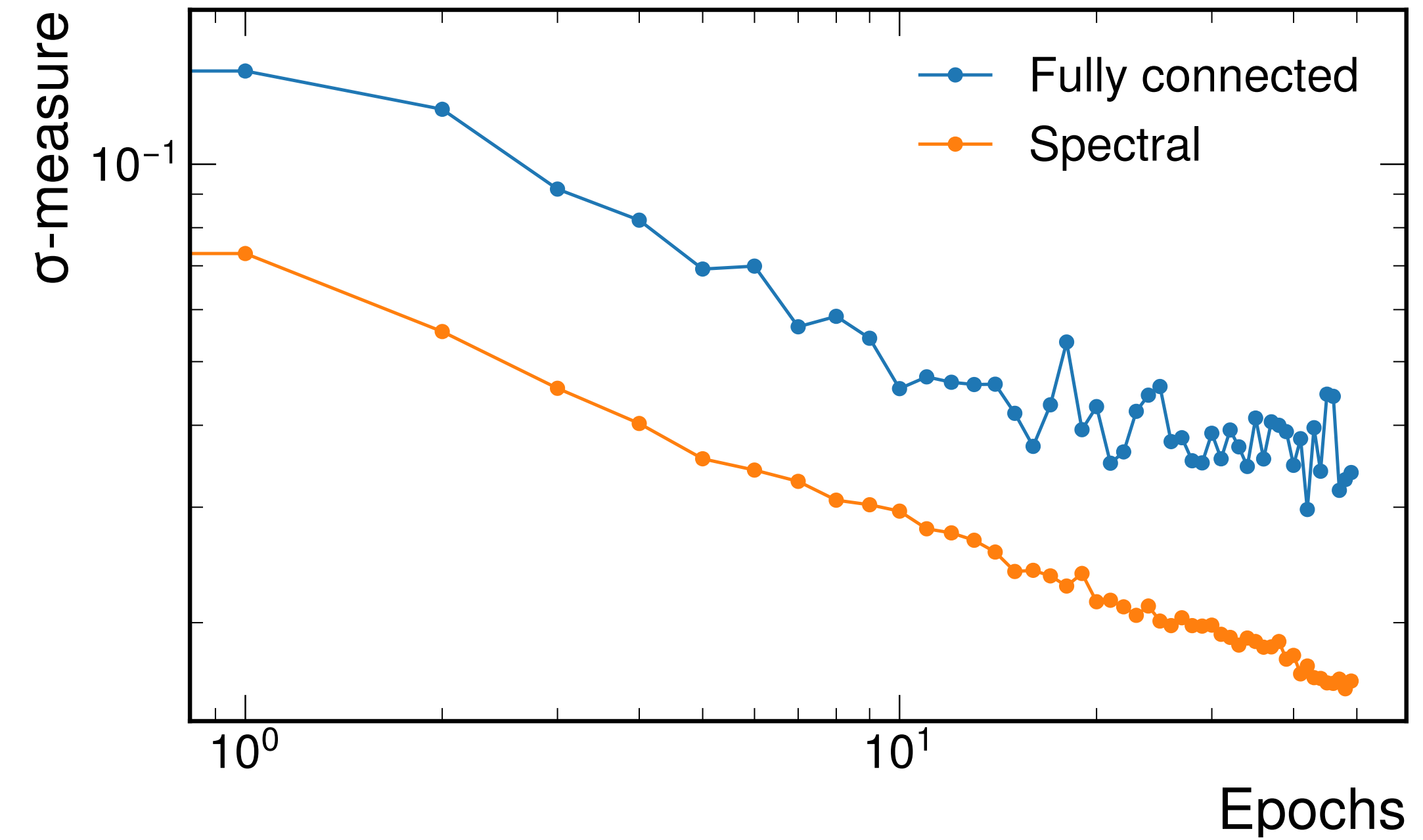
$\lambda = 3$

$$\chi = \int_{X_s} e(J) + 2 |\text{Sing}(X)| = 24 .$$

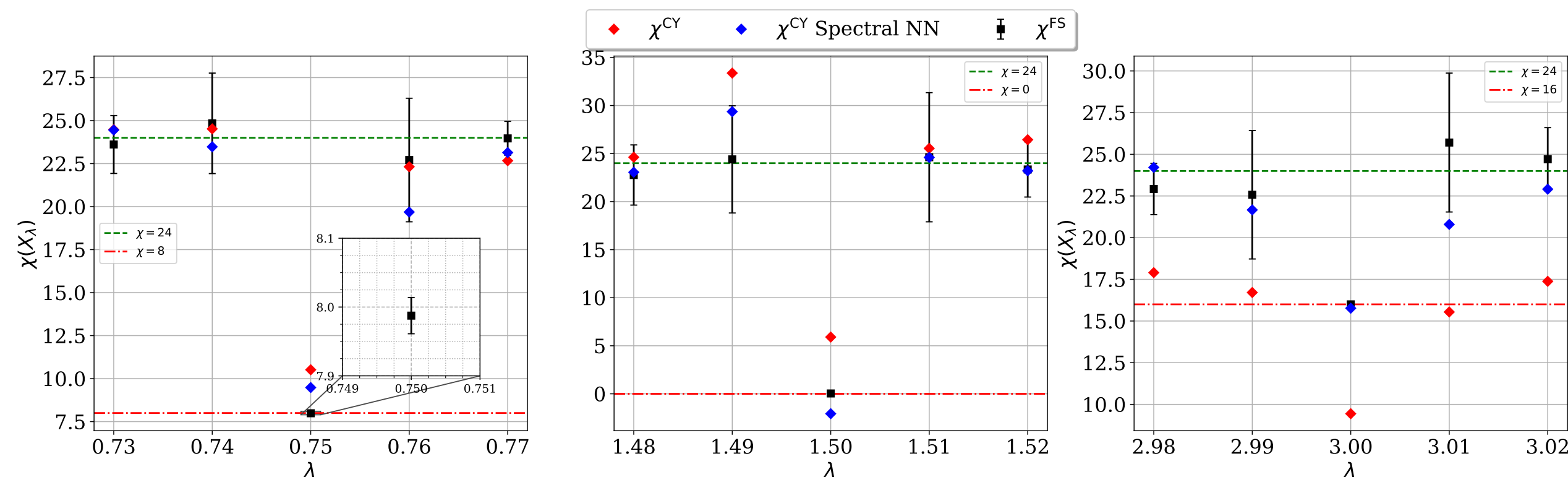
Calabi–Yau Twofold



$\lambda = 1$



$\lambda = 1.00$



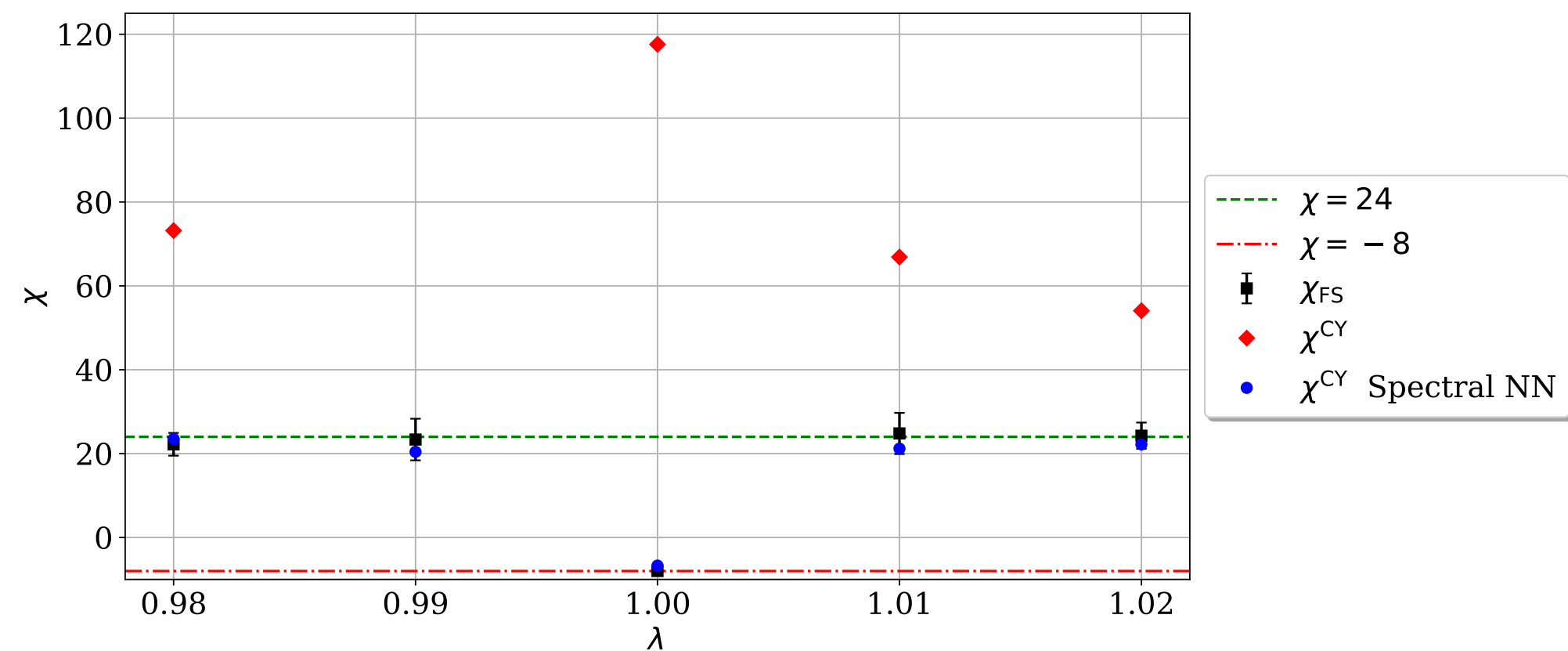
$\lambda = 0.75$

$\lambda = 1.5$

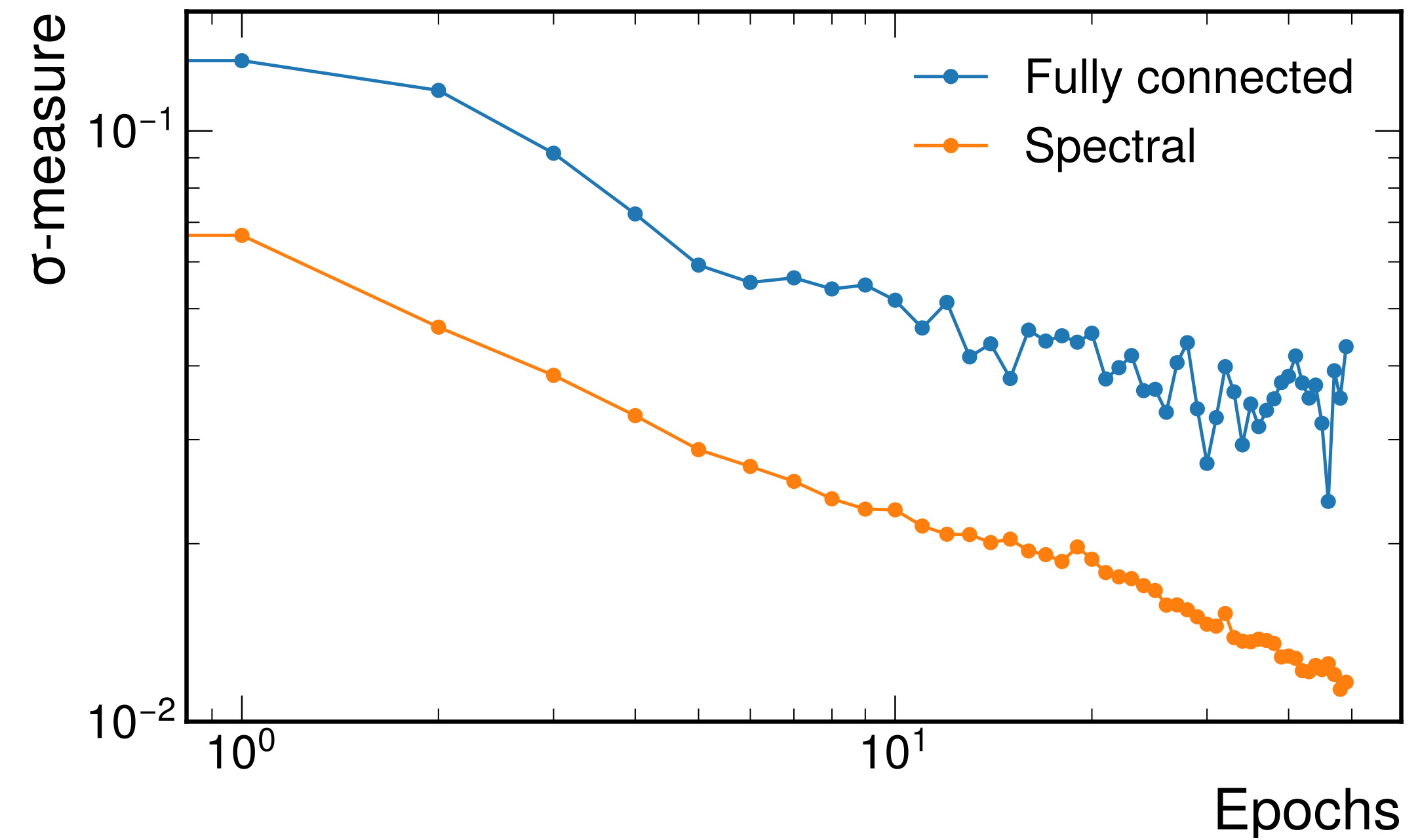
$\lambda = 3$

$$\chi = \int_{X_s} e(J) + 2 |\text{Sing}(X)| = 24 .$$

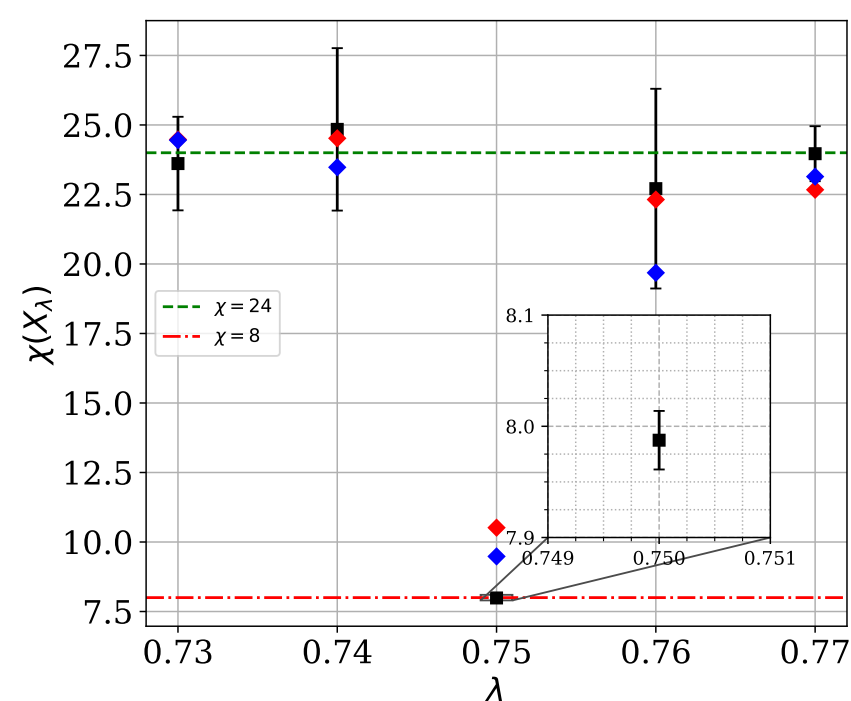
Calabi–Yau Twofold



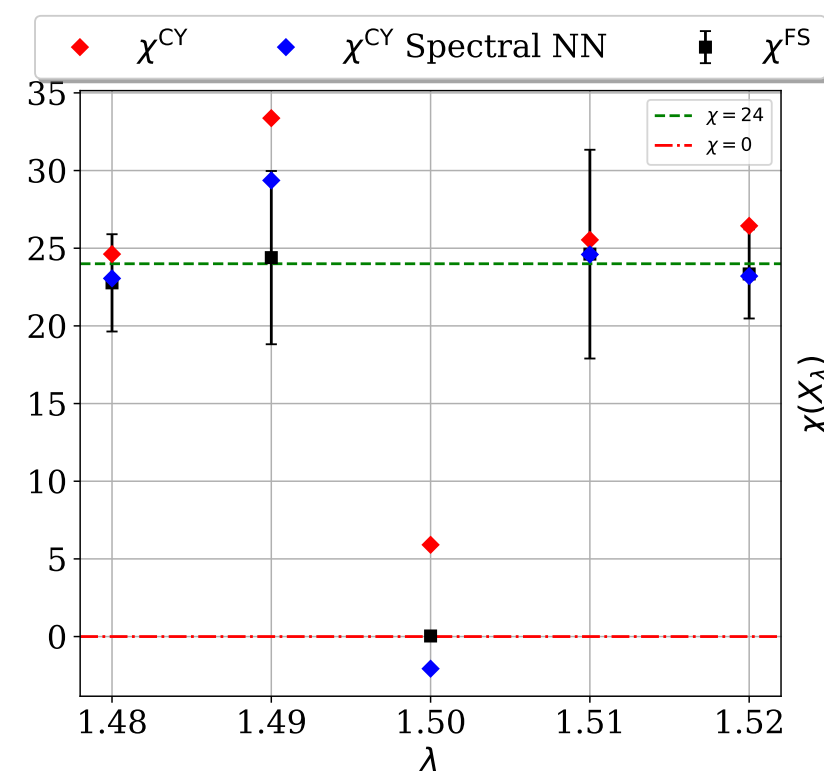
$\lambda = 1$



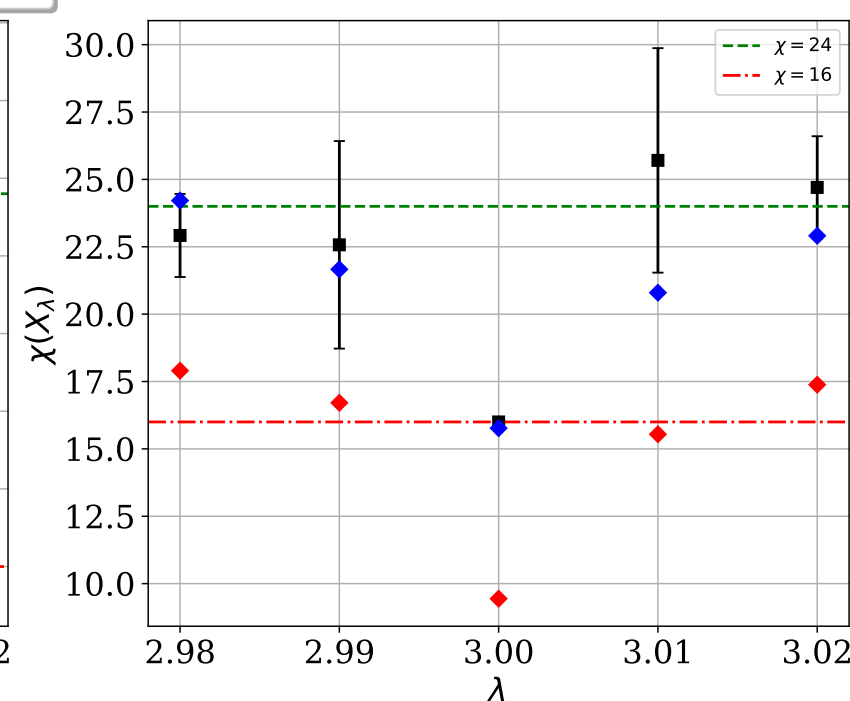
$\lambda = 1.01$



$\lambda = 0.75$



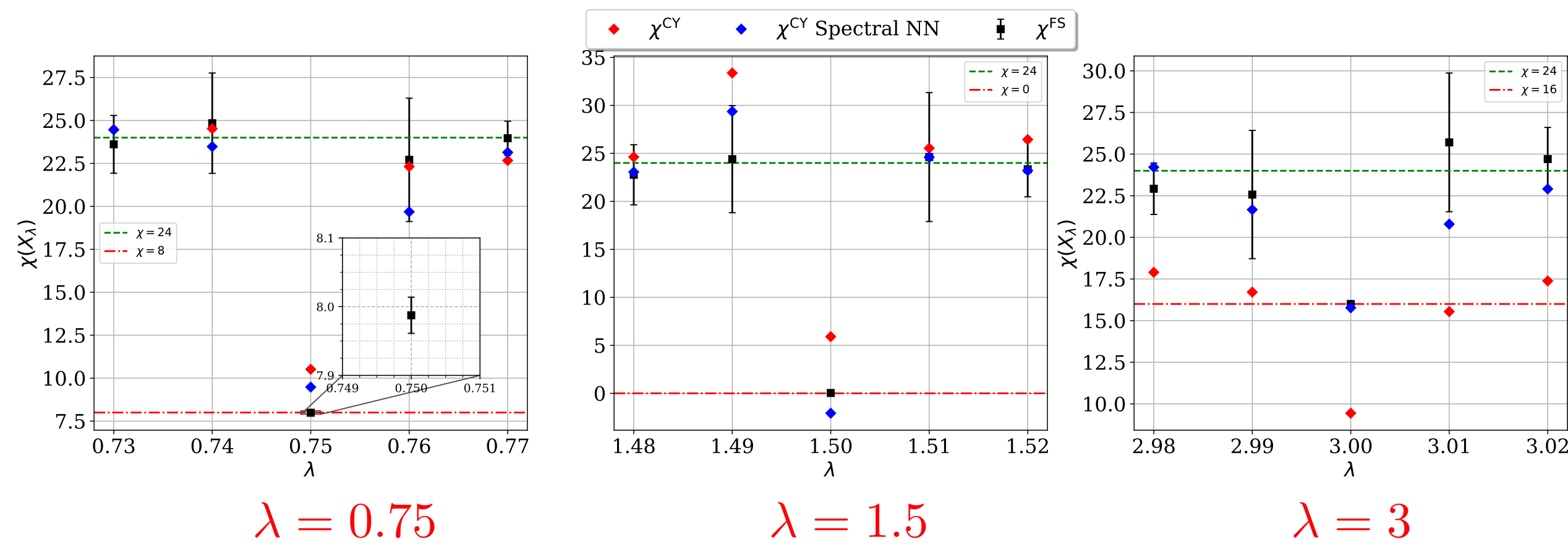
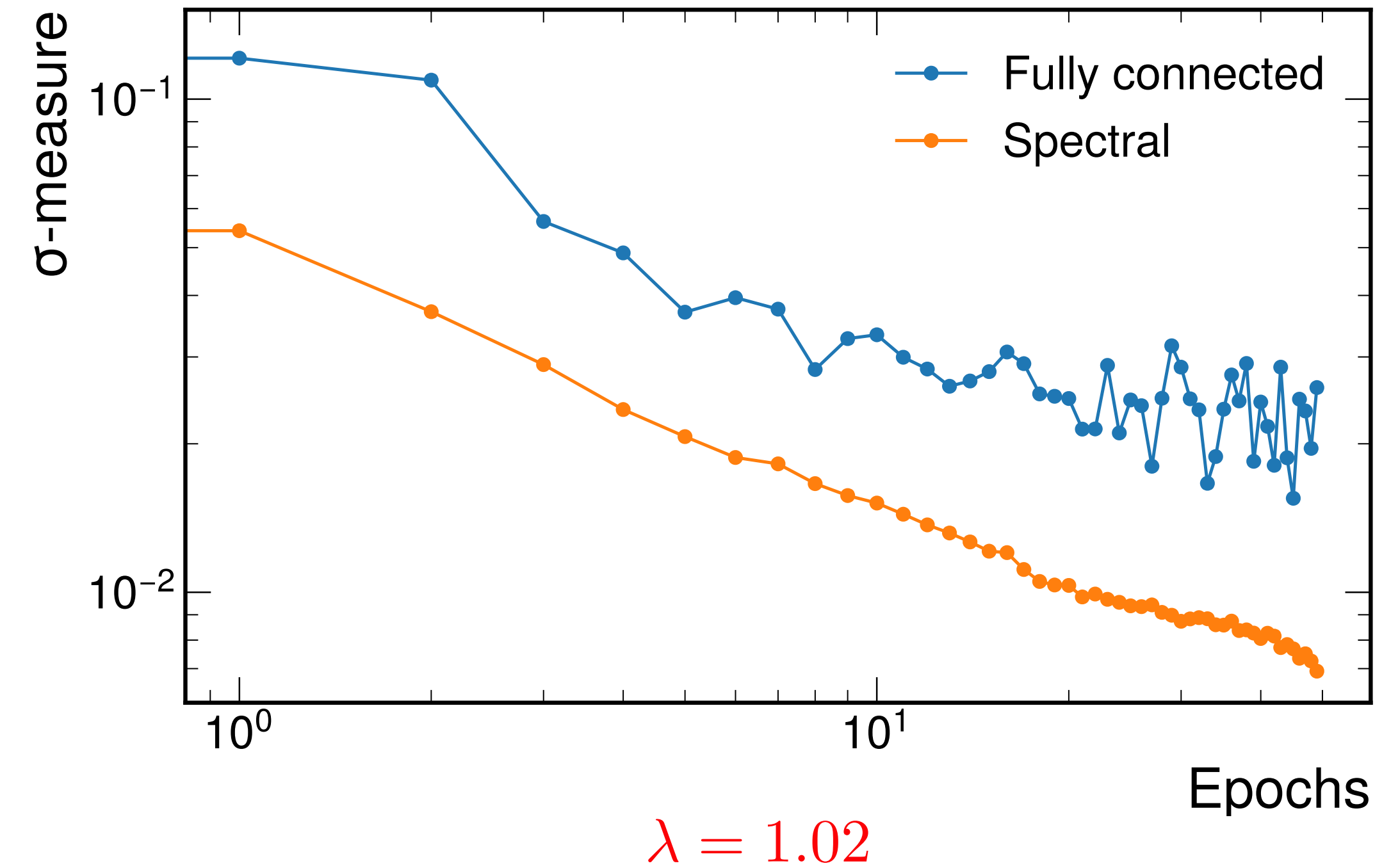
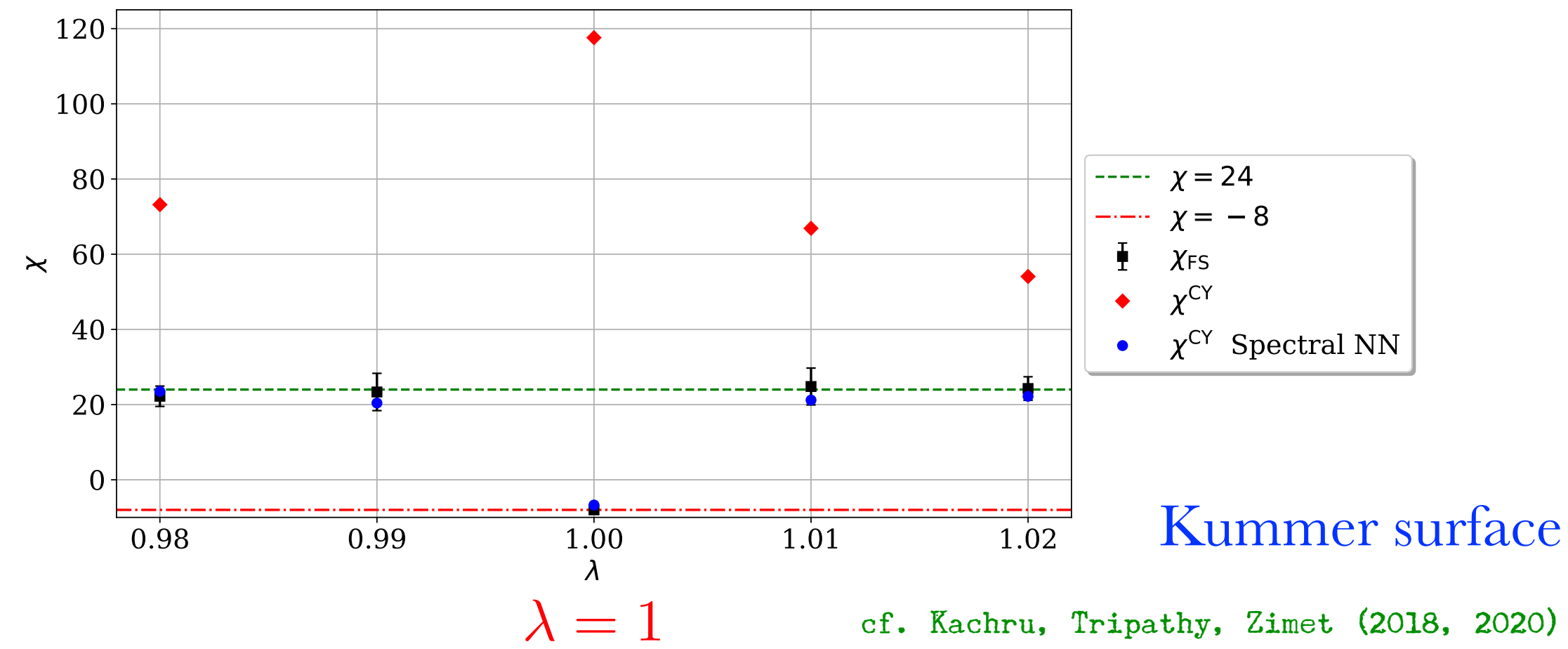
$\lambda = 1.5$



$\lambda = 3$

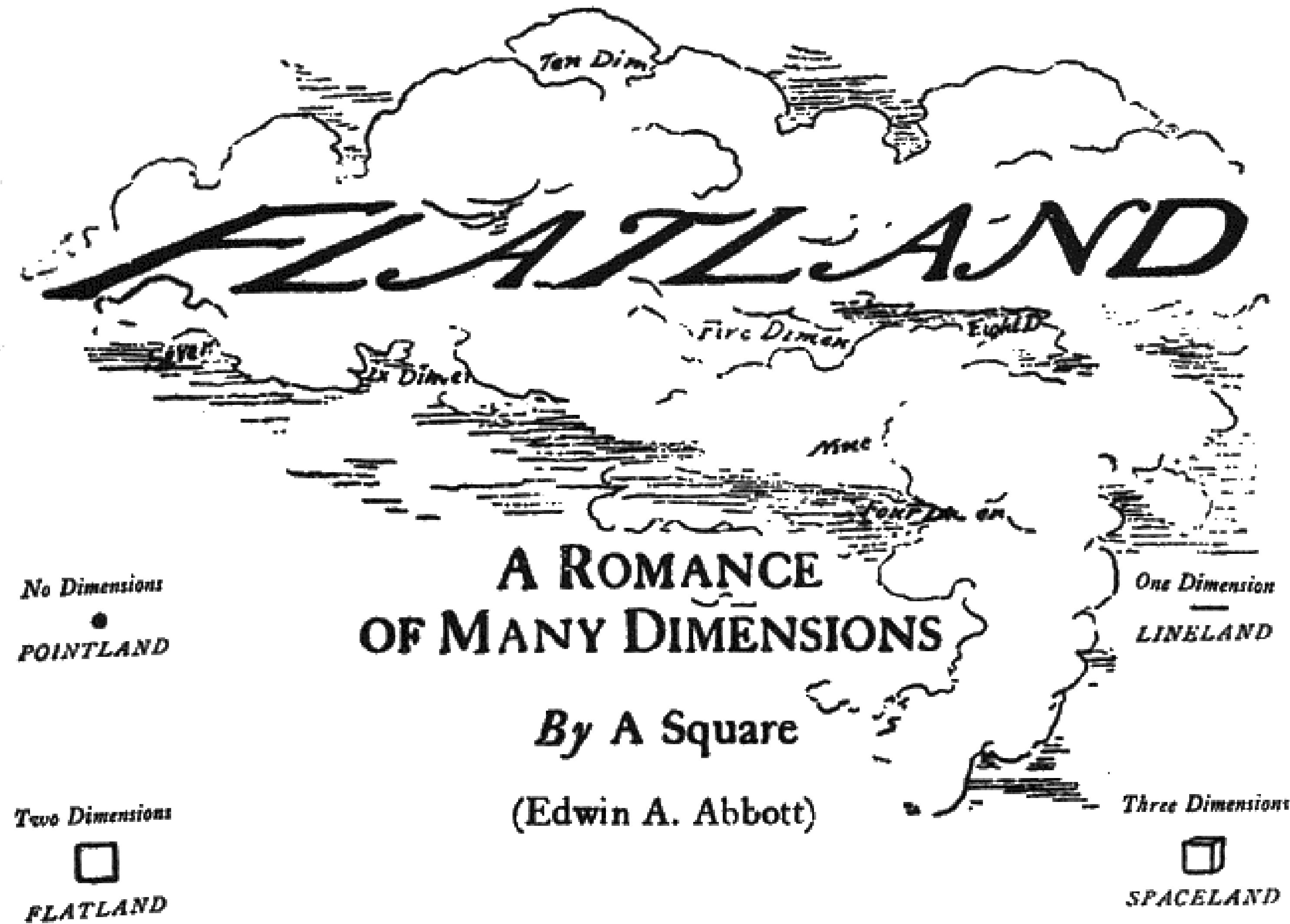
$$\chi = \int_{X_s} e(J) + 2 |\text{Sing}(X)| = 24 .$$

Calabi–Yau Twofold

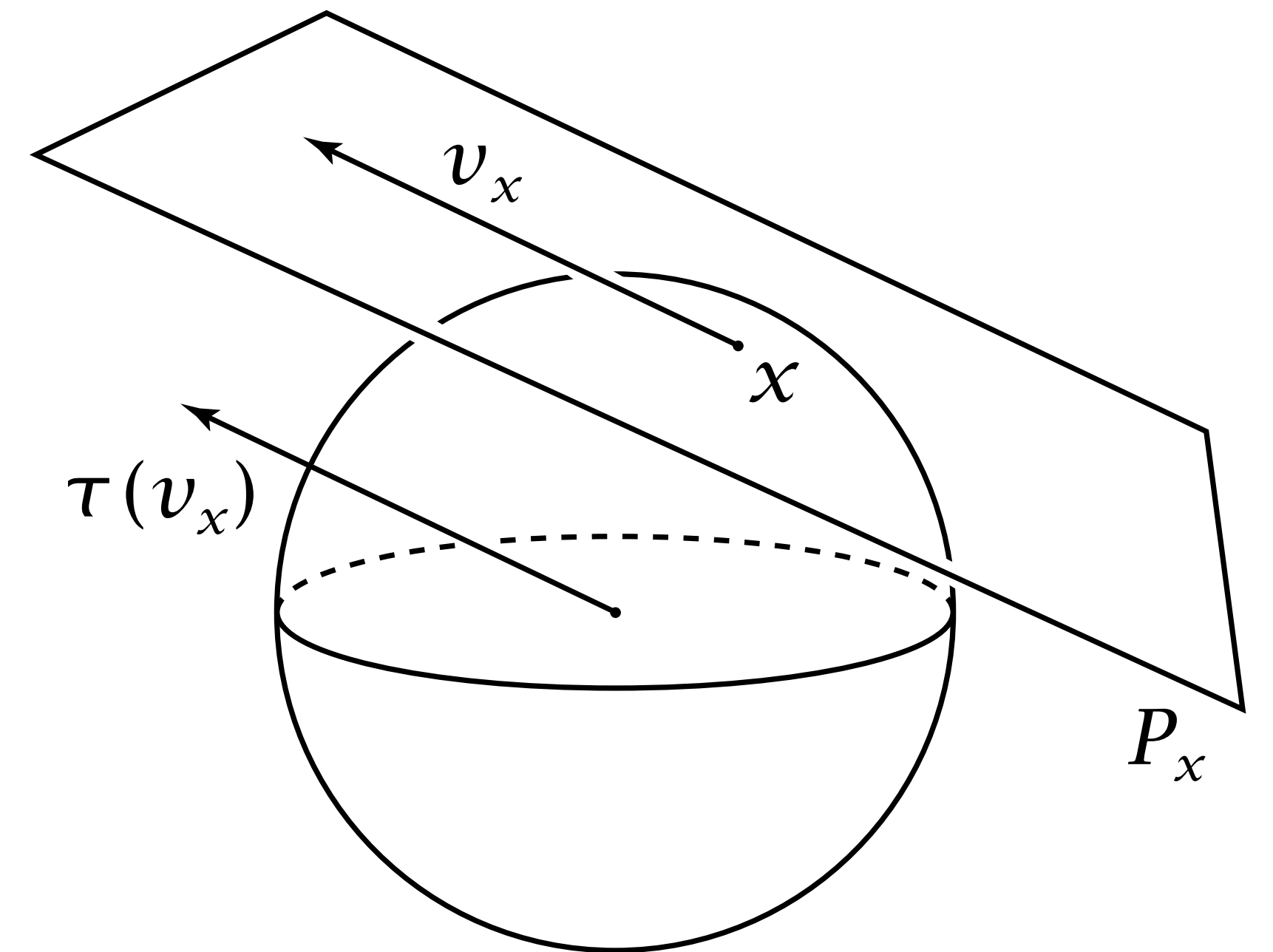
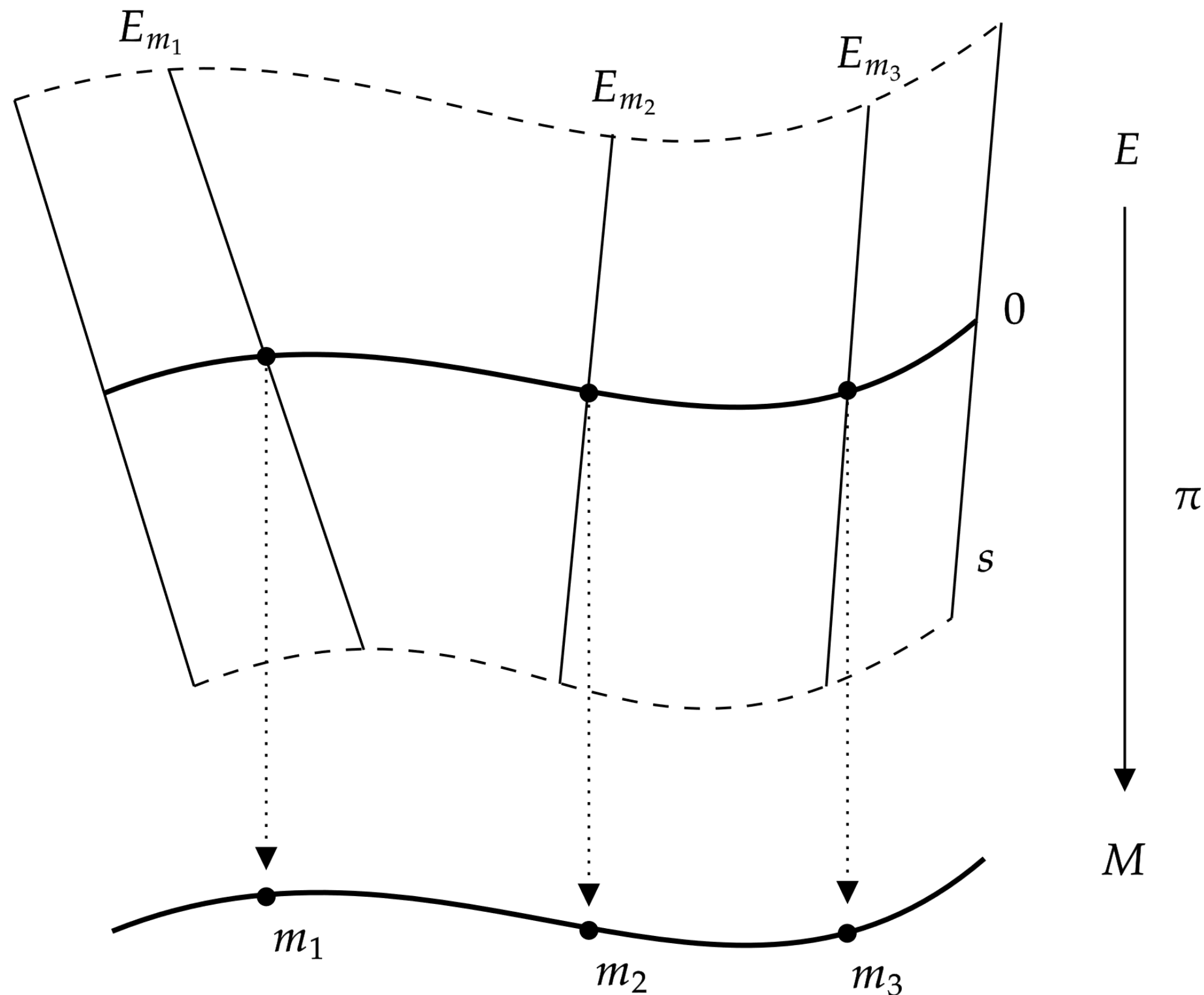


$$\chi = \int_{X_s} e(J) + 2 |\text{Sing}(X)| = 24 .$$

"O day and night, but this is wondrous strange"



Vector Bundles



holomorphic transition functions, μ -stable, connection solves Hermitian Yang–Mills equation

$$F \subset E \implies \mu(F) = \deg(F)/\text{rk}(F) < \mu(E)$$

$$\deg(F) = \int_X c_1(F) \wedge \omega^{n-1}, \quad \text{rk}(F) = \dim(\text{fiber})$$

Hermitian metric with connection
 $(0, 2)$ part of curvature vanishes
 constant mean curvature

Low-energy Matter Spectrum

- Charged matter multiplets are counted by $h^1(V)$, $h^1(V^*) = h^2(V)$
- We specialize to the case of E_6 , the commutant of $SU(3)$ holonomy in E_8
- $(1,1)$ -forms and $(2,1)$ -forms correspond to number of generations, anti-generations
- Yukawa couplings of $(1,1)$ -forms are topological (without worldsheet instantons) $(\overline{27})^3$ couplings
- There is a nowhere vanishing top holomorphic form $\Omega = \frac{1}{3!} \Omega_{ijk} dz^i \wedge dz^j \wedge dz^k$
- Given a $(2,1)$ -form, there is an equivalent $(0,1)$ -form: $\omega = \frac{1}{2} \omega_{ij\bar{k}} dz^i \wedge dz^j \wedge dz^{\bar{k}} \longleftrightarrow \omega^i = \frac{1}{2} \overline{\Omega}^{ijk} \omega_{j\bar{k}\bar{m}} dz^{\bar{m}}$
- Unnormalized Yukawa couplings computed as $\kappa = \int_X \Omega \wedge \omega^i \wedge \omega^j \wedge \omega^k \Omega_{ijk}$ $(27)^3$ couplings

quasi-topological: depends on cohomology class of ω but not actual representative

Normalized Yukawa Couplings

- We have unnormalized trilinear couplings $\kappa_{ijk} = \int_X \Omega \wedge \tilde{\Omega}(a_i, a_j, a_k)$
- Generically, normalizations are computed using inner product that depends on Ricci-flat metric

$$(a, b)_V := \int_X a \wedge \overline{\star_V} \mathcal{H}b \quad \text{with} \quad \mathcal{H} : H^1(V) \rightarrow \mathcal{H}^1(V) \quad \text{denoting harmonic projection}$$

- The physical Yukawa couplings are $\lambda_{ijk} = \frac{\kappa_{ijk}}{\|a_i\|_V \|a_j\|_V \|a_k\|_V} \left(\int_X \Omega \wedge \overline{\Omega} \right)^{-1}$
- Here, $\{a_i\}_{i=1}^{h^{2,1}}$ is orthogonal basis for $H^1(V)$: $K_{i\bar{j}} = \int_X J \wedge J \wedge a_i \wedge \bar{a}_{\bar{j}} = \delta_{i\bar{j}}$
- In standard embedding, because normalization can be computed independent of Ricci-flat metric and only requires knowledge of metric on complex structure moduli space, the Yukawa couplings are same in every Kähler class

- Non-standard embeddings

Standard Embedding

- Let holomorphic vector bundle that solves Hermitian Yang–Mills equations be tangent bundle so that metric on X does double duty as the fiber metric
- When $V \cong T_X$, the number of generations $N_{\text{gen}} = \text{ind } \mathcal{D} = \frac{1}{2}|\chi| = |h^{1,1} - h^{2,1}|$, where $\mathcal{D}^2 = \square$
- Deformations of V correspond to complex structure moduli of X
- Metric on complex structure moduli space is Weil–Petersson metric
- Using Weil–Petersson geometry, can compute Yukawa couplings without knowing Ricci-flat metric
- In standard embedding, will compute Yukawa couplings in three different ways
 - period integrals
 - Kodaira–Spencer map
 - explicit harmonic forms using Ricci-flat metric

} Weil–Petersson geometry

Machine Learning Sections

- Start from harmonic projection $\mathcal{H} : H^1(T_X) \rightarrow \mathcal{H}^1(T_X)$
 $\psi \in \rho(\partial_t) \mapsto \mathcal{H}\psi$
- Find smooth section s of T_X such that $\Delta(\psi + \bar{\partial}s) = 0$ — *i.e.*, $\mathcal{H}\psi - \psi = \bar{\partial}s$
- Strategy is to model s with a neural network using the machine learned Ricci-flat metric
 - start with an over-complete basis of sections
 - parametrize coefficients to get globally defined object
- In principle, can do this for any holomorphic vector bundle on a Kähler manifold
- We can adapt the idea to the general non-Abelian vector bundle case
 - this is the setup for “realistic” string constructions of Standard Model

Douglas (2023)
cf. Constantin, Fraser-Taliente, Harvey, Lukas, Ovrut (2024)

Butbaia, Mayorga Peña, Tan, Berglund, Hübsch, VJ, Mishra (2024)
Berglund, Butbaia, Hübsch, VJ, Mayorga Peña, Mishra, Tan (2024)
Mishra, Tan (2025)

Machine Learning Sections

- Given a reference form $\psi \in H^1(X; V)$, approximate cohomologous harmonic projection

$\mathcal{H}\psi \approx \eta := \psi + \bar{\partial}_V s_{\text{NN}}$, where s_{NN} is given by a spectral network

- In particular, $s_{\text{NN}} = \sum_{ijkl} \psi_{ijkl}^{(\text{NN})} \frac{\overline{\alpha_\nu^{ij}} z^k z^l}{|z|^4} g^{\mu\bar{\nu}} \cdot \frac{\partial}{\partial z^\mu} \in \Gamma(T_X)$

output of spectral network

$$\alpha_\nu^{ij} dz^\nu = \iota^*(z^i dz^j - z^j dz^i), \text{ where } \iota : X \hookrightarrow A$$

Greek Latin

Ashmore (2023)

- Harmonic forms minimize the inner product $(\eta, \Delta_V \eta)$

- $\bar{\partial}_V$ closed by construction

- Minimize loss function $\ell_H = (\bar{\partial}_V^\dagger \eta, \bar{\partial}_V^\dagger \eta)$

- Methods apply to any $h^{2,1}$

Quintic

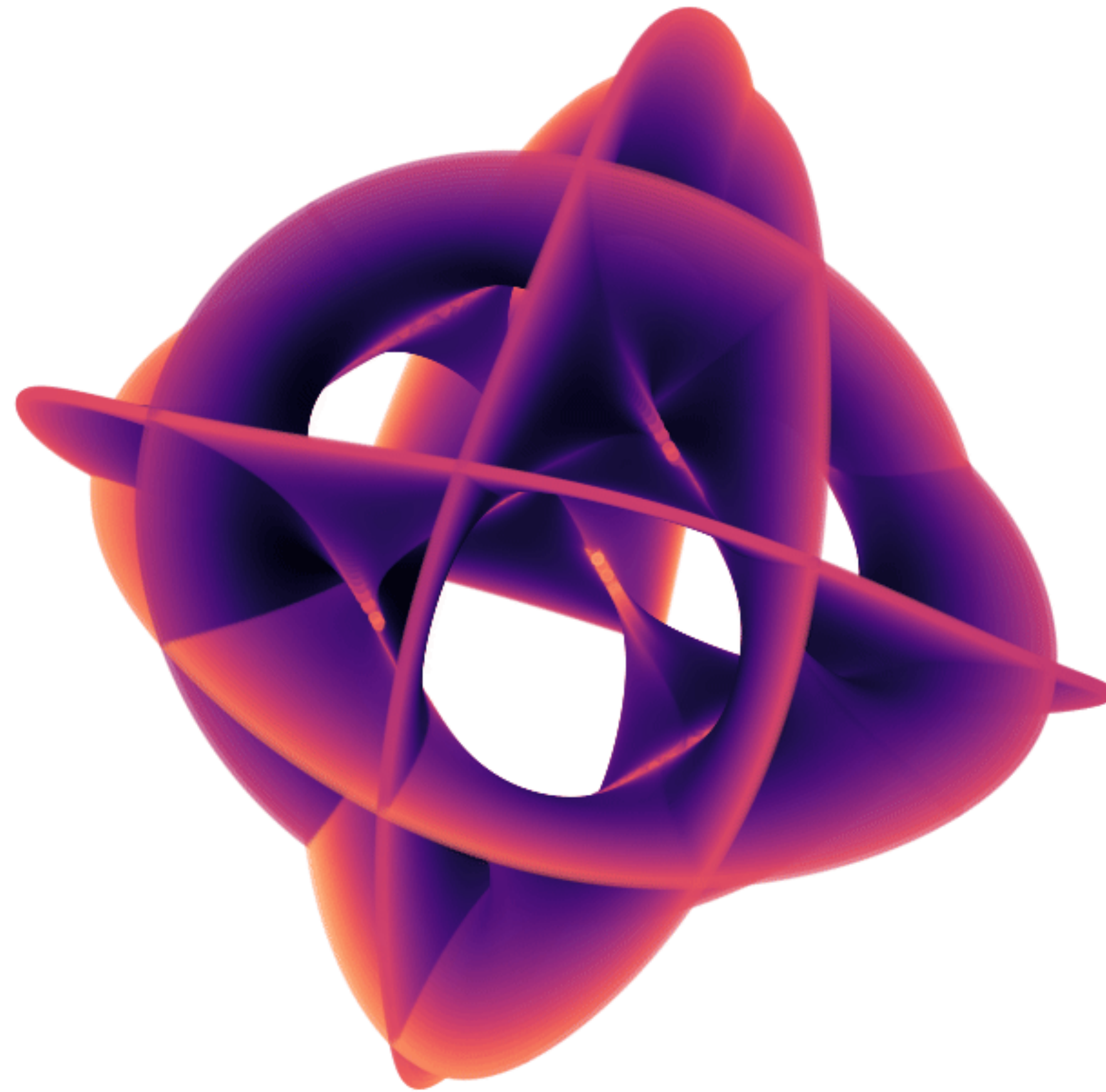
- Fermat quintic, $\psi = 0$

$$\sum_{i=1}^5 z_i^5 - 5\psi z_1 z_2 z_3 z_4 z_5 = 0 \subset \mathbb{P}^4$$

singular when $\psi^5 = 1$

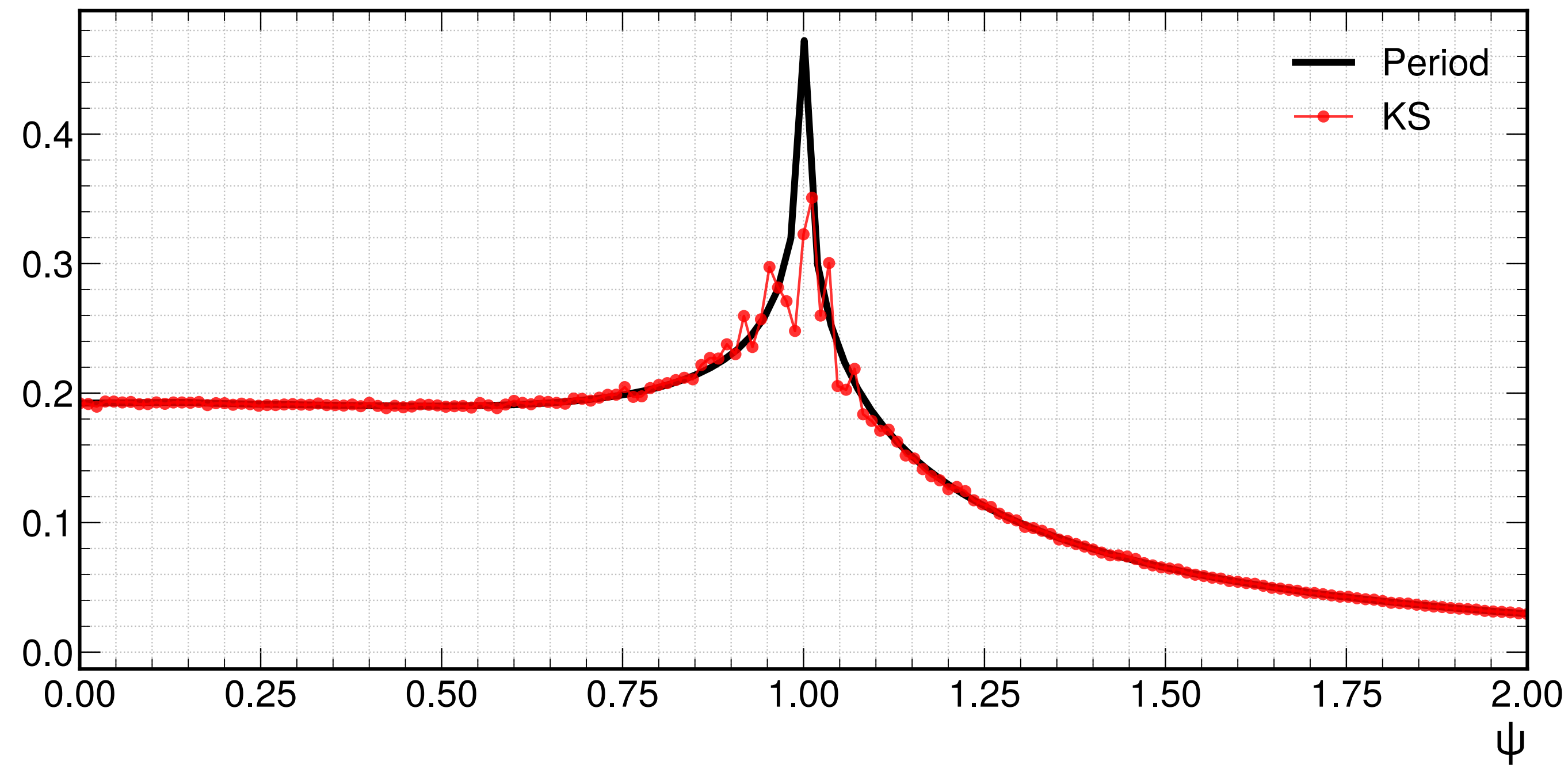
$$h^{1,1} = 1, \quad h^{2,1} = 101$$

ϕ



Quintic

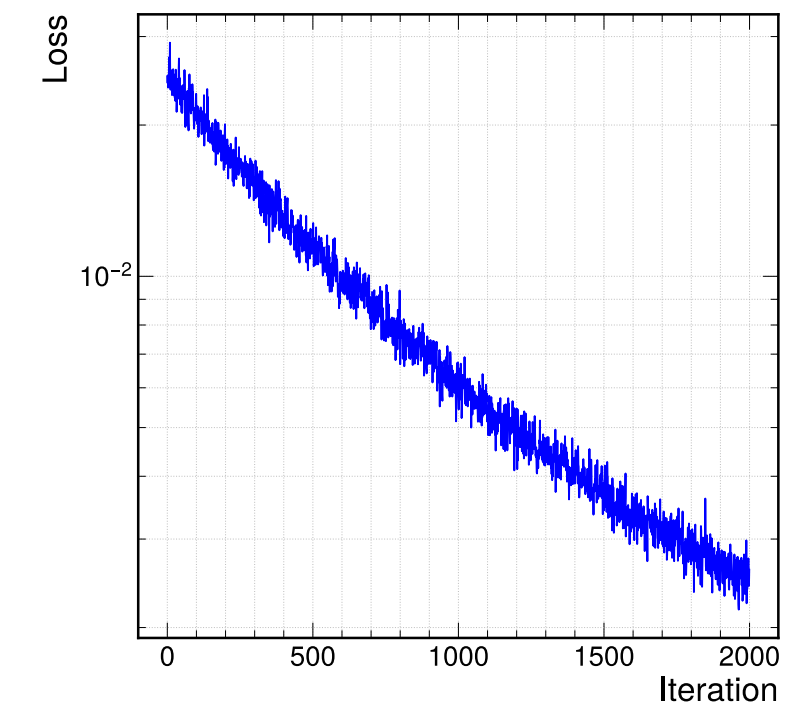
- Mirror of Dwork quintic



$$\sum_{i=1}^5 z_i^5 - 5\psi z_1 z_2 z_3 z_4 z_5 = 0 \subset \mathbb{P}^4$$

singular when $\psi^5 = 1$

$$h^{1,1} = 1, \quad h^{2,1} = 101$$



loss for finding harmonic representative

$$g_{\psi\bar{\psi}}(\psi=0) : 0.1914 \quad \text{vs.} \quad 25 \left(\frac{\Gamma(\frac{4}{5})\Gamma(\frac{2}{5})}{\Gamma(\frac{1}{5})\Gamma(\frac{3}{5})} \right)^5 \approx 0.1923$$

ML

periods

Quintic

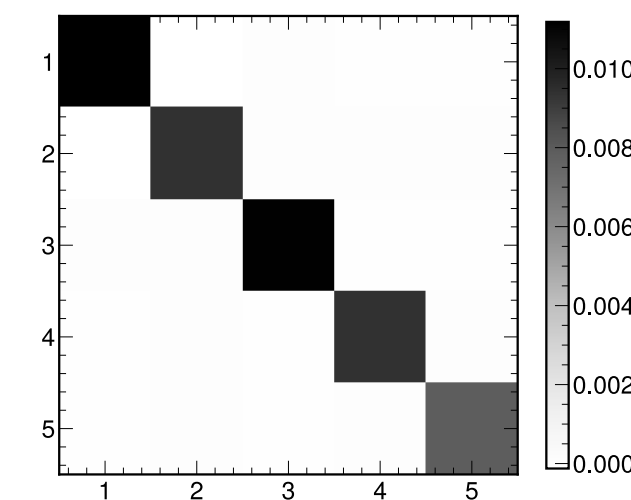
$$\sum_{i=1}^5 z_i^5 - 5\psi z_1 z_2 z_3 z_4 z_5 = 0 \subset \mathbb{P}^4$$

- Gepner model $Y_{4;5}$ is specific point in moduli space
mod Fermat quintic by $\mathbb{Z}_5 \times \mathbb{Z}_5$ to get manifold with $h^{1,1} = 1$, $h^{2,1} = 5$

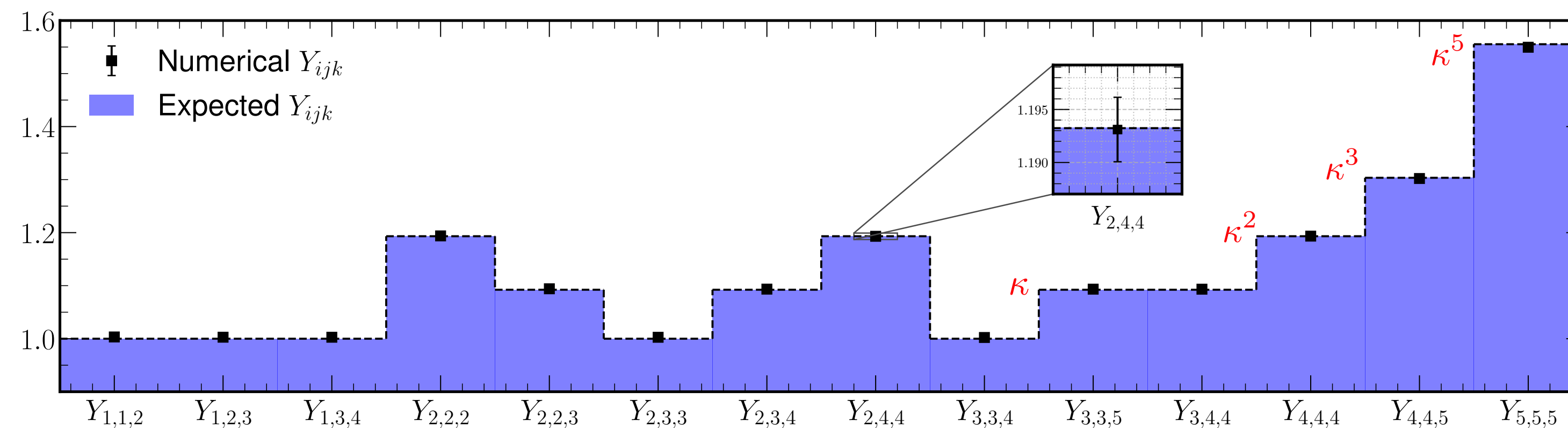
- Yukawa couplings are powers of $\kappa = \sqrt{\frac{\Gamma(3/5)^3 \Gamma(1/5)}{\Gamma(2/5)^3 \Gamma(4/5)}} \approx 1.092$

Distler, Greene (1988)

- Normalization with respect to Weil–Petersson metric



- Normalized Yukawa couplings



Butbaia, Mayorga Peña, Tan, Berglund, Hübsch, VJ, Mishra (2024)

Quarti-Quadric

- Two Kähler classes descending from projective factors

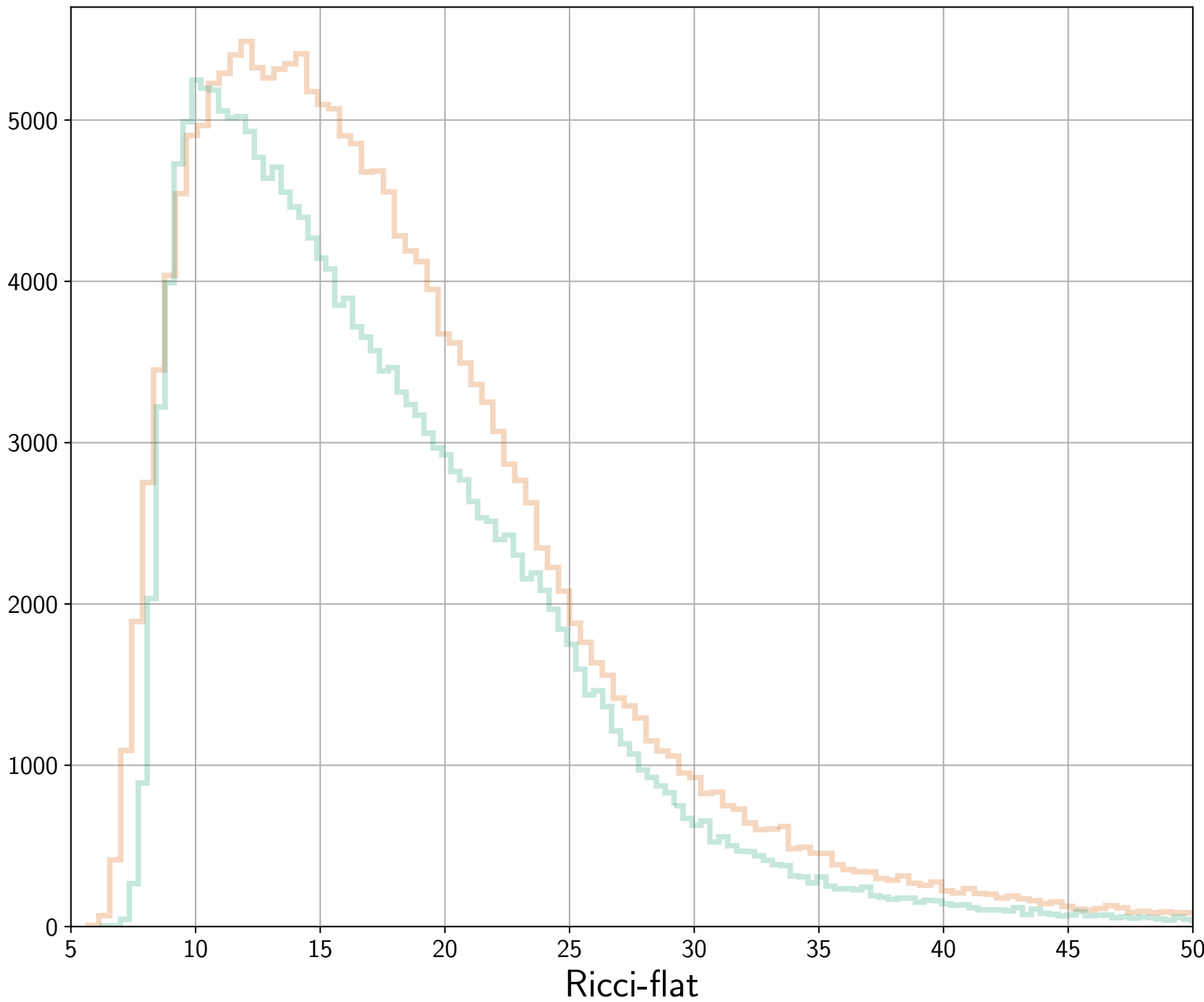
$$J = t_1 J_1 + t_2 J_2$$

- Kretschmann scalar $(R_{a\bar{b}c\bar{d}} R^{a\bar{b}c\bar{d}})^{\frac{1}{2}}$

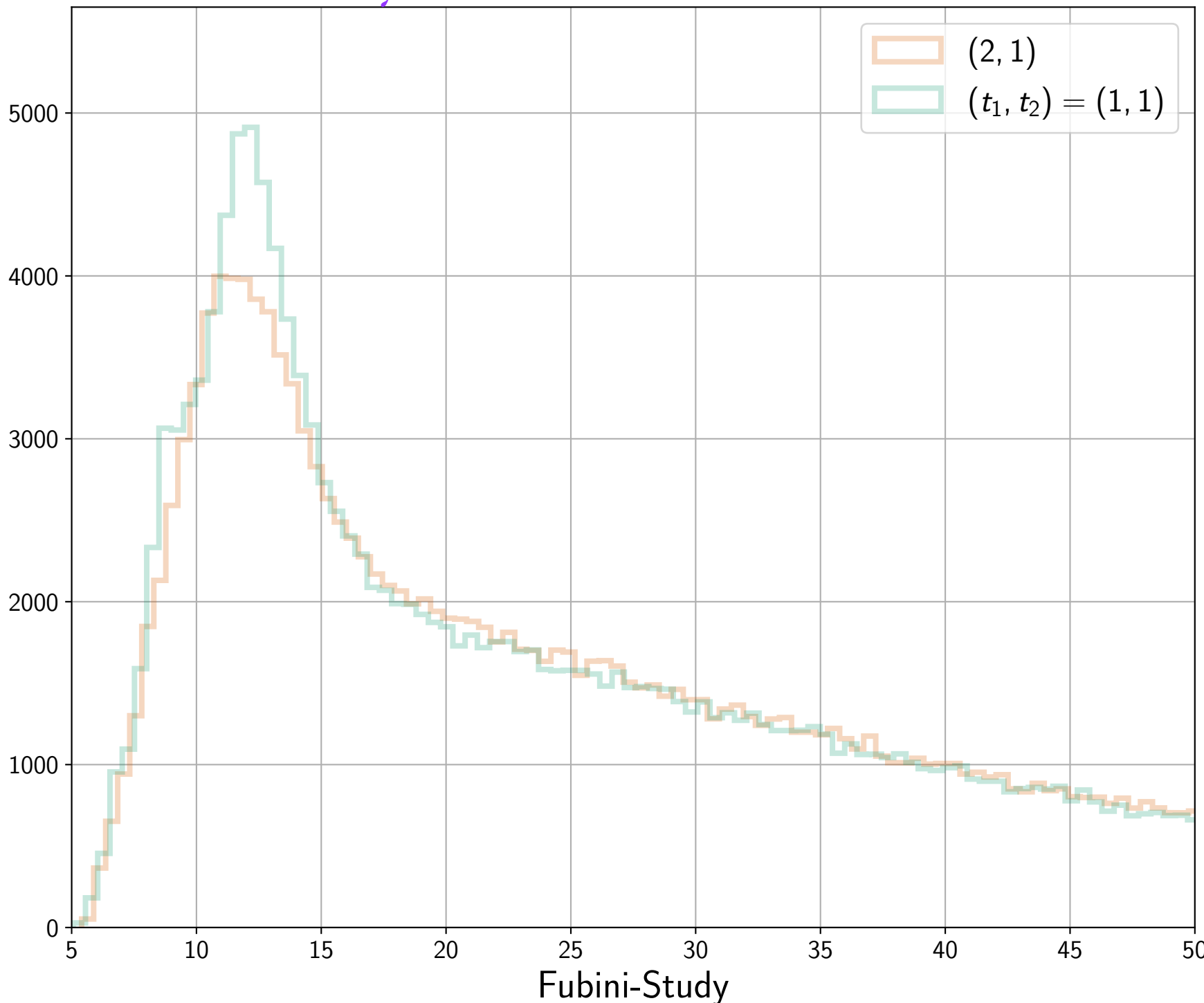
$$\left[\begin{array}{c|c} \mathbb{P}^3 & 4 \\ \mathbb{P}^1 & 2 \end{array}\right]_{\chi=-168}^{2,86}$$

$$x_1^4(y_1^2+y_2^2) + x_2^4(y_1^2+2y_2^2) + x_3^4(y_1^2-y_2^2) + x_4^4(y_1^2-2y_2^2) - 4\psi x_1x_2x_3x_4y_1y_2 = 0$$

Ricci-flat



Fubini-Study



singular when $\psi^4 = -1, -9$

$\psi = 0.2$

Quarti-Quadric

- Two Kähler classes descending from projective factors

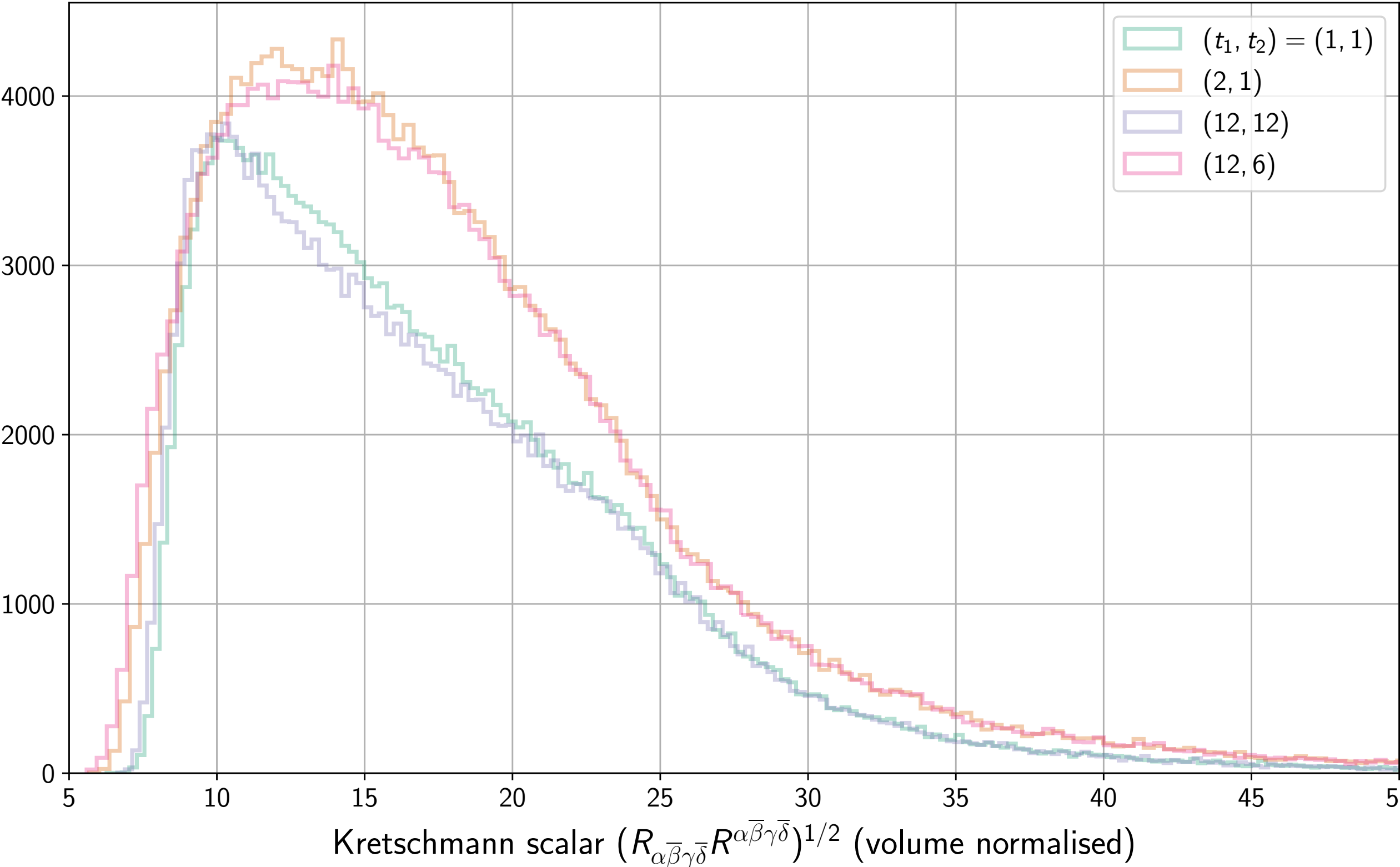
$$J = t_1 J_1 + t_2 J_2$$

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$$\text{singular when } \psi^4 = -1, -9$$



$$\psi = 0.2$$

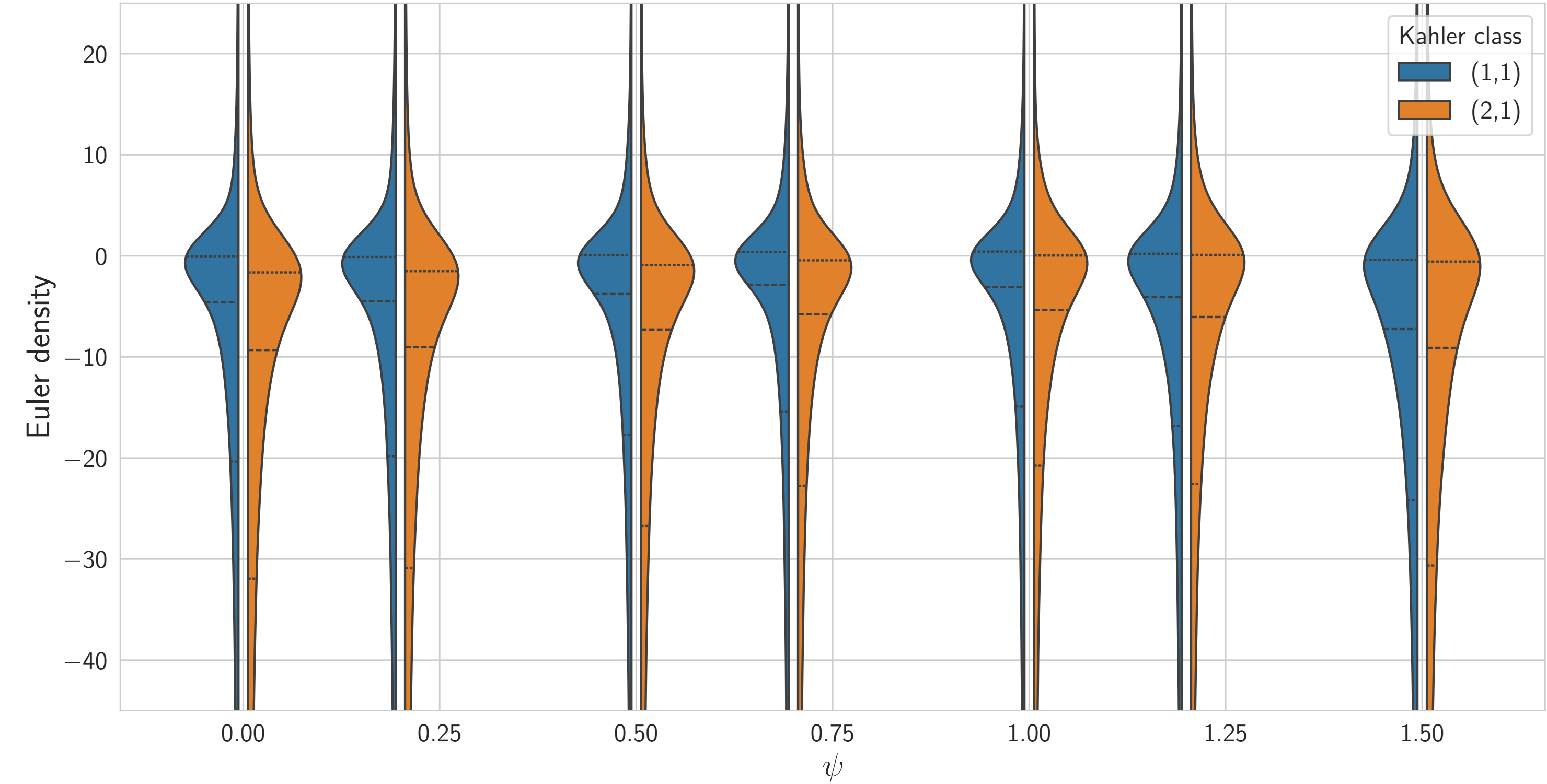
different distributions in two Kähler classes
 same in different descriptions of same Kähler class

Quarti-Quadric

- Two Kähler classes descending from projective factors

$$J = t_1 J_1 + t_2 J_2$$

- Euler density



$$\left[\begin{array}{c|c} \mathbb{P}^3 & 4 \\ \mathbb{P}^1 & 2 \end{array} \right]_{\chi=-168}^{2,86}$$

$$x_1^4(y_1^2+y_2^2) + x_2^4(y_1^2+2y_2^2) + x_3^4(y_1^2-y_2^2) + x_4^4(y_1^2-2y_2^2) - 4\psi x_1 x_2 x_3 x_4 y_1 y_2 = 0$$

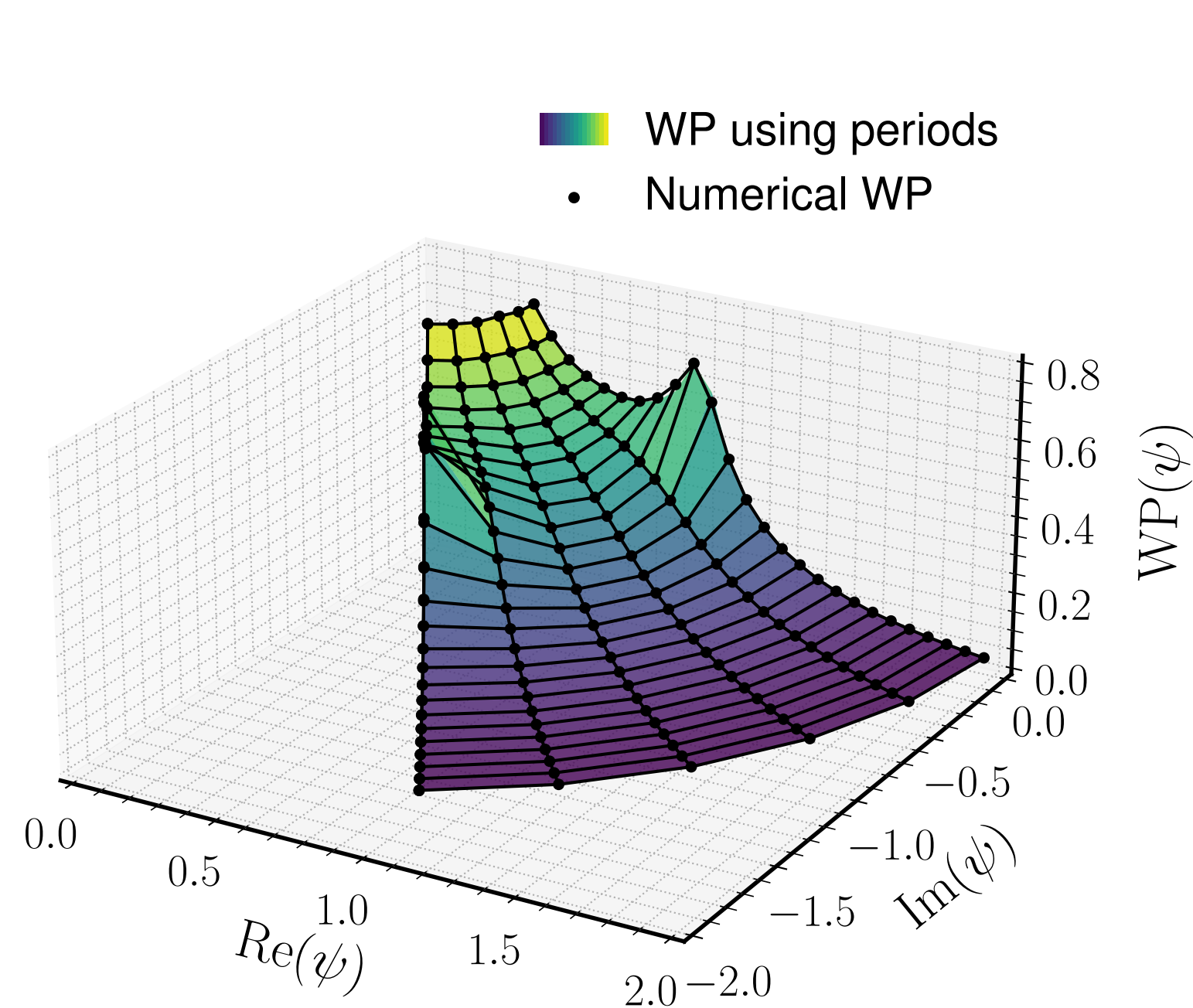
singular when $\psi^4 = -1, -9$

Intersection of Two Cubics

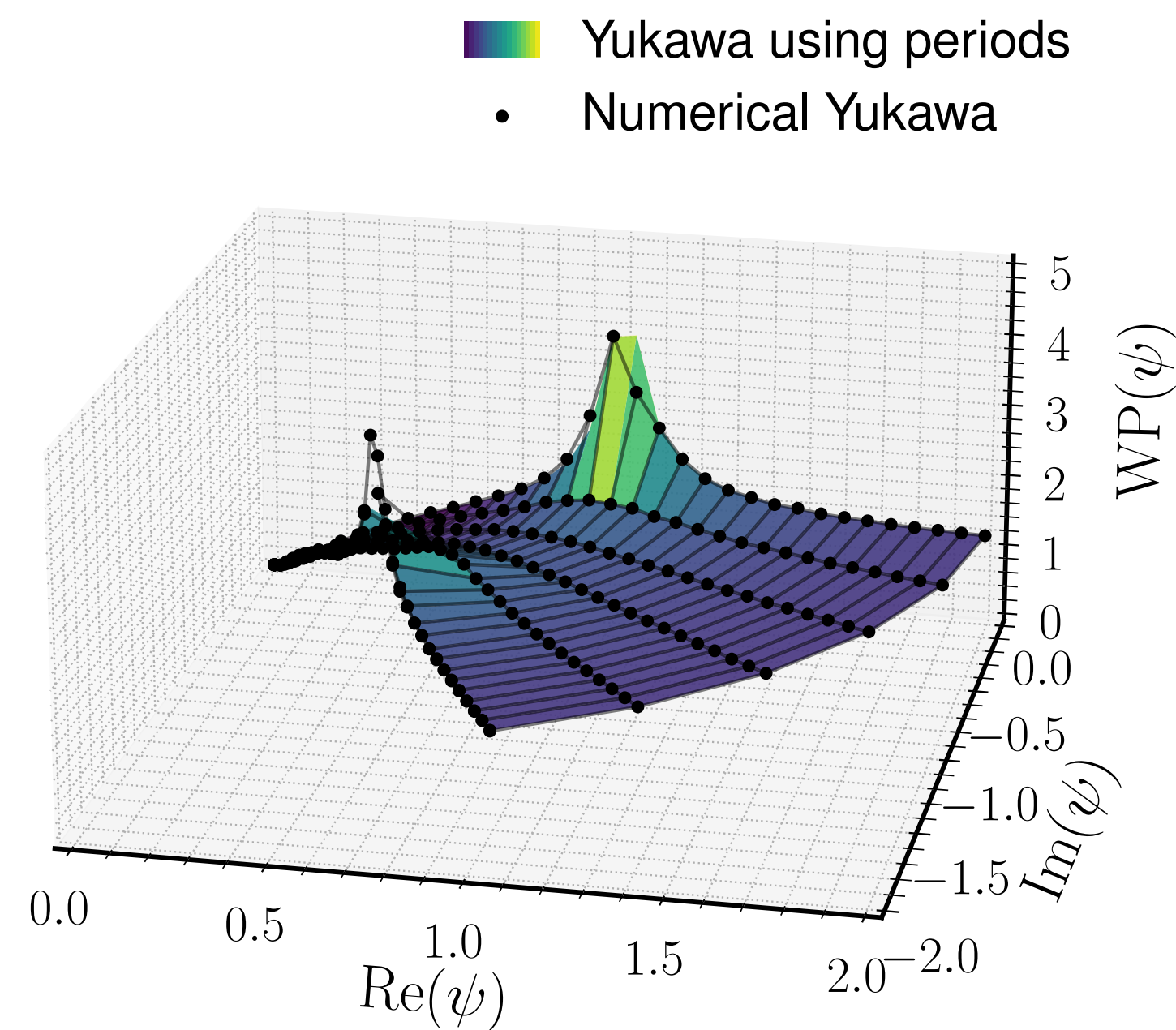
- Mirror of $\mathbb{P}^5[3, 3]$

$$\sum_{i=1}^3 x_i^3 - 3\psi x_4 x_5 x_6 = \sum_{i=4}^6 x_i^3 - 3\psi x_1 x_2 x_3 = 0$$

$$h^{1,1} = 1, \quad h^{2,1} = 73$$



(a) Weil–Petersson metric



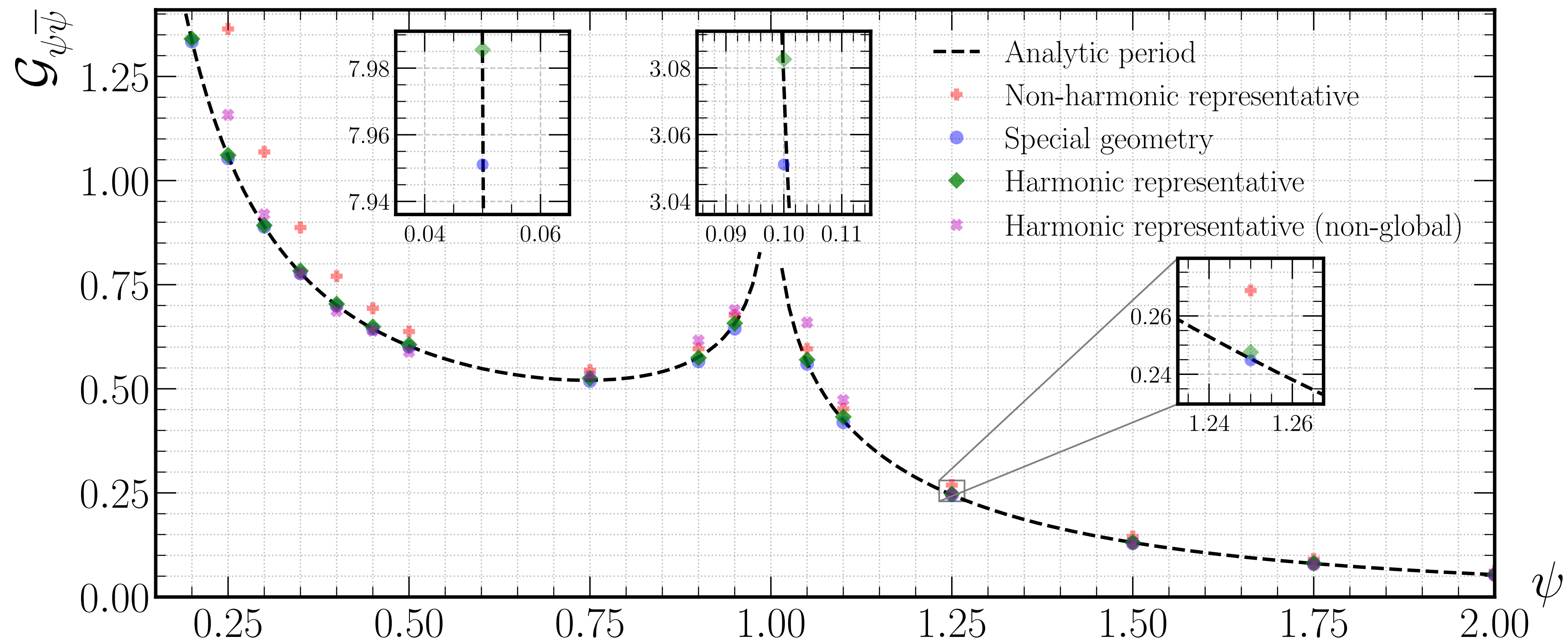
(b) Normalized Yukawa coupling

Intersection of Two Cubics

- Mirror of $\mathbb{P}^5[3, 3]$

$$\sum_{i=1}^3 x_i^3 - 3\psi x_4 x_5 x_6 = \sum_{i=4}^6 x_i^3 - 3\psi x_1 x_2 x_3 = 0$$

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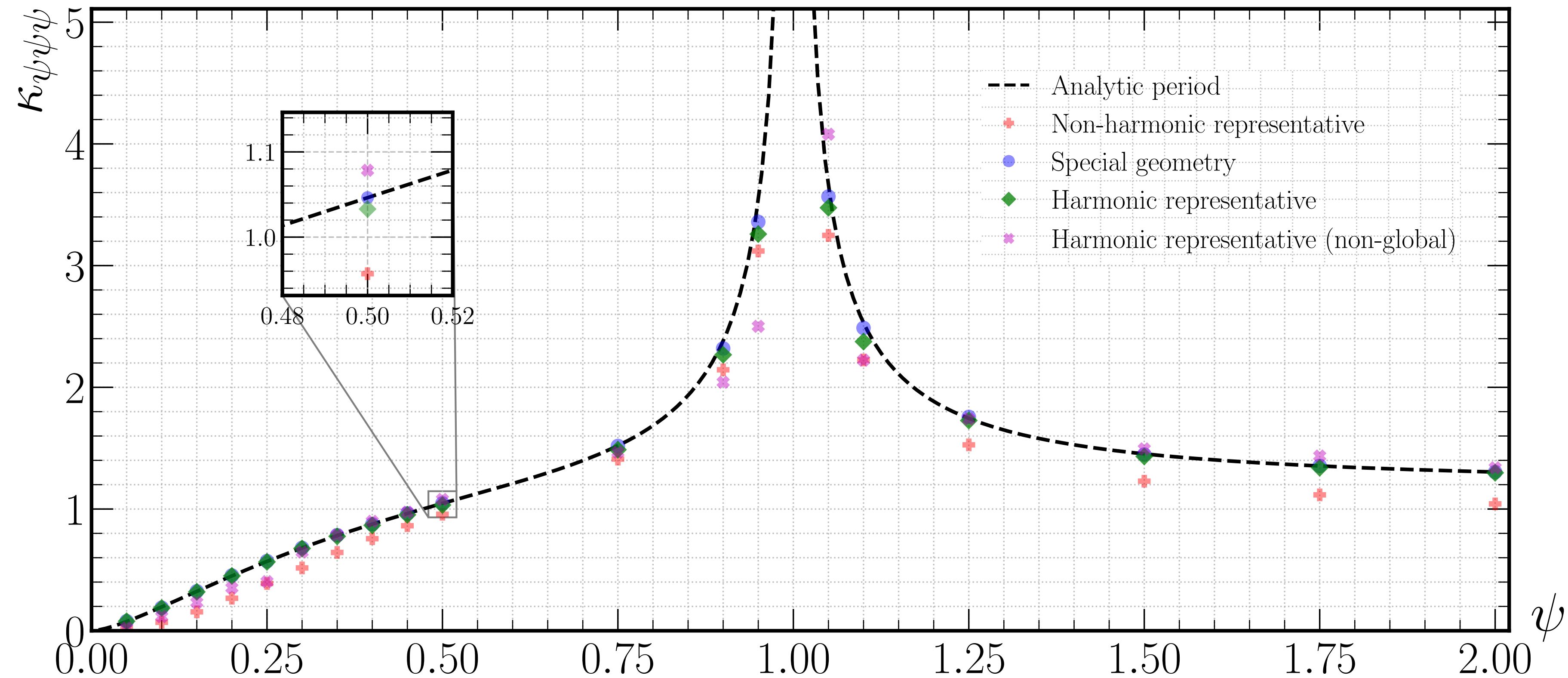


Intersection of Two Cubics

- Mirror of $\mathbb{P}^5[3, 3]$

$$\sum_{i=1}^3 x_i^3 - 3\psi x_4 x_5 x_6 = \sum_{i=4}^6 x_i^3 - 3\psi x_1 x_2 x_3 = 0$$

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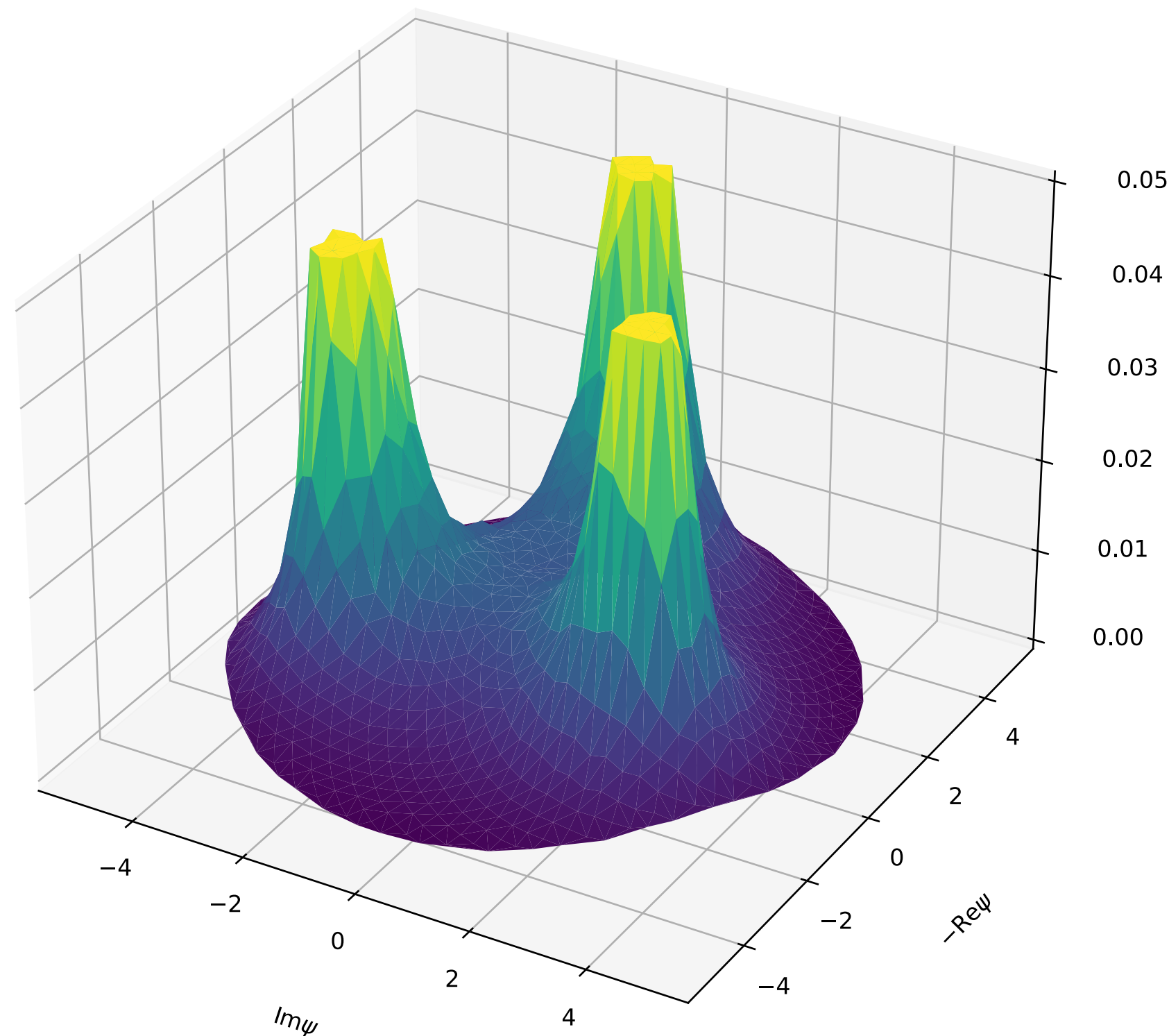
Tian–Yau

- Tian–Yau quotient

- Weil–Petersson metric along one direction of moduli

$$\frac{1}{3} \sum_{i=1}^4 x_i^3 + \psi x_1 x_2 x_3 = 0, \quad \epsilon = 0$$

$$g_{\lambda_1 \bar{\lambda}_1} (p_1 \rightarrow p_1 + \psi x_0 x_1 x_2)$$



$$\left[\begin{array}{c} \mathbb{P}^3 \\ \mathbb{P}^3 \end{array} \middle| \begin{array}{ccc} 3 & 0 & 1 \\ 0 & 3 & 1 \end{array} \right]_{\chi=-18}^{14, 23}$$

$$\frac{1}{3} \sum_{i=1}^4 x_i^3 = \frac{1}{3} \sum_{i=1}^4 y_i^3 = \sum_{i=1}^4 x_i y_i + \epsilon \sum_{i=3}^4 x_i y_i = 0$$

$$\begin{aligned} \mathbb{Z}_3 : \quad (x_1, x_2, x_3, x_4) &\mapsto (x_1, \omega_3^{-1} x_2, \omega_3 x_3, \omega_3 x_4) \\ (y_1, y_2, y_3, y_4) &\mapsto (y_0, \omega_3 y_2, \omega_3^{-1} y_3, \omega_3^{-1} y_4) \end{aligned}$$

$$h^{1,1} = 6, \quad h^{2,1} = 9, \quad \chi = -6$$

$$SU(3)^3 \text{ model, } 9\lambda\text{s, } 7Q\text{s, } 7Q^c\text{s}$$

Tian, Yau (1986)
Kalara, Mohapatra (1987)

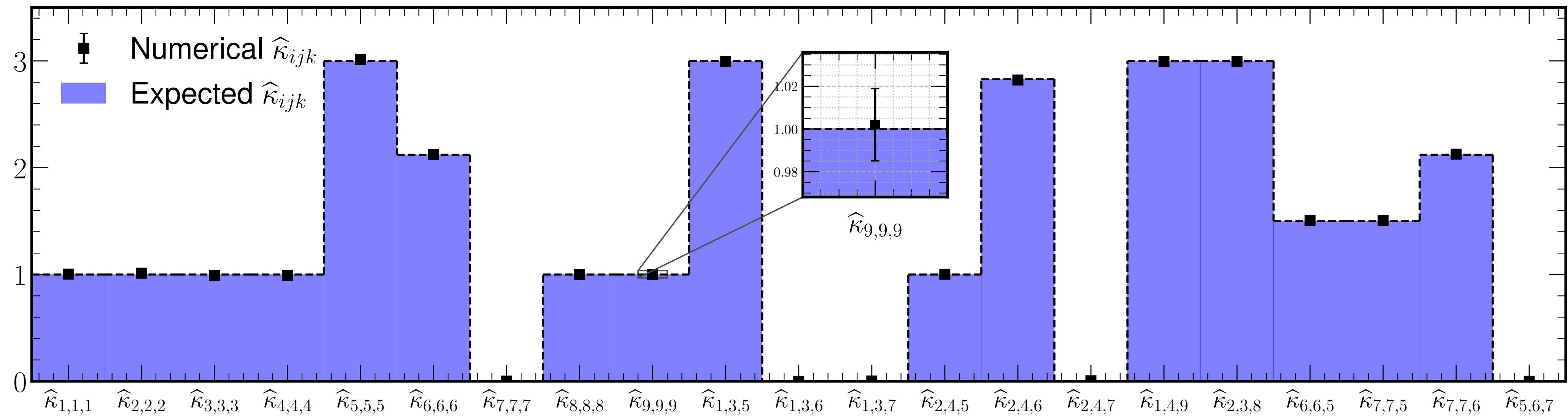
cf. Greene, Kirklan, Myron, Ross (1986)

Tian–Yau

- Tian–Yau quotient

$$\left[\begin{array}{c} \mathbb{P}^3 \\ \mathbb{P}^3 \end{array} \middle| \begin{array}{ccc} 3 & 0 & 1 \\ 0 & 3 & 1 \end{array} \right] / \mathbb{Z}_3$$

- Unnormalized Yukawa couplings



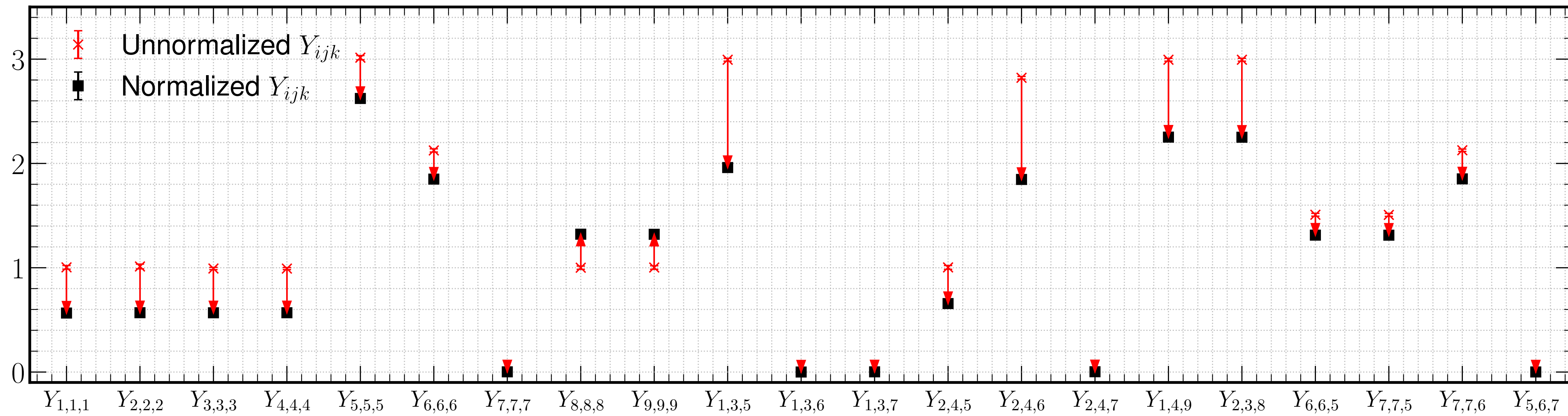
Candelas, Kalara (1988)

Tian–Yau

- Tian–Yau quotient

$$\left[\begin{array}{c} \mathbb{P}^3 \\ \mathbb{P}^3 \end{array} \middle| \begin{array}{ccc} 3 & 0 & 1 \\ 0 & 3 & 1 \end{array} \right] / \mathbb{Z}_3$$

- Physical Yukawa couplings



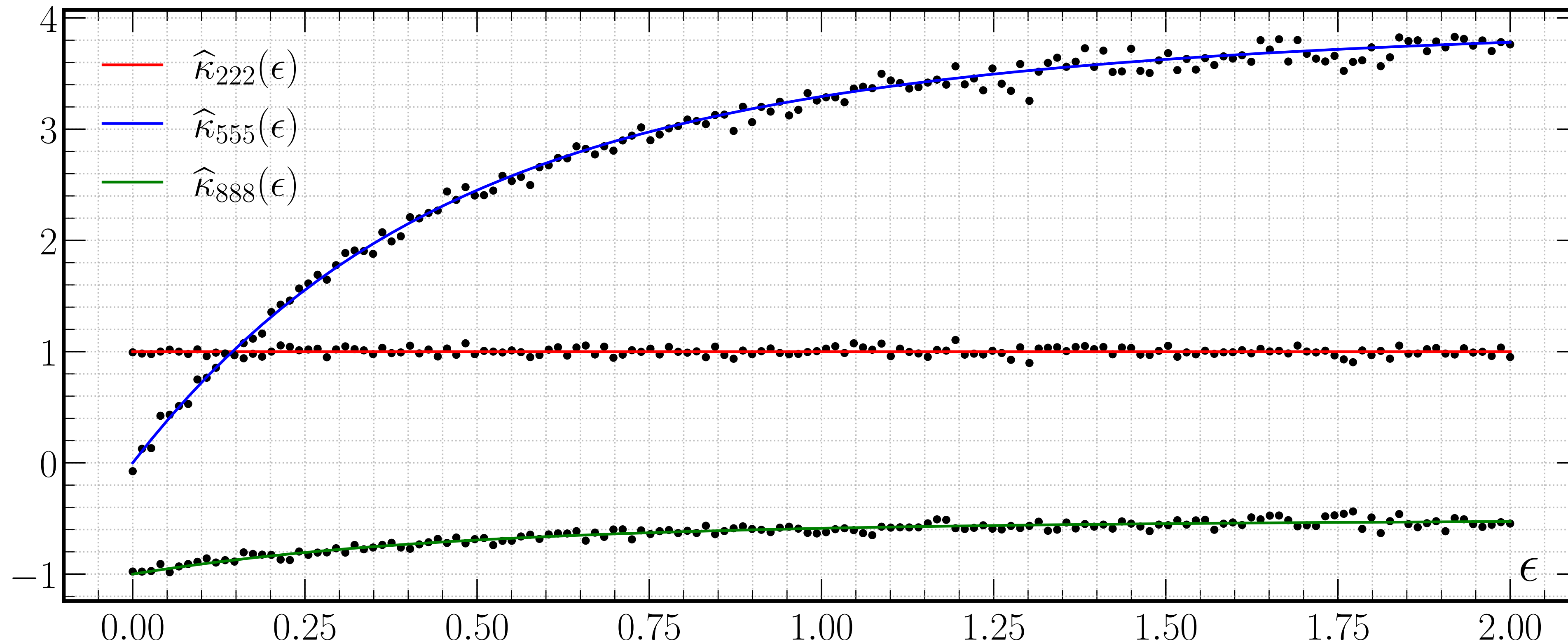
Tian–Yau

- Tian–Yau quotient

$$\left[\begin{array}{c} \mathbb{P}^3 \\ \mathbb{P}^3 \end{array} \middle| \begin{array}{ccc} 3 & 0 & 1 \\ 0 & 3 & 1 \end{array} \right] / \mathbb{Z}_3$$

- Physical Yukawa couplings as a function of moduli

$$x_0 y_0 + x_1 y_1 + (1 + \epsilon)(x_2 y_2 + x_3 y_3) = 0$$



Yukawa couplings

$$\hat{\kappa}_{iii} = \frac{\lambda_{iii}}{\lambda_{111}}$$

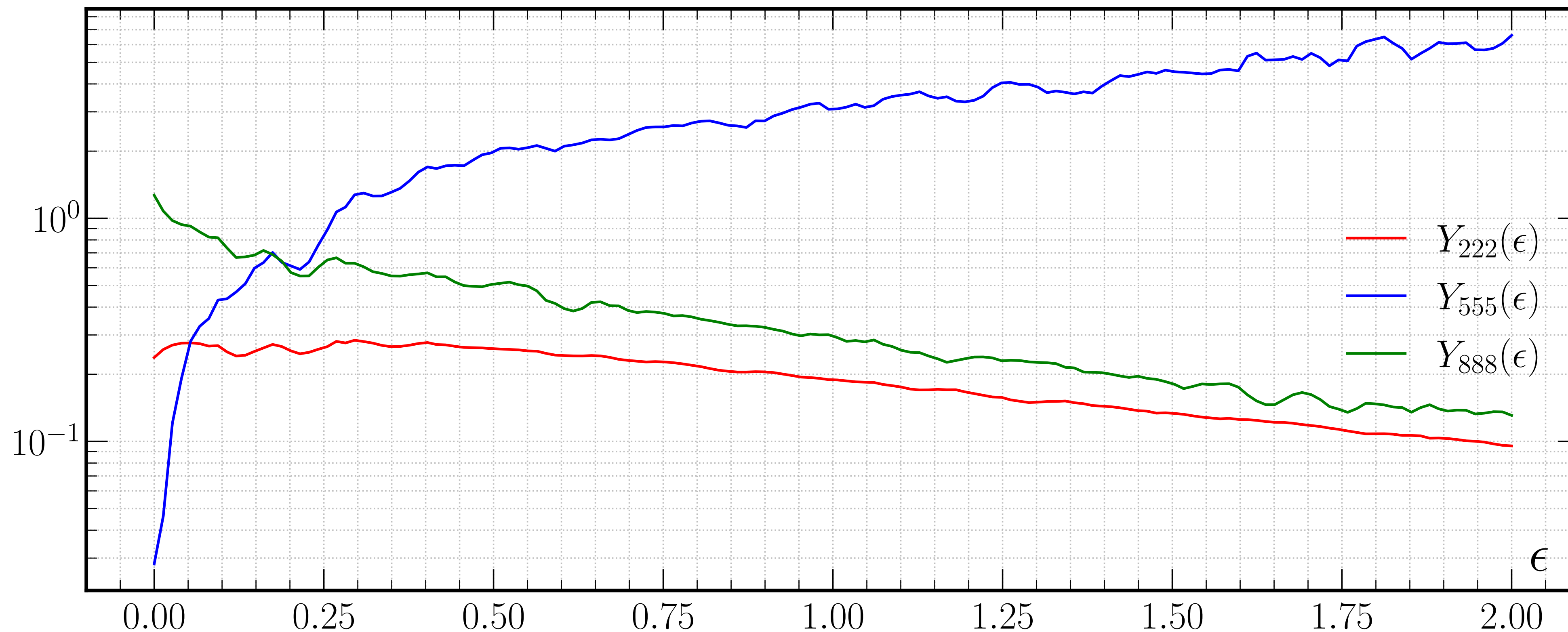
Tian–Yau

- Tian–Yau quotient

$$\left[\begin{array}{c} \mathbb{P}^3 \\ \mathbb{P}^3 \end{array} \middle| \begin{array}{ccc} 3 & 0 & 1 \\ 0 & 3 & 1 \end{array} \right] / \mathbb{Z}_3$$

- Physical Yukawa couplings as a function of moduli

$$x_0 y_0 + x_1 y_1 + (1 + \epsilon)(x_2 y_2 + x_3 y_3) = 0$$



Yukawa couplings

hierarchies arise upon
moving away from
symmetric points in
complex structure
moduli space

Large Complex Structure Limit

- Consider Dwork family: $X_\psi : \sum_i z_i^n - n\psi \prod_i z_i = 0$

$$t = \log |n\psi| \rightarrow \infty, \quad q = (n\psi)^{-n} \rightarrow 0 \quad \implies \quad X_\infty : \prod_i z_i = 0$$

- this is the maximally unipotent boundary point; on the mirror it is the large volume limit
- in this limit, we have a union of coordinate hyperplanes

- SYZ limit: $T^d \longrightarrow X_\psi \longrightarrow B$, $d = \dim_{\mathbb{C}} X$

Strominger, Yau, Zaslow (1996)
Li (2022, 2026)

- near LCSL the torus fibers collapse and base B carries limiting integral affine/tropical geometry

- Ambient monomial basis turns balanced metric entries into asymptotic observables

$|H_{ij}| \sim A_{ij} |n\psi|^{-\beta_{ij}}$ allowed entries survive as base data; forbidden entries decay as collapsing fiber data

organizes SYZ/tropical data

- Exponents assemble into a convex function $F(\xi)$; its Legendre transform is the tropical/SYZ potential

Modified Donaldson Algorithm

- What is right basis in which asymptotics become visible?
- In standard Donaldson algorithm, we work with basis of $H^0(X, \mathcal{O}(k)) \cong H^0(\mathbb{P}^n, \mathcal{O}(k)) / \ker \rho$
when $X \subset \mathbb{P}^n$ this space is obtained from ambient sections modulo defining relation
- For large enough k , the ambient monomials are no longer independent upon restriction to X ;
quotient basis natural for Donaldson's algorithm but geometrically hides canonical ambient monomials
- We keep ambient monomial basis $H^0(X, \mathcal{O}(k))$ and encode restriction to X with matrix $R : s^{\mathbb{P}}|_X = R s^X$
- Given ordinary balanced metric H^X on X , lift to ambient matrix $H^{\mathbb{P}} = (R^+)^{\dagger} H^X R^+$

Moore–Penrose pseudoinverse

$$\begin{aligned} RR^+R &= R \\ R^+RR^+ &= R^+ \\ (RR^+)^{\dagger} &= RR^+ \\ (R^+R)^{\dagger} &= R^+R \end{aligned}$$
- Retains the same Kähler potential on X : $R^{\dagger} H^{\mathbb{P}} R = H^X$

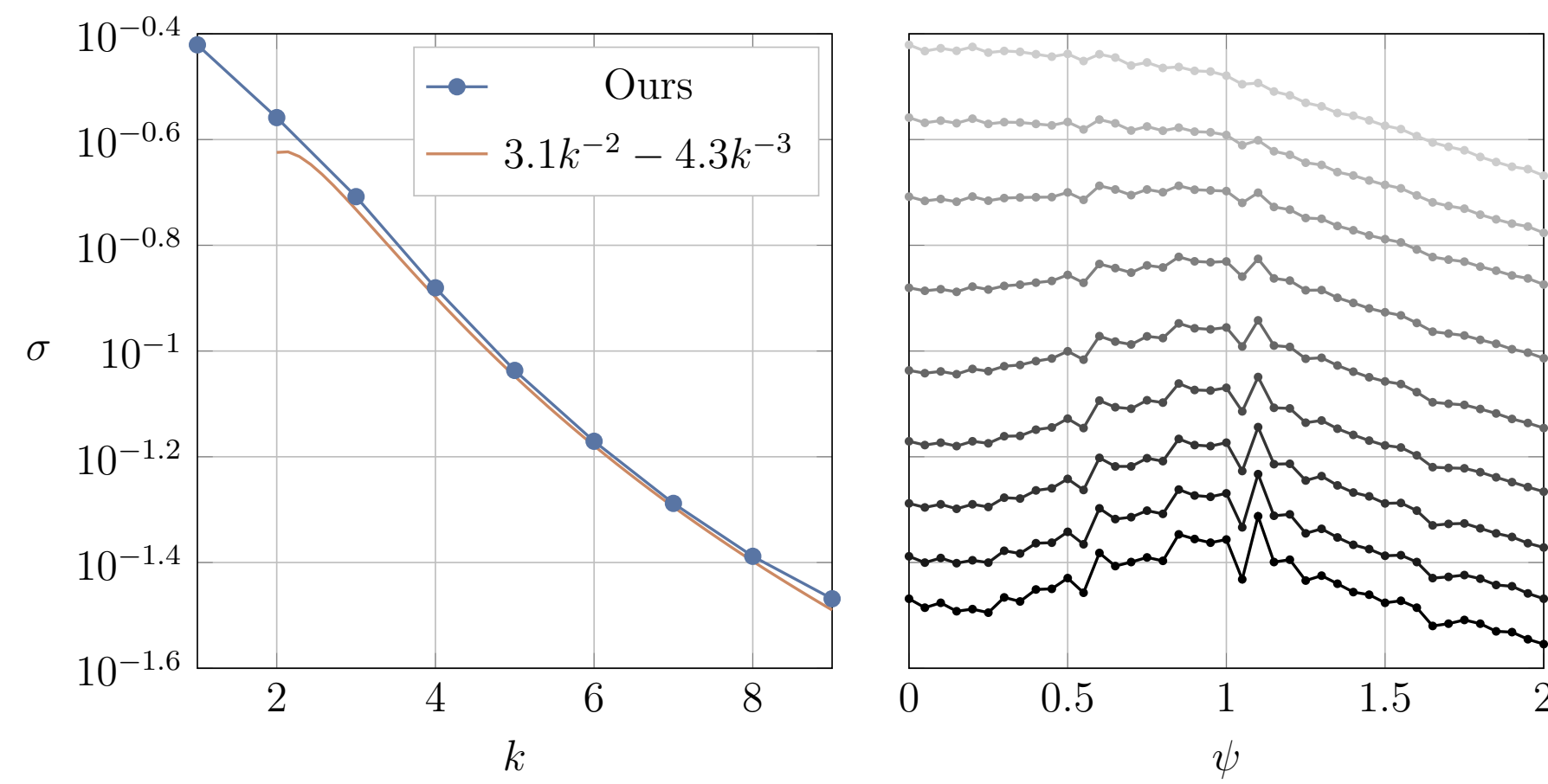
Modified Donaldson Algorithm

- We not only lift the final balanced metric, but the T operator itself: $T_X^+(G) = \frac{N^X}{\text{Vol}_\Omega} \int_X \text{dVol}_\Omega \frac{s^\mathbb{P}(s^\mathbb{P})^\dagger}{(s^\mathbb{P})^\dagger G + s^\mathbb{P}}$
 $G = H^{-1}$
- Key structural element is: $\lambda \circ T = T_X^+ \circ \lambda$, $\lambda(G) = RGR^\dagger$

- this means we have the same fixed point set after restriction to X as usual Donaldson iteration
- ambient method is not changing balanced metric problem, it is changing the representation in which we read off data

- Computationally, ambient matrices are larger, so we look at symmetry invariant sectors

i.e., $\mathbb{Z}_n^{n-2} \rtimes S_n$ for Dwork together with complex conjugation when $\psi \in \mathbb{R}$



σ measure scaling of quintic as sanity check
 9800 vs. 510 parameters for H^X vs. $H^\mathbb{P}$
 zerofold agrees to machine precision

cf. Douglas, Karp, Lukic, Reinbacher (2006)
 Headrick, Nasser (2009)

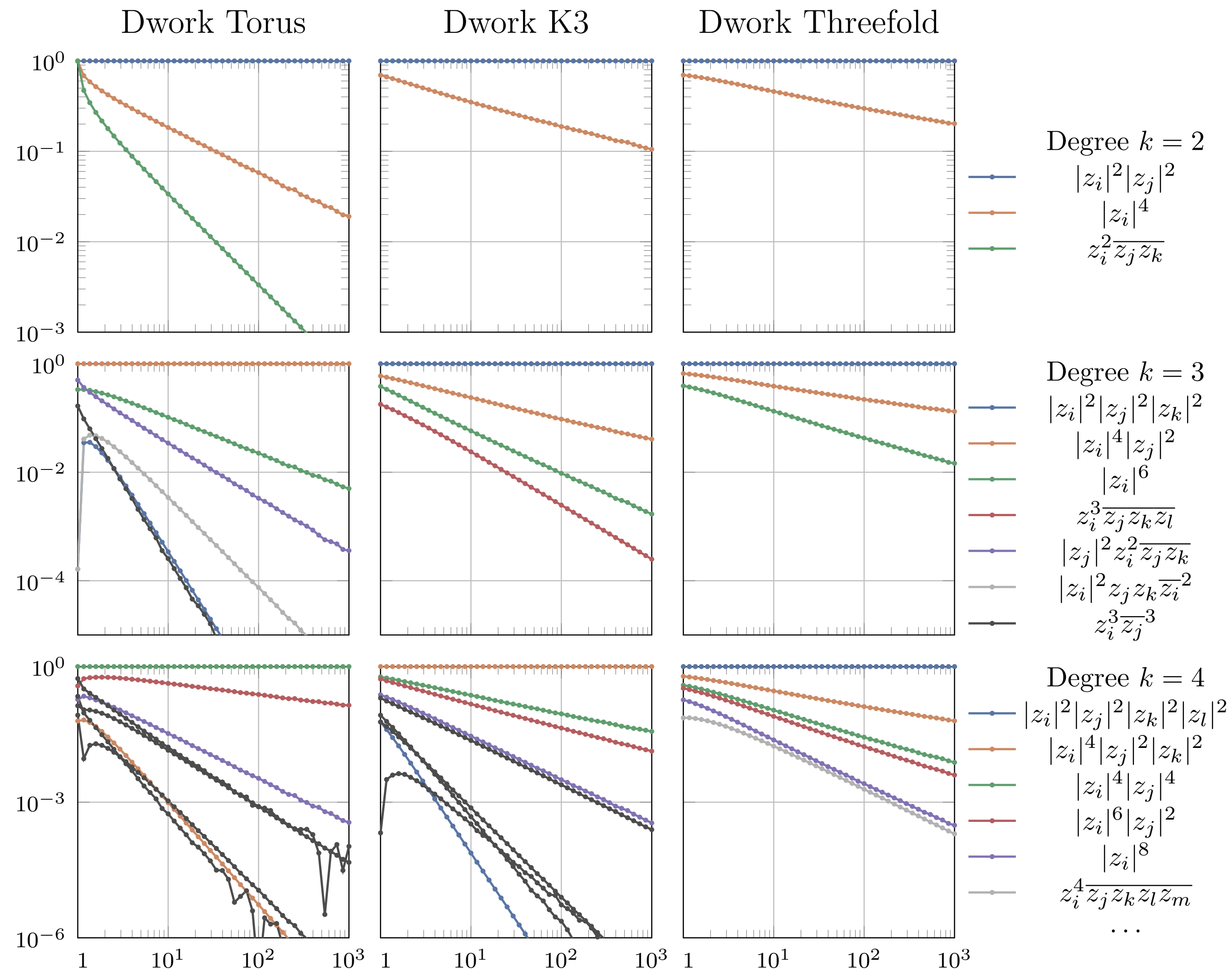
Power Laws

- Near $|\psi| \rightarrow \infty$, ambient coefficients obey $\log |H_{ij}| = a + b \log |n\psi|$, $|H_{ij}| \sim A |n\psi|^{-\beta}$
– *i.e.*, approximately linear behavior on log–log plot
- Key point is not that there are power laws but that exponents organize according to monomial labels

The entry $H[z^\alpha \overline{z}^\beta]$ is *allowed* if the vectors α and β are equal (so the entry is diagonal), and if there is some index i for which $\alpha_i = \beta_i = 0$ (so $z^\alpha \overline{z}^\beta$ does not vanish). Otherwise, the entry is *forbidden*.

- Allowed entries are those compatible with central degeneration $X_\infty : \prod_i z_i = 0$
- Forbidden entries either vanish on central fiber or correspond to off-diagonal/fiber directions
– decay as fast as $O(|\psi|^{-\min_i(\alpha_i + \beta_i)})$ or $O(|\psi|^{-1})$, whichever is stronger

Power Laws



- Consistent with SYZ expectation, base survives, fiber directions collapse

N.B.: $\psi \rightarrow \infty$, fixed k

all entries but one decay to zero

we take $k \rightarrow \infty$, Ricci-flat

- Equivalently, balanced metrics know about SYZ collapse because ambient coefficients scale differently in base and fiber directions

Allowed entries decay slowly — base data

Forbidden entries decay rapidly — collapsing fibers

SYZ/Tropical Potential

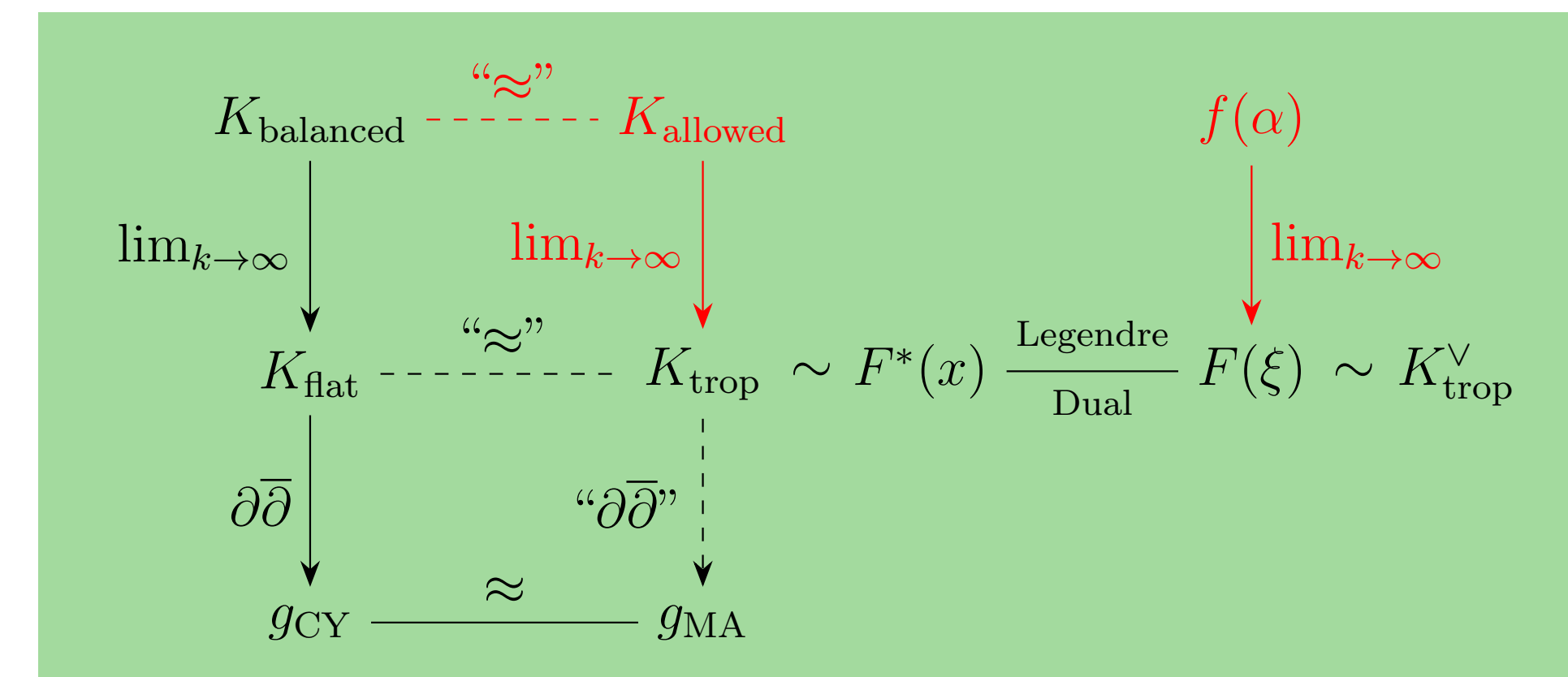
- Restrict attention to allowed coefficients because these encode limiting base geometry
- Allowed part of Kähler potential takes schematic form $K_{\text{allowed}} = \frac{1}{k} \log \sum_{|\alpha|=k}^{\text{allowed}} |n\psi|^{-f(\alpha)} |z^\alpha|^2$
- This looks like weighted sum over monomials; in large k -limit, treat monomial labels as continuous

$$\xi_i = \alpha_i/k, \quad u_i = \log |z_i|, \quad t = \log |n\psi|$$

$$F(\xi) = \lim_{k \rightarrow \infty} \frac{f(k\xi)}{k} \quad \longleftarrow \text{exponents define limiting convex function}$$

- Logarithm of sum becomes tropical maximum

$$K_{\text{trop}} = \max_{\xi} \{2u \cdot \xi - tF(\xi)\} = tF^*\left(\frac{2u}{t}\right)$$



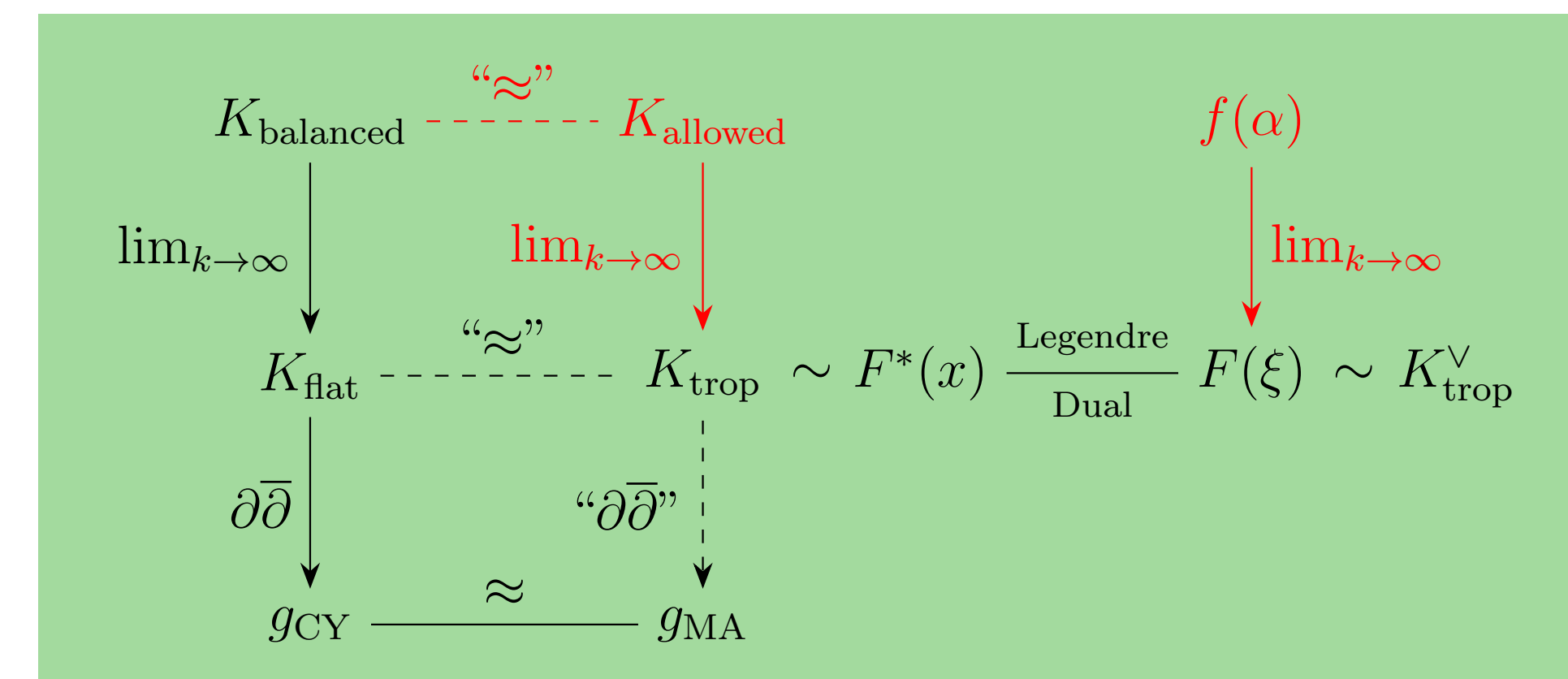
SYZ/Tropical Potential

- From the previous slide: $K_{\text{trop}} = tF^*\left(\frac{2u}{t}\right)$, where F^* is Legendre transform
- For Dwork torus $f(\alpha_i, \alpha_j) = \frac{\lfloor (\alpha_i - \alpha_j)^2/4 \rfloor}{\alpha_i + \alpha_j}$, which reproduces flat base behavior

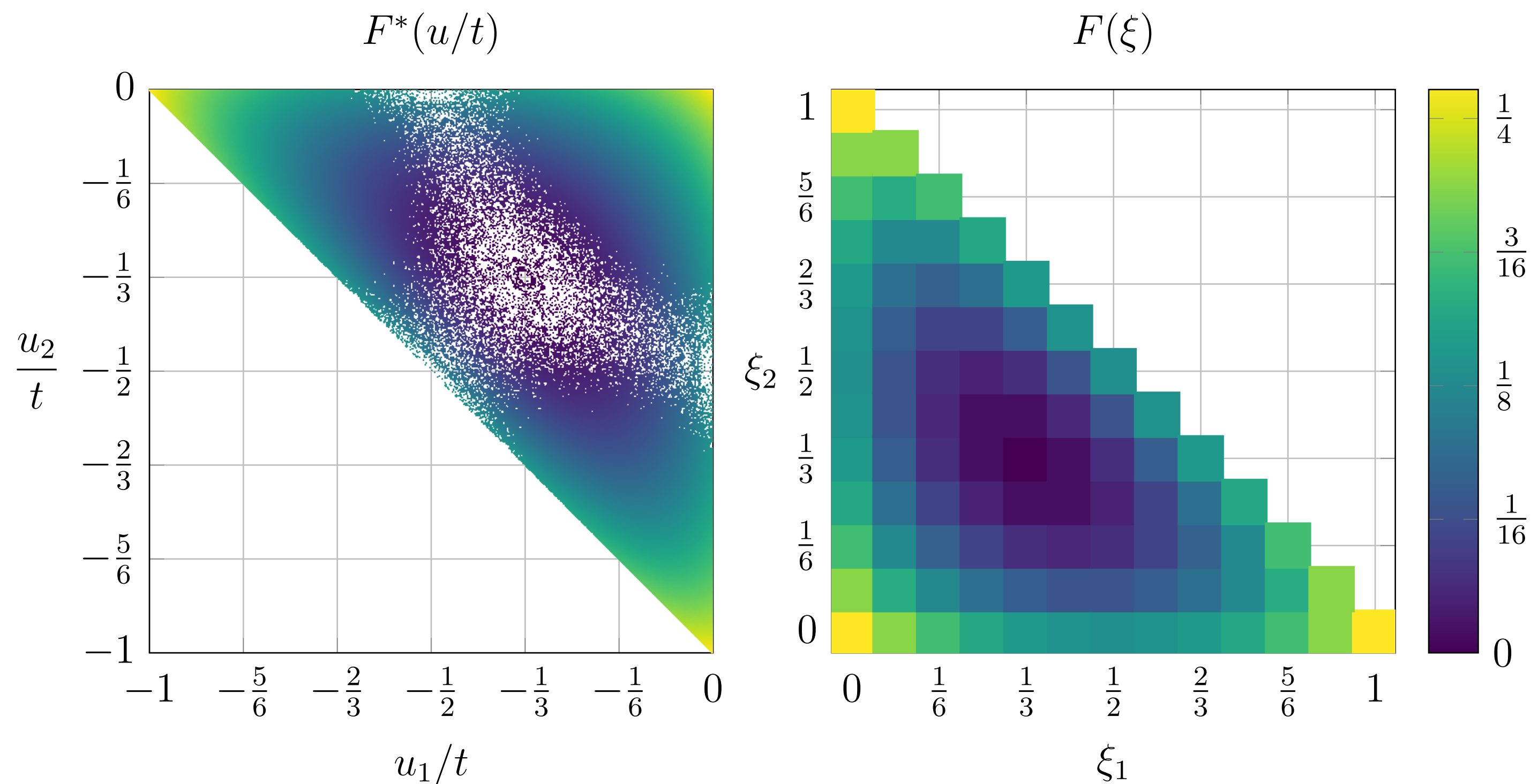
SLag fibers collapse at rate $D_{\text{fiber}}/D_{\text{base}} \sim 1/\log |\psi|$

- For K3 near midpoint singularity, $F(\xi, 1 - \xi, 0) = \frac{1}{8} + \frac{2}{3}r^2 + \frac{1}{6}r^2 \log |r| + O(r^4)$
 $r = \xi - \frac{1}{2}$

– the logarithm is the kind of singular correction one expects from 24 singular fibers in SYZ



Mirror Symmetry



- Two sides look similar, both close to quadratic; claim is Legendre duality, not equality
- See similar behavior (*viz.*, allowed/forbidden) power law structure for Cefalú quartic and bicubic threefold

Left: balanced $k = 12$ metric on X_{1000}

Right: power laws from balanced exponents from dual potential

Balanced Metrics Probe SYZ Limit

- When balanced metrics are expressed in ambient monomial basis, large complex structure asymptotics organize according to SYZ geometry
- Balanced metrics are not only numerical approximations, they encode degeneration data
- Pseudoinverse ambient algorithm does two things: preserves Donaldson's balanced metric construction and makes SYZ/tropical geometry visible
- To do: prove observed power laws from lifted T -map
 - identify limiting convex function F with known non-Archimedean or mirror potential
 - extend these observations to broader CICY and toric families

Balanced Metrics Probe SYZ Limit

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 - extend these observations to broader CICY and toric families

Code

```
import jax
from cymc.calabi_yau import *

key = jax.random.key(0)
keys = jax.random.split(key, 3)

cy = DworkQuintic(0.0)
pts = cy.sample_points(
    key = keys[1],
    max_pts = int(1e5))

metric = RicciFlatMetric(
    key = keys[2],
    cy = cy,
    tx = optax.adamw(1e-4))

metric.fit(pts, epochs=10)

e_X = metric.chern3(
    pts.generator(
        batch_size = 100,
        include_pullbacks = True),
    pts.kappa)
print(cy.integrate(pts = pts, func_vals = e_X))
~
~
~
~
~
~
~
```

Define CY
Sample points

Solve for
Ricci-flat metric

Compute χ

```
(cymc) gpu@gpu-P55A-UD3 ~ $ python3 metric.py
```

x 20

```
TERMINAL /bin/bash
```

```
in:1/171
```

```
[2] 0:vim*
```

```
"gpu-P55A-UD3" 20:25 10-Dec-24
```

cymc



Dwork quintic

Prospectus

What to do with a Ricci-flat Calabi–Yau metric?

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E-mail: per.berglund@unh.edu, thubsch@howard.edu, v.jejjala@wits.ac.za

ABSTRACT: Numerical approximations to Ricci-flat Calabi–Yau metrics make it possible to move beyond the topological and holomorphic data that have traditionally dominated explicit string compactifications. This article explains what new physics and mathematics become accessible once the metric, and eventually the associated Hermitian Yang–Mills bundle data, can be computed. In heterotic compactifications, such data are needed to determine matter Kähler metrics, canonically normalized Yukawa couplings, Kaluza–Klein spectra, threshold effects, soft terms, and other non-holomorphic ingredients of the four-dimensional effective action. More broadly, numerical Calabi–Yau geometry provides quantitative input for moduli stabilization, α' -corrected backgrounds, de Sitter model building, axion physics, swampland distance tests, and compactifications in which the internal geometry varies over spacetime. Geometric data permit a computational approach to long-standing mathematical questions involving special Lagrangian submanifolds, SYZ fibrations, mirror symmetry, calibrated geometry, metric degeneration, restrictions of Ricci-flat metrics to fibers, and the search for analytic or semi-analytic structures. We present these directions as a roadmap for future work.

THANK YOU!



science, technology
& innovation

Department:
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National
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Fundamental Interactions

