Universität Münster

Scalar quantum fields on 4-dimensional noncommutative geometry

Raimar Wulkenhaar

living.knowledge

CRC 1442 GEOMETRY: DEFORMATIONS AND RIGIDITY

MM Mathematics Münster Cluster of Excellence

Prehistory

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- In 1999, being postdoc at CPT Marseille, I considered applying for a Marie Curie Fellowship to continue the work on quantum fields and noncommutative geometry.
- This needed a partner. Clearly, an ideal host would be Harald Grosse because of his work with John Madore, Peter Prešnajder and others on this topic and his expertise in quantum physics in general.
- So I contacted Harald. Harald invited me to a visit at ESI in April 1999. We submitted in May 1999 a project "Noncommutative geometry and quantum field theory" in which we proposed to study the short-distance structure of space-time in the framework of noncommutative geometry and its impact on renormalisation of quantum fields.
- We got Marie Curie, and it was the starting point of an extremely pleasant and also quite successful collaboration.

In our project there have been many critical situations where Harald's advice and his remarkable ability to connect different research fields rescued me from making a wrong turn. In long and always fruitful discussions during workshops and mutual visits, Harald raised the central questions which always gave a fresh boost to our project.

Input from string theory

Recent developments changed the Marie Curie project completely:

- In August 1999, Seiberg and Witten published "String theory and noncommutative geometry", extending earlier work by Volker Schomerus. Their main messages are:
  - Open strings ending on D-branes in the presence of a B-field can effectively be described as Yang-Mills on noncommutative Moyal space.
  - There is a field redefinition between ordinary and noncommutative Yang-Mills.

  The Seiberg–Witten paper had an enormous impact; many people jumped into the field.

- Thomas Krajewski and I, and a few other groups. understood before that QFT models on noncommutative geometry are one-loop renormalisable. Only planar diagrams need to be renormalised; non-planar diagrams are UV-finite.

- Minwalla, van Raamsdonk and Seiberg noticed in November 1999 that, although finite, the amplitude of non-planar graphs grows when momenta approach zero (IR divergence). Inserting sufficiently many of them as subgraphs in a UV-superficially convergent graph makes the whole graph divergent.

They coined the term "UV/IR-mixing" for that phenomenon.

UV/IR-mixing

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- Take the action functional

$$S[\phi] = \int_{\mathbb{R}^4} dx \Big(\frac{1}{2}\phi(x)((-\Delta + m^2)\phi)(x) + \frac{\lambda}{4}(\phi \star \phi \star \phi \star \phi)(x) \Big)$$

  where $(f \star g)(x) = \int_{\mathbb{R}^4} \frac{dp}{(2\pi)^4} \frac{dq}{(2\pi)^4}\, e^{\mathrm{i}\langle p+q,x\rangle - \frac{\mathrm{i}}{2}\langle p,\Theta q\rangle} \hat{f}(p)\hat{g}(q)$ is the Moyal product.

- The phase $e^{-\frac{\mathrm{i}}{2}\langle p,\Theta q\rangle}$ cancels in planar graphs and leaves them unchanged. In non-planar graphs, the oscillations lead to convergence, but only for non-exceptional external momenta. But these are realised in loop integrations.
- Two impressive articles by Chepelev and Roiban proved that UV/IR-mixing is robust at any loop order. Only theories with at most logarithmic divergences, such as supersymmetric models, have a chance to be renormalisable.
- In Vienna we tried something in this direction, with limited success.

Seiberg-Witten map

- The situation changed when Harald was contacted by Manfred Schweda from TU Vienna. Schweda always had many young students and gave them hot arXiv topics.

  One group around Andreas Bichl, Jesper Grimstrup and Lukas Popp got noncommutative QFT as topic, and Manfred Schweda suggested to join forces.
- The young people came into contact with a group around Julius Wess which actively studied the Seiberg-Witten map.
- We had the idea to study loop correction in this setting. Harald obtained a distinguished Leibniz guest professorship at the University of Leipzig in that time and participated remotely in our effort.
- We produced 6 papers in 2001 on this topic which were relatively well received. We proved that some sectors of SW-expanded Yang-Mills theory are indeed renormalisable. But in the very end it became clear that it does not work for the whole theory.

A last attempt

- In 2002 I started a follow-up postdoc position at MPI Leipzig. After finishing the Vienna projects, Harald and I tried a last idea to escape the UV/IR-problem.
- Thomas Krajewski advertised Polchinski's renormalisation of usual ϕ^4_4 via renormalisation group methods. It proves rigorous bounds by induction, thus needs the right guess.
- One cannot hope to combine it with the oscillatory phase factors on Moyal space. We learned from José Gracia-Bondía that the Moyal algebra is isomorphic to an infinite matrix algebra, and the transformation is explicit by Laguerre polynomials.
- This brought us to guess scaling laws for matrix amplitudes and to check them by induction. The difficulty was that the Laplacian is no longer diagonal, and we needed bounds for the matrix kernel of $(-\Delta + m^2)^{-1}$.

The obtained bounds were insufficient for renormalisation — UV/IR-mixing reconfirmed!

Good news in November 2002

Lieber Harald,
es funktioniert wieder. Nicht wie bisher, dort bleibt \phi^4 nichtrenormierbar. Die Lösung steckt in den ortsabhängigen Massenrenormierungen, auf die ich zuvor gestoßen war. Wenn die Massenrenormierung sowieso ortsabhängig ist, dann kann man natürlich auch eine andere Normierungsbedingung im IR stellen. Die einfachste Wahl, die einem einfällt, ist die Ortsabhängigkeit, die einem Magnetfeldterm in der Wirkung entspricht. Und damit wird die Theorie renormierbar! Das Magnetfeld kann beliebig klein sein, aber eben nicht Null.

QFT on Moyal space with harmonic propagation

It is not quite a magnetic field, but a harmonic oscillator potential:

$$S[\phi] = \int_{\mathbb{R}^4} dx \Big(\frac{1}{2} \phi(x) \big((-\Delta + \Omega^2 \|2\Theta^{-1} x\|^2 + m^2) \phi \big)(x) + \frac{\lambda}{4} (\phi \star \phi \star \phi \star \phi)(x) \Big)$$

Remarks and connections

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$$S[\phi] = \int_{\mathbb{R}^4} dx \Big(\frac{1}{2}\phi(x)((-\Delta\Omega^2\|2\Theta^{-1}x\|^2 + m^2)\phi)(x) + \frac{\lambda}{4}(\phi\star\phi\star\phi\star\phi)(x) \Big)$$

- Later we understood that the appearance of $\|2\Theta^{-1}x\|^2$ is suggested by a duality between direct and Fourier space found by Edwin Langmann and Richard Szabo in 2002. This duality leaves $\phi\star\phi\star\phi\star\phi$ invariant and exchanges $\phi(-\Delta)\phi$ and $\phi\|2\Theta^{-1}x\|^2\phi$.
- In 2003, Edwin Langmann, Richard Szabo and Konstantin Zarembo studied the magnetic field situation and came to a complex matrix model

$$\mathcal{Z}[M,M^\dagger] = \int_{H_N} dMdM^\dagger e^{-S[M,M^\dagger]}\;, \qquad S[M,M^\dagger] = \mathrm{Tr}(EM^\dagger M + \tilde{E}MM^\dagger + P(M^\dagger M))$$

which they proved to be exactly solvable. Years later $\Omega = 1$ brought us close to this.
- In 2005/06, Harald Grosse and Harold Steinacker pointed out that $\Omega = 1$ together with ϕ^3-potential can be mapped to the Kontsevich model with added exact renormalisation in dimension 2,4,6.
- We (HG+RW) revisited this model with Akifumi Sako in 2016/17 and Alex Hock in 2019.

Completing the renormalisation

- In May 2003 we finished a Polchinski-type renormalisation proof for perturbative amplitudes characterised by genus of a Riemann surface, number of boundary components and specified number of legs attached to each boundary component. Only input was a general scaling behaviour of the partially summed matrix covariance.
- Shortly later we completed a paper which established this scaling behaviour for Φ^4 on 2D Moyal and proved its renormalisability. One can even send $\Omega \to 0$ in the end.
- The scaling alone is not enough to renormalise Φ^4 on 4D Moyal. One needs to prove that the difference of two (divergent) planar 4-point functions (with different matrix indices) is convergent! Similar for the planar 2-point function.
- Harald attended a talk at ESI where Jacobi matrices appeared, i.e special infinite tridiagonal matrices. Harald understood immediately that this is important for us.
- During my visit at ESI we found exact formulae for our matrix covariance in terms of Meixner orthogonal polynomials. Permitted to prove convergence of above differences.

In January 2004 the renormalisation of Φ^4 on 4D Moyal was complete.

The β-function

- Φ^4 on 4D Moyal is characterised by (in RG-language) the relevant mass m and marginal wave function Z, coupling constant λ and oscillator frequency Ω.
- Thanks to Meixner polynomials we were able to compute the one-loop RG-equations exactly. These implied that $\frac{\lambda}{\Omega^2}$ is constant!

Since Ω runs into the fixed point $\Omega_{UV} = 1$, this proves that λ has a finite UV-fixed point $\lambda_{UV} = \frac{\lambda_{IR}}{\Omega^2_{IR}}$. Thus, $\Omega_{IR} \neq 0$ to cure UV/IR also tames the Landau ghost!

- Vincent Rivasseau became excited about an asymptotically safe scalar 4D QFT and joined this project with many of his students. With Margherita Disertori he confirmed that for $\Omega \equiv 1$, which simplifies enormously, λ_{UV} is unchanged to 3-loop order.
- They conjectured some algebraic structure. Using techniques they learned from Vieri Mastropietro, in December 2006, Disertori, Gurau, Magnen and Rivasseau found a Ward identity that allows to prove (still perturbatively) that the divergent part of the 4-point function comes alone from the divergent wave function renormalisation. The coupling constant does not need a renormalisation.

Progress in solving a NCQFT in four dimensions

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When I showed Harald the preprint, Harald said: Now we must solve the model!

- We discussed this and made modest progress every time we met. Decisive progress came during my first sabbatical in 2009 which I spent at ESI.
- We showed that the techniques of Disertori, Gurau, Magnen & Rivasseau can be refined to obtain a closed non-linear equation for the renormalised planar two-point function alone!
- Our preprint in September 2009 (never submitted) already suggested a limit to continuous matrix indices and integral equations which later became instrumental.
- We later understood that one better keeps renormalisation functions $M(\Lambda), Z(\Lambda)$ (divergent for $\Lambda \to \infty$). The non-linear integral equation then reads ($\varrho_0(t) = t$ for 4D Moyal at large deformation)

$$\Big(\zeta+\eta+M^2+\lambda\int_0^{\Lambda^2} dt\,\varrho_0(t) Z G^{(0)}(\zeta,t)\Big) Z G^{(0)}(\zeta,\eta) = 1+\lambda\int_0^{\Lambda^2} dt\,\varrho_0(t)\frac{Z\big(G^{(0)}(t,\eta)-G^{(0)}(\zeta,\eta)\big)}{t-\zeta}$$

Further development

- Until 2012 we developed a more analytical approch which gave
  - equations for any correlation function at any genus,
  - the proof that some of them satisfy an algebraic recursion and
  - a representation of the planar 2-point function as a Hilbert transform.
- The latter allowed to reduce the problem to a non-linear equation for $G^{(0)}(\zeta, 0)$. Via fixed point theorems we proved existence of a solution for $\lambda > 0$ and later in 2015 for $\lambda < 0$.
- In a Research-in-Teams project at ESI in 2013 with Gandalf Lechner we translated the results to position space and investigated under which condition the 2-point function (for infinite deformation) is reflection positive. This favoured $\lambda < 0$.
- Much effort was put into the investigation of reflection positivity, with little outcome.
- With Erik Panzer in 2018 I succeeded in solving the equation for the planar 2-point function of 2D Moyal space exactly in terms of the Lambert-W function.
- With my students Jins de Jong and Alex Hock I found in 2019 the solution of the algebraic recursion of all planar correlation functions in terms of nested Catalan tables.

Solution of all quartic matrix models

- Alex Hock pointed out that an intermediate step of my solution with Erik Panzer is of the form $\mathrm{Re}(-f(z)+\lambda\log(-f(z)))$, where $f(z)$ solves $z=f(z)+\lambda\log(1+f(z))$
- I understood immediately which ansatz should be made in the general quartic case in any dimension

Theorem [Panzer-W 18 for $\varrho_0=1$, Grosse-Hock-W 19a]

1. Ansatz $G^{(0)}(x,y)=\dfrac{e^{\mathcal{H}_x[\tau_y(\bullet)]}\sin\tau_y(x)}{Z\lambda\pi\varrho_0(x)}$, $\qquad \mathcal{H}_x[f]:=\frac{1}{\pi}⨍_0^{\Lambda^2}\frac{dp\,f(p)}{p-x}$ finite Hilbert transf.
2. $\tau_y(x)=\mathrm{Im}\log\big(y+I(x+\mathrm{i}\epsilon)\big)$ with $I(\zeta)=-R_D(-m^2-R_D^{-1}(\zeta))$
3. $$\boxed{R_D(z)=z-\lambda(-z)^{D/2}\int_0^\infty\frac{dt\;\varrho_\lambda(t)}{(m^2+t)^{D/2}(t+m^2+z)}}\qquad D=2[\tfrac{d_{\mathrm{spectral}}}{2}]$$
4. ϱ_λ is implicit solution of $\varrho_0(R_D(z))=\varrho_\lambda(z)$.

$D=4$ Moyal space: $\varrho_0(t)=t$ [Grosse-Hock-W 19b]

We planned to announce the result at a workshop in York organised by Kasia Rejzner.

- But we seemed to run into a problem on 4D Moyal space where
$$\varrho_\lambda(z)\equiv\varrho_0(R_4(z))=R_4(z)=z-\lambda z^2\int_0^\infty\frac{dt\;\varrho_\lambda(t)}{(m^2+t)^2(m^2+t+z)}\,.$$
If $\varrho_\lambda(t)\sim\varrho_0(t)=t$, then $R_4(z)$ bounded above. Consequently, R_4^{-1} would not be globally defined: triviality!

- Both via differential equations pursued by Harald and Alex' resummation of the perturbative series we proved that the Fredholm equation is solved by

$$R_4(z)\equiv\varrho_\lambda(z)=z\cdot\;{}_2F_1\left(\begin{matrix}\alpha_\lambda,1-\alpha_\lambda\\2\end{matrix}\middle|-\frac{z}{m^2}\right)\qquad\alpha_\lambda=\begin{cases}\dfrac{\arcsin(\lambda\pi)}{\pi} & \text{for } |\lambda|\leq\frac{1}{\pi}\\[1ex]\frac{1}{2}+\mathrm{i}\dfrac{\operatorname{arcosh}(\lambda\pi)}{\pi} & \text{for } \lambda\geq\frac{1}{\pi}\end{cases}$$

Corollary

The interaction alters the *effective* spectral dimension to $4-\frac{2}{\pi}\arcsin(\lambda\pi)$ and thus avoids the triviality problem (in the planar sector).

(Blobbed) topological recursion

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- Eynard and Orantin in 2007 understood that many matrix models are governed by a universal structure which they called topological recursion. Starting from ramified coverings $x, y : \Sigma \to \mathbb{P}^1$ of Riemann surfaces and the Bergman kernel $\omega_{0,2}$ $\left(= \frac{dz_1 dz_2}{(z_1-z_2)^2}\right.$ for $\left.\Sigma = \mathbb{P}^1\right)$ one recursively builds a family $\omega_{g,n}$ of meromorphic differentials on Σ^n.
- My students Johannes Branahl, Alex Hock and I established in 2020 that for the quartic model (and $N \times N$-matrices) the appropriate definition is:

$$\Omega^{(g)}_{q_1,\dots,q_n} := \frac{(-N)^{n-1}\partial^{n-1}\left(\frac{1}{N}\sum_{k=1}^{N} G^{(g)}_{|kq_1|} + G^{(g-1)}_{|q_1|q_1|}\right)}{\partial E_{q_2}\cdots\partial E_{q_n}} + \frac{\delta_{g,0}\delta_{n,2}}{(E_{q_1}-E_{q_2})^2}$$

$$G_{|ab|} = \frac{N}{\mathcal{Z}}\int_{H_N} d\Phi\, e^{-N\,\mathrm{Tr}(E\Phi^2+\frac{\lambda}{4}\Phi^4)}\Phi_{ab}\Phi_{ba} = \sum_{g=0}^{\infty} N^{-2g}\, G^{(g)}_{|ab|}$$

$$G_{|a|b|} = \frac{N^2}{\mathcal{Z}}\int_{H_N} d\Phi\, e^{-N\,\mathrm{Tr}(E\Phi^2+\frac{\lambda}{4}\Phi^4)} = \sum_{g=0}^{\infty} N^{-2g}\, G^{(g)}_{|a|b|}$$

- A translation of $\Omega^{(g)}_{q_1,\dots,q_n}$ to meromorphic differentials $\omega_{g,n}(z_1,\dots,z_n)$ obeys an extension called blobbed topological recursion. The $\omega_{g,n}(z_1,\dots,z_n)$ are, in principle, computable. From them any correlation function of the quartic model is found.

What next?

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- Blobbed topological recursion breaks down in the limit $N \to \infty$ and 4D. The spectral curve is no longer algebraic, there are phase transitions with the ramification points of x.
- We think that analytic methods must be used. We perfectly understand the planar sector of the model: All planar cumulants are known explicitly, and they behave effectively as a scalar theory in dimension $4 - \frac{2}{\pi}\arcsin(\lambda\pi)$.
  - Can one understand this planar sector as equilibrium limit of a stochastic process?
  - Is there a corresponding SPDE for this process?
  - Does its regularity reflect the dimension drop to $4 - \frac{2}{\pi}\arcsin(\lambda\pi)$?
  - Can the powerful SPDE methods Hairer, Gubinelli and others be used to construct the full theory as perturbation of the planar sector?

This will probably keep us busy until the 88th birthday.

Happy birthday, Harald